COUPLING COEFFICIENTS FOR CUBIC GROUPS. III. THE DOUBLE TETRAHEDRAL GROUP

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A method for the determination of Clebsch-Gordan coefficients in terms of spherical $3-j$ Wigner symbols is applied to the double tetrahedral group T′. A complete set of $3j\Gamma\gamma$ symbols, which allows one to determine the numerical value of any Clebsch-Gordan coefficient for this group, is proposed and appropriate permutation matrices are derived.

1. Introduction

In a previous paper Lulek [1] proposed a general algorithm for the determination of Clebsch-Gordan coefficients (CGC's) for any point group G in terms of CGC's for the group SU_2. This method has been applied in a paper of Lulek et al. [2] (denoted as I) to the double octahedral group O′. Next, in the paper of Lulek and Lulek [3] (denoted as II), permutational properties of appropriate symmetric coupling coefficients (the so called $3\Gamma\gamma$ symbols) were discussed. The aim of the present paper is a similar discussion for the double tetrahedral group T′, which essentially completes the case of cubic groups. A new feature characterising the group T′ in comparison with O′ is existence of complex representations which stimulates a careful distinguishing between Γ and Γ^*, and consequently between labels for a CGC $(\Gamma_1\Gamma_2\Gamma_3)$ and for the corresponding $3j\Gamma\gamma$ or $3\Gamma\gamma$ symbol $(\Gamma_1\Gamma_2\Gamma_3^*)$. In the following we use notation introduced in I and II.

2. Standard bases for irreducible representations and metric tensors

We use, according to Griffith [4], the following notation for irreducible representations of the group T′: A, $E1$, $E2$, T, E', E'', E'''. The standard bases $|j\Gamma\gamma\rangle$ (cf. Eq. (1) of I) are defined by

$$|0Aa\rangle = |0, 0\rangle,$$

$$|2E1e_1\rangle = \frac{1}{\sqrt{2}}|2, 0\rangle + \frac{i}{2}(|2, -2\rangle + |2, 2\rangle),$$

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$$|2E2e_2\rangle = \frac{1}{\sqrt{2}}|2,0\rangle - \frac{i}{2}(|2,-2\rangle + |2,2\rangle),$$

$$|1T-1\rangle = |1,-1\rangle, \quad |1T0\rangle = |1,0\rangle, \quad |1T1\rangle = |1,1\rangle,$$

$$|\tfrac{1}{2}E'\alpha'\rangle = |\tfrac{1}{2},\tfrac{1}{2}\rangle, \quad |\tfrac{1}{2}E'\beta'\rangle = |\tfrac{1}{2},-\tfrac{1}{2}\rangle,$$

$$|\tfrac{3}{2}E''\alpha''\rangle = -\frac{1}{\sqrt{2}}(i|\tfrac{3}{2},-\tfrac{3}{2}\rangle + |\tfrac{3}{2},\tfrac{1}{2}\rangle),$$

$$|\tfrac{3}{2}E''\beta''\rangle = \frac{1}{\sqrt{2}}(|\tfrac{3}{2},-\tfrac{1}{2}\rangle + i|\tfrac{3}{2},\tfrac{3}{2}\rangle),$$

$$|\tfrac{3}{2}E'''\alpha'''\rangle = \frac{1}{\sqrt{2}}(i|\tfrac{3}{2},-\tfrac{3}{2}\rangle - |\tfrac{3}{2},\tfrac{1}{2}\rangle),$$

$$|\tfrac{3}{2}E'''\beta'''\rangle = \frac{1}{\sqrt{2}}(|\tfrac{3}{2},-\tfrac{1}{2}\rangle - i|\tfrac{3}{2},\tfrac{3}{2}\rangle). \tag{1}$$

TABLE I

Metric tensors $[-1]^{\Gamma-\gamma}$ for the standard bases of irreducible representations of T′

	A	$E1$	$E2$	T			E'		E''		E'''	
γ	a	e_1	e_2	-1	0	1	α'	β'	α''	β''	α'''	β'''
$-\gamma$	a	e_2	e_1	1	0	-1	β'	α'	β'''	α'''	β''	α''
$[-1]^{\Gamma-\gamma}$	1	1	1	1	-1	1	1	-1	1	-1	1	-1

Components $[-1]^{\Gamma-\gamma}$ of the corresponding metric tensors together with the definitions of $-\gamma$ are given in Table I (cf. Eqs. (2) of I). Eq. (1) provides the decomposition coefficients $a^{jm}_{\Gamma v\gamma}$ for the lowest value of j for which a given Γ appears. We can assume that for any j

$$a^{jm}_{\Gamma v\gamma} = \sum_{\gamma_0} a^{jm}_{\Gamma_0 v_0 \gamma_0} a^{\Gamma_0\gamma_0}_{\Gamma\gamma}, \tag{2}$$

where the index 0 denotes the double octahedral group O′, which is the intermediate subgroup in the chain $SU_2 \to O' \to T'$. In this case $v = (\Gamma_0, v_0)$, i.e. repetition indices of the first kind for T′ are related to representations and appropriate repetition indices for the intermediate subgroup O′. The bases

$$|\Gamma_0\Gamma\gamma\rangle = \sum_{\gamma_0} a^{\Gamma_0\gamma_0}_{\Gamma\gamma}|\Gamma_0\gamma_0\rangle \tag{3}$$

are given by

$$|A_1Aa\rangle = |A_1a_1\rangle,$$

$$|A_2Aa\rangle = |A_2a_2\rangle,$$

$$|EE1e_1\rangle = \frac{1}{\sqrt{2}}(|E\theta\rangle + i|E\varepsilon\rangle),$$

$$|EE2e_2\rangle = \frac{1}{\sqrt{2}}(|E\theta\rangle - i|E\varepsilon\rangle),$$

$$|T_iT\gamma\rangle = |T_i\gamma\rangle, \quad i = 1, 2, \quad \gamma = 0, \pm 1,$$

$$|E'E'\alpha'\rangle = |E'\alpha'\rangle, \quad |E'E'\beta'\rangle = |E'\beta'\rangle,$$

$$|E''E'\alpha'\rangle = |E''\alpha''\rangle, \quad |E''E'\beta'\rangle = |E''\beta''\rangle,$$

$$|U'E''\alpha''\rangle = -\frac{1}{\sqrt{2}}(|U'\lambda\rangle + i|U'\nu\rangle),$$

$$|U'E''\beta''\rangle = \frac{1}{\sqrt{2}}(i|U'\kappa\rangle + |U'\mu\rangle),$$

$$|U'E'''\alpha'''\rangle = -\frac{1}{\sqrt{2}}(|U'\lambda\rangle - i|U'\nu\rangle),$$

$$|U'E'''\beta'''\rangle = \frac{1}{\sqrt{2}}(-i|U'\kappa\rangle + |U'\mu\rangle). \tag{4}$$

Eqs (2)—(4) allow one to determine the decomposition coefficients $a^{jm}_{\Gamma v\gamma}$ for the chain $SU_2 \to T'$ in terms of known coefficients $a^{jm}_{\Gamma_0 v_0 \gamma_0}$ for $SU_2 \to O'$ (Griffith [4]). E.g. the basis $|2T\gamma\rangle$, which is necessary for the determination of CGC's for the triad TTT by the method proposed in [1], is given by

$$|2T-1\rangle = -|2, 1\rangle, \quad |2T0\rangle = \frac{1}{\sqrt{2}}(-|2, -2\rangle + |2, 2\rangle), \quad |2T1\rangle = |2, -1\rangle. \tag{5}$$

TABLE II

Decomposition coefficients $a^{jm}_{\Gamma v\gamma}$ for the chain $SU_2 \to O' \to T'$ for $j = 3$

$j=3$ \ m	A	$T_{v=T_1}$			$T_{v=T_2}$		
	a	-1	0	1	-1	0	1
-3				$-\sqrt{5}/2\sqrt{2}$	$-\sqrt{3}/2\sqrt{2}$		
-2	$-1/\sqrt{2}$					$1/\sqrt{2}$	
-1		$-\sqrt{3}/2\sqrt{2}$					$\sqrt{5}/2\sqrt{2}$
0			1				
1				$-\sqrt{3}/2\sqrt{2}$	$\sqrt{5}/2\sqrt{2}$		
2	$1/\sqrt{2}$					$1/\sqrt{2}$	
3		$-\sqrt{5}/2\sqrt{2}$					$-\sqrt{3}/2\sqrt{2}$

Complete sets of decomposition coefficients for $j = 3$ and $5/2$ are given in Table II and III, respectively. These Tables are useful for numerical checking of the orthogonality properties of $3j\Gamma\gamma$ symbols for the group T′. Together with Eqs. (1) and (5) they provide a complete set for the reduction related to the chain $SU_2 \to O' \to T'$.

TABLE III

Decomposition coefficients $a^{jm}_{\Gamma\gamma}$ for the chain $SU_2 \to O' \to T'$ for $j = 5/2$

$j = 5/2$ \ m	E'		E''		E'''	
	α'	β'	α''	β''	α'''	β'''
$-5/2$		$1/\sqrt{2\cdot 3}$		$-i\sqrt{5}/2\sqrt{3}$		$i\sqrt{5}/2\sqrt{3}$
$-3/2$	$-\sqrt{5}/\sqrt{2\cdot 3}$		$-i/2\sqrt{3}$		$i/2\sqrt{3}$	
$-1/2$				$-1/\sqrt{2}$		$-1\sqrt{2}$
$1/2$			$1/\sqrt{2}$		$-1/\sqrt{2}$	
$3/2$		$-\sqrt{5}/\sqrt{2\cdot 3}$		$-i/2\sqrt{3}$		$i/2\sqrt{3}$
$5/2$	$1/\sqrt{2\cdot 3}$		$-i\sqrt{5}/2\sqrt{3}$		$i\sqrt{5}/2\sqrt{3}$	

3. 3jΓγ symbols and Clebsch-Gordan coefficients

The $3j\Gamma\gamma$ symbols, defined by the formula

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ \Gamma_1 v_1 & \Gamma_2 v_2 & \Gamma_3 v_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix} = \sum_{m_1,m_2,m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} (a^{j_1m_1}_{\Gamma_1v_1\gamma_1} a^{j_2m_2}_{\Gamma_2v_2\gamma_2} a^{j_3m_3}_{\Gamma_3v_3\gamma_3})^*, \tag{6}$$

possess, by the definition, the following properties under a permutation $\sigma(B/A)$ of their columns:

$$\begin{pmatrix} j_b & j_{b'} & j_{b''} \\ \Gamma_b v_b & \Gamma_{b'} v_{b'} & \Gamma_{b''} v_{b''} \\ \gamma_b & \gamma_{b'} & \gamma_{b''} \end{pmatrix} = (-1)^{f(\sigma(B/A),j_1,j_2,j_3)} \begin{pmatrix} j_a & j_{a'} & j_{a''} \\ \Gamma_a v_a & \Gamma_{a'} v_{a'} & \Gamma_{a''} v_{a''} \\ \gamma_a & \gamma_{a'} & \gamma_{a''} \end{pmatrix}, \tag{7}$$

where

$$f(\sigma(B/A), j_1, j_2, j_3) = \begin{cases} j_1+j_2+j_3 & \text{when } \sigma \text{ is an odd permutation} \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

and under time reversal:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ \Gamma_1 v_1 & \Gamma_2 v_2 & \Gamma_3 v_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix} = [-1]^{\Gamma_1-\gamma_1+\Gamma_2-\gamma_2+\Gamma_3-\gamma_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ \Gamma_1^* v_1 & \Gamma_2^* v_2 & \Gamma_3^* v_3 \\ -\gamma_1 & -\gamma_2 & -\gamma_3 \end{pmatrix}^* . \tag{9}$$

A complete set of $3j\Gamma\gamma$ symbols, which are required for the determination of CGC's by the method of [1], is given in Table IV. Each triad $\Gamma_1\Gamma_2\Gamma_3 \neq TTT$ is related to a single set of $(j_1j_2j_3)$ corresponding to the smallest value of j for which $n(j, \Gamma) = 0$, and the triad TTT is associated with two such sets: (1 1 1) and (1 1 2).

TABLE IV

$3j\Gamma\gamma$ symbols for the group T′. The Table contains all independent non-zero symbols for the arguments arranged according to the following criterions: (*i*) $\Gamma_1 \leqslant \Gamma_2 \leqslant \Gamma_3$, where the relation $\leqslant$ is defined by the sequence $A \leqslant E1 \leqslant E2 \leqslant T \leqslant E' \leqslant E'' \leqslant E'''$; (*ii*) within a fixed triad $\Gamma_1\Gamma_2\Gamma_3$ the symbols are ordered according to increasing j_3; (*iii*) for a fixed $\Gamma_1\Gamma_2\Gamma_3 j_3$ symbols are ordered first according to increasing γ_1, and then (under fixed γ_1) — according to increasing γ_2, γ_i being ordered according to the sequence given in Table I; (*iv*) if $(j_i\Gamma_i) = (j_{i+1}\Gamma_{i+1})$, then $\gamma_i \leqslant \gamma_{i+1}$. All other non vanishing symbols can be easily obtained from Eqs (7)—(9)

Γ_1	Γ_2	Γ_3	j_1	j_2	j_3	γ_1	γ_2	γ_3	$3j\Gamma\gamma$
A	A	A	0	0	0	a	a	a	1
A	$E1$	$E2$	0	2	2	a	e_1	e_2	$1/\sqrt{5}$
A	T	T	0	1	1	a	-1	1	$1/\sqrt{3}$
							0	0	$-1/\sqrt{3}$
A	E'	E'	0	1/2	1/2	a	α'	β'	$1/\sqrt{2}$
A	E''	E'''	0	3/2	3/2	a	α''	β'''	1/2
							β''	α'''	$-1/2$
$E1$	$E1$	$E1$	2	2	2	e_1	e_1	e_1	$-2/\sqrt{5\cdot 7}$
$E1$	T	T	2	1	1	e_1	-1	-1	$-i/2\sqrt{5}$
								1	$1/2\sqrt{3\cdot 5}$
							0	0	$1/\sqrt{3\cdot 5}$
$E1$	E'	E'''	2	1/2	3/2	e_1	α'	β'''	$-1/\sqrt{2\cdot 5}$
							β'	α'''	$1/\sqrt{2\cdot 5}$
$E1$	E''	E''	2	3/2	3/2	e_1	α''	β''	$-1/\sqrt{2\cdot 5}$
T	T	$T1$	1	1	1	-1	0	1	$1/\sqrt{2\cdot 3}$
T	T	$T2$	1	1	2	-1	-1	0	$1/\sqrt{2\cdot 5}$
							0	-1	$1/\sqrt{2\cdot 5}$
T	E'	E'	1	1/2	1/2	-1	α'	α'	$-1/\sqrt{3}$
						0	α'	β'	$1/\sqrt{2\cdot 3}$
T	E'	E''	1	1/2	3/2	-1	α'	α''	$-1/2\sqrt{2\cdot 3}$
							β'	β''	$i/2\sqrt{2}$
						0	α'	β''	$-1/2\sqrt{3}$
							β'	α''	$-1/2\sqrt{3}$
						1	α'	α''	$i/2\sqrt{2}$
							β'	β''	$-1/2\sqrt{2\cdot 3}$
T	E'	E'''	1	1/2	3/2	-1	α'	α'''	$-1/2\sqrt{2\cdot 3}$
							β'	β'''	$-i/2\sqrt{2}$
						0	α'	β'''	$-1/2\sqrt{3}$
							β'	α'''	$-1/2\sqrt{3}$
						1	α'	α'''	$-i/2\sqrt{2}$
							β'	β'''	$-1/2\sqrt{2\cdot 3}$
T	E''	E''	1	3/2	3/2	-1	α''	α''	$1/\sqrt{2\cdot 3\cdot 5}$
							β''	β''	$i/\sqrt{2\cdot 5}$

TABLE IV (continued)

Γ_1	Γ_2	Γ_3	j_1	j_2	j_3	γ_1	γ_2	γ_3	$3j\Gamma\gamma$
A	A	A	0	0	0	a	a	a	1
						0	α''	β''	$1/\sqrt{3\cdot 5}$
						1	α''	α''	$i/\sqrt{2\cdot 5}$
							β''	β''	$1/\sqrt{2\cdot 3\cdot 5}$
T	E''	E'''	1	3/2	3/2	-1	α''	α'''	$1/\sqrt{2\cdot 3\cdot 5}$
						0	α''	β'''	$-1/2\sqrt{3\cdot 5}$
							β''	α'''	$-1/2\sqrt{3\cdot 5}$
						1	β''	β'''	$1/\sqrt{2\cdot 3\cdot 5}$

TABLE V

The coefficients $\alpha(\Gamma_1\Gamma_2\Gamma_3)$ for the group T′ for $c(\Gamma_1\Gamma_2\Gamma_3) = 1$

Γ_1	Γ_2	Γ_3	$\alpha(\Gamma_1\Gamma_2\Gamma_3)$	Γ_1	Γ_2	Γ_3	$\alpha(\Gamma_1\Gamma_2\Gamma_3)$
A	A	A	1	T	E'	E'	$\sqrt{2}$
	$E1$	$E1$	$\sqrt{5}$			E''	2
	$E2$	$E2$	$\sqrt{5}$			E'''	2
	T	T	$\sqrt{3}$		E''	E'	2
	E'	E'	$\sqrt{2}$			E''	$2\sqrt{5}$
	E''	E''	2			E'''	$\sqrt{5}$
	E	E	2		E'''	E'	2
$E1$	$E1$	$E2$	$\sqrt{5\cdot 7/2}$			E''	$\sqrt{5}$
	$E2$	A	$\sqrt{5}$			E'''	$2\sqrt{5}$
	T	T	$\sqrt{3\cdot 5}$	E'	E'	A	1
	E'	E''	$\sqrt{2\cdot 5}$			T	$\sqrt{3}$
	E''	E'''	$\sqrt{2\cdot 5}$		E''	$E1$	$\sqrt{5}$
	E'''	E'	$\sqrt{2\cdot 5}$			T	$\sqrt{2\cdot 3}$
$E2$	$E2$	$E1$	$\sqrt{5\cdot 7/2}$		E'''	$E2$	$\sqrt{5}$
	T	T	$\sqrt{3\cdot 5}$			T	$\sqrt{2\cdot 3}$
	E'	E'''	$\sqrt{2\cdot 5}$	E''	E''	$E2$	$\sqrt{5}$
	E''	E'	$\sqrt{2\cdot 5}$			T	$\sqrt{3\cdot 5}/\sqrt{2}$
	E'''	E''	$\sqrt{2\cdot 5}$		E'''	A	$\sqrt{2}$
T	T	A	1			T	$\sqrt{2\cdot 3\cdot 5}$
		$E1$	$\sqrt{5}$	E'''	E''	$E1$	$\sqrt{5}$
		$E2$	$\sqrt{5}$			T	$\sqrt{3\cdot 5}/\sqrt{2}$

CGC's are given by the formula

$$\begin{bmatrix} \Gamma_1 & \Gamma_2 & \Gamma_3 & w \\ \gamma_1 & \gamma_2 & \gamma_3 & \end{bmatrix} = [-1]^{\Gamma_3-\gamma_3} \sum_{j_3} \alpha_{wj_3}(\Gamma_1\Gamma_2\Gamma_3) \begin{pmatrix} j_1 & j_2 & j_3 \\ \Gamma_1 & \Gamma_2 & \Gamma_3^* \\ \gamma_1 & \gamma_2 & -\gamma_3 \end{pmatrix}, \qquad (10)$$

where the sum runs over $c(\Gamma_1\Gamma_2\Gamma_3)$ smallest values of those j_3's which enclose Γ_3. Assuming, according to I, the convention of nonnegative reduction coefficients, we obtain for $c(\Gamma_1\Gamma_2\Gamma_3) = 1$ that

$$\alpha(\Gamma_1\Gamma_2\Gamma_3) = \alpha(\Gamma_2\Gamma_1\Gamma_3) = \left[\frac{2j_3+1}{N(j_1\Gamma_1, j_2\Gamma_2, j_3\Gamma_3)}\right]^{1/2} \quad \text{for} \quad \Gamma_1\Gamma_2\Gamma_3 \neq TTT. \qquad (11)$$

A complete set of coefficients $\alpha(\Gamma_1\Gamma_2\Gamma_3)$ for this case is given in Table V.

In order to derive CGC's for TTT we must first assume the reduction coefficients $[1T, 1T, w|j_3T]$. They can form an arbitrary two-dimensional unitary matrix, since $c(TTT) = 2$. The simplest permutation properties are associated with the choice

$$[1T, 1T, w|j_3T] = \delta_{w,j_3} \qquad (12)$$

(cf. the next Section). According to the terminology introduced in II, this choice defines a symmetrised system of repetition indices. The matrix $\{\alpha_{wj_3}(TTT)\}$ for this system is given by Table VI.

TABLE VI

The matrix $\alpha(TTT)$ for the symmetrised system of repetition indices

TTT	$j_3 = 1$	$j_3 = 2$
$w = a$	$\sqrt{3}$	0
$w = s$	0	$\sqrt{5}$

The Tables I, IV—VI and Eq. (10) provide numerical determination of any CGC for the group T′ (cf. appropriate remarks in I, Section 4).

4. *The permutation symmetry of CGC's for the group T′*

Accoding to paper II, properties of CGC's with respect to a permutation of their columns are fully determined by permutation matrices $m(\sigma, \Gamma_a\Gamma_{a'}\Gamma_{a''})$, whose elements are given by

$$m_{xw}(\sigma(B/A), \Gamma_a\Gamma_{a'}\Gamma_{a''}) = \sum_{\gamma_1\gamma_2\gamma_3} \begin{pmatrix} \Gamma_b & \Gamma_{b'} & \Gamma_{b''} \\ \gamma_b & \gamma_{b'} & \gamma_{b''} \end{pmatrix}_x \begin{pmatrix} \Gamma_a & \Gamma_{a'} & \Gamma_{a''} \\ \gamma_a & \gamma_{a'} & \gamma_{a''} \end{pmatrix}_w^*, \qquad (13)$$

where the two-row symbols in parentheses are $3\Gamma\gamma$ symbols, related to CGC's by the formula

$$\begin{bmatrix} \Gamma_1 & \Gamma_2 & \Gamma_3 & w \\ \gamma_1 & \gamma_2 & \gamma_3 & \end{bmatrix} = [-1]^{\Gamma_3-\gamma_3}[\Gamma_3]^{1/2}\begin{pmatrix} \Gamma_1 & \Gamma_2 & \Gamma_3^* \\ \gamma_1 & \gamma_2 & -\gamma_3 \end{pmatrix}_w^* . \tag{14}$$

For all the cases, except of the triad TTT, the permutation matrices are one-dimensional. Using Eqs (10) and (11) we obtain from Eq. (13) that

$$m(\sigma, \Gamma_1\Gamma_2\Gamma_3) = \begin{cases} (-1)^{j_1+j_2+j_3} & \text{for } \sigma \text{ -- an odd permutation} \\ +1 & \text{otherwise.} \end{cases} \tag{15}$$

It follows from Eq. (12) that

$$\alpha_{wj_3}(TTT) = (2j_3+1)^{1/2}\delta_{wj_3}, \tag{16}$$

implying that

$$m_{xw}(\sigma, TTT) = \delta_{xw}\begin{cases} (-1)^x & \text{for } \sigma \text{ -- an odd permutation} \\ 1 & \text{otherwise.} \end{cases} \tag{17}$$

The choice (12) corresponds therefore to a symmetrised system of repetition indices, $x = j_3 = 1$ and $x = j_3 = 2$ being related to the antisymmetric and symmetric cube of the representation T.

5. *Final remarks and conclusions*

We proposed in this paper a complete set of coupling coefficients for the double tetrahedral group T′. It follows that the permutation symmetry of these coefficients can be fully determined by the corresponding symmetry of $3j\Gamma\gamma$ symbols for the group SU_2.

We also proposed in Section 2 a relation between repetition indices of the first kind (v-type) for $SU_2 \to T'$ and appropriate repetition indices and irreducible representations of the intermediate subgroup O′. In particular, it is easy to see that the repetition indices of the second kind w-type for the triad TTT are related to the repetition indices of the first kind, associated with the reduction of the resulting representation $D^{(j_3)}$ via the group O′: $j_3vT = 1T_1T$ and $2T_2T$ for $w = a$ and s, respectively.

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