RENORMALIZED MAGNONS AND PHONONS IN A STRONGLY ANHARMONIC HEISENBERG FERROMAGNETIC CRYSTAL. II

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The theory of a previous paper which describes the effects of spin-phonon interactions in a highly anharmonic Heisenberg ferromagnet is applied to evaluate the reduced magnetization and the renormalization factors of the magnon and phonon energies. The numerical calculations of these quantities have been performed for temperatures close to the Curie point and for various spin quantum numbers as well as for several values of the dynamic parameters being a measure of the strength of the spin-phonon coupling.

1. *Introduction*

In an earlier paper [1] we presented the perturbation method of describing the thermodynamical properties of a strongly anharmonic Heisenberg ferromagnet. This method has enabled us to consider up to infinite orders the anharmonicity effects concerning both the spin-phonon and phonon-phonon interactions. For simplicity our treatment has been limited to the case of the monoatomic cubic crystal. The procedure used in [1] has employed the Matsubara thermodynamic perturbation calculus, which gives the perturbational expansion of the partition function in terms of the Feynman graphs. We have restricted our calculations to the class of diagrams deficient in the energy denominators (Hartree-Fock approximation).

In this paper, we present a numerical and graphical analysis of the formal results derived in [1]. For describing the lattice vibrations we introduce the parameters which are a measure of the strength of the spin-phonon coupling. We allow these parameters to change within a wide range of values. Moreover, we allow for various spin quantum numbers. The computations including evaluation of the reduced magnetization and the renormalization factors of the magnon and phonon energies we carry out for temperatures close to the Curie point. We would like to note that the numerical analysis of the results obtained by other authors using different approaches to investigate the strongly

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anharmonic effects in a Heisenberg ferromagnet turned out to be rather involved (see e. g. [2]). For this reason, those authors achieved qualitative results only, whereas our method [1] enables us to carry out in a simple way the computations involving the anharmonic effects of all orders.

2. Computational method

Before beginning our calculations, let us briefly recall those results from reference [1] which will be applied troughout this paper. Our treatment in [1] was limited to the Hartree--Fock approximation. In this approximation the graphs resulting from the perturbational expansion of the partition function were summed up yielding the following set of self--consistent equations

$$Y^{(m)} = N^{-1} \sum_{\lambda} (1-x_\lambda)\tilde{n}_\lambda^{(m)}, \tag{2.1}$$

$$\tilde{n}_\lambda^{(m)} = \frac{1}{e^{\beta\left[L+\varepsilon_\lambda\left(1-\frac{Y^{(p)}}{S}\right)\left(1-\frac{Y^{(m)}}{S}\right)\right]}-1}, \tag{2.2}$$

$$Y^{(p)} = \frac{1}{NJ\gamma_0} \sum_{\lambda} \hbar\omega_{\lambda 3}(\tilde{n}_{\lambda 3}^{(p)}+\tfrac{1}{2})c_{\lambda 3}^{(1)}, \tag{2.3}$$

$$\tilde{n}_{\lambda s}^{(p)} = \frac{1}{e^{\beta\hbar\omega_{\lambda s}\left[a_{\lambda s}-c_{\lambda s}Y^{(m)}\left(1-\frac{Y^{(m)}}{2S}\right)\right]}-1}, \tag{2.4}$$

$$a_{\lambda s} = (a_{\lambda 3}^{(1)}+1)\delta_{s,3}+a_{\lambda s}^{(2)}, \tag{2.5}$$

$$c_{\lambda s} = c_{\lambda 3}^{(1)}\delta_{s,3}+c_{\lambda s}^{(2)}, \tag{2.6}$$

$$a_{\lambda 3}^{(1)} = \tfrac{1}{2}\left\{\exp\left[\frac{\hbar}{NM}\sum_{\mu s}\frac{(\vec{\lambda}\cdot\vec{e}_{\mu s})^2}{\omega_{\mu s}}(\tilde{n}_{\mu s}^{(p)}+\tfrac{1}{2})\right]-1\right\}, \tag{2.7}$$

$$a_{\lambda s}^{(2)} = \frac{1}{NM\omega_{\lambda s}^2}\sum_{\mu} \hbar\omega_{\mu 3}(\tilde{n}_{\mu 3}^{(p)}+\tfrac{1}{2})(\vec{\mu}\cdot\vec{e}_{\lambda s})^2(a_{\mu 3}^{(1)}+\tfrac{1}{2}), \tag{2.8}$$

$$c_{\lambda 3}^{(1)} = \frac{JS\gamma_\lambda \lambda^2}{M\omega_{\lambda 3}^2}\exp\left[\frac{\hbar}{NM}\sum_{\mu s}\frac{(\vec{\lambda}\cdot\vec{e}_{\mu s})^2}{\omega_{\mu s}}(\tilde{n}_{\mu s}^{(p)}+\tfrac{1}{2})\right], \tag{2.9}$$

$$c_{\lambda s}^{(2)} = \frac{1}{NM\omega_{\lambda s}^2}\sum_{\mu} \hbar\omega_{\mu 3}(\tilde{n}_{\mu 3}^{(p)}+\tfrac{1}{2})(\vec{\mu}\cdot\vec{e}_{\lambda s})^2 c_{\mu 3}^{(1)}. \tag{2.10}$$

In the above equations we have employed the same notation as in [1], i. e.,

$$L = g\mu_B H, \tag{2.11}$$

with g being Lande's factor, μ_B — Bohr's magneton and H the external magnetic field applied along z axis;

$$\varepsilon_\lambda = JS(\gamma_0 - \gamma_\lambda), \tag{2.12}$$

where J is the nearest neighbour exchange integral, S denotes spin quantum number and

$$\gamma_\lambda = \sum_\eta e^{i\vec{\lambda}\cdot\vec{\eta}}, \tag{2.13}$$

$$x_\lambda = \gamma_\lambda/\gamma_0, \tag{2.14}$$

with η's being the vectors reaching to all nearest neighbours, and $\vec{\lambda}$ being the wave vector. Quantities $\omega_{\lambda s}$ and $\vec{e}_{\lambda s}$ are the phonon frequency and the polarization vector for a phonon of branch s. N is the number of atoms in the crystal, and M denotes the atomic mass.

The method of investigating the lattice vibrations, as formulated in [1], makes use of the isotropic model, in which each branch of the dispersion relation is purely longitudinal or purely transverse. We have shown that this model leads to the following dispersion relation for the longitudinal branch ($s = 3$):

$$\omega_{\lambda 3} = \left[\frac{1}{M}(J_\lambda S^2 - U_\lambda)\right]^{\frac{1}{2}} \lambda, \tag{2.15}$$

where J_λ and U_λ denote the Fourier components of the interatomic exchange integral and potential energy, respectively. The above relation holds true in the long-wavelength limit only. Furthermore, as follows from [1], the transverse parts ($s = 1, 2$) of the zeroth order Hamiltonian vanish. However, it is possible to test the transverse vibrations of the anharmonic crystal lattice. Indeed, taking into account that the phonon frequencies for the transverse branches ($s = 1, 2$) can be determined with the aid of the dynamical matrix [3], we considered the importance of the anharmonic transverse terms of the Hamiltonian for describing the dynamic and magnetic properties of a ferromagnet, see [1]. Generally, our theory describes the longitudinal lattice vibrations quite well, whereas the transverse lattice vibrations are less satisfactorily described.

From Eqs. (2.2) and (2.4), which define the renormalized magnon and phonon average occupation numbers $\tilde{n}_\lambda^{(m)}$ and $\tilde{n}_{\lambda s}^{(p)}$, we see that the renormalized magnon and phonon excitation energies can be written in the form

$$\tilde{\varepsilon}_\lambda = \alpha^{(m)}\varepsilon_\lambda, \tag{2.16}$$

$$\hbar\tilde{\omega}_{\lambda s} = \alpha_{\lambda s}\hbar\omega_{\lambda s}, \tag{2.17}$$

where the renormalization energy factors $\alpha^{(m)}$ and $\alpha_{\lambda s}^{(p)}$ are

$$\alpha^{(m)} = \left(1 - \frac{Y^{(p)}}{S}\right)\left(1 - \frac{Y^{(m)}}{S}\right), \tag{2.18}$$

$$\alpha_{\lambda s}^{(p)} = a_{\lambda s} - c_{\lambda s} Y^{(m)}\left(1 - \frac{Y^{(m)}}{2S}\right). \tag{2.19}$$

Taking into consideration (2.5)–(2.10) and (2.19) we notice that for each branch of the dispersion relation the renormalized phonon frequency $\tilde{\omega}_{\lambda s}$ becomes infinite as the wave vector approaches zero (independent of direction). As the phonon frequencies should be finite for all values of the wave vector, it turns out that the Hartree-Fock approximation is not sufficiently precise for describing the dynamic properties of an anharmonic crystal and our considerations should be extended to include some classes of graphs containing the energy denominators. Then, all divergent terms due to diagrams containing and not containing the energy denominators have to cancel each other, and so the renormalized phonon frequencies become finite. However, as we shall see, even in the Hartree-Fock approximation one can avoid the difficulty mentioned above.

We proceed now to discuss the method of the numerical analysis of Eqs. (2.1)–(2.10). The computations are performed for the case of s. c. lattice and for $L = 0$. Besides, the normal frequencies are assumed to be of the form

$$\omega_{\lambda s} = v_s \lambda, \tag{2.20}$$

where v_s denotes the sound velocity for branch s.

To remove the divergence of the renormalized phonon frequencies, we modify expressions (2.3)–(2.10) taking advantage of an approximate formula for the free phonon average occupation numebr $\bar{n}_{\lambda s}^{(p)}$. Namely, in the expansion (see [4])

$$(e^{t_{\lambda s}}-1)^{-1} = \sum_{n=0}^{\infty} B_n \frac{t_{\lambda s}^{n-1}}{n!} = t_{\lambda s}^{-1} - \tfrac{1}{2} + \frac{1}{2!}\tfrac{1}{6}\, t_{\lambda s} - \frac{1}{4!}\tfrac{1}{30}\, t_{\lambda s}^3 + \ldots, \tag{2.21}$$

where $t_{\lambda s} = \beta\hbar v_s \lambda$ and B_n is the n-th Bernoulli number, for sufficiently small $t_{\lambda s}$ we can retain only two terms, i. e.,

$$\bar{n}_{\lambda s}^{(p)} \approx (\beta\hbar v_s \lambda)^{-1} - \tfrac{1}{2}. \tag{2.22}$$

Accordingly, expression

$$\begin{aligned} m_{\lambda s}^{(p)i} &= (\beta\hbar v_s \lambda)^i \sum_{n=0}^{\infty} n^{i-1} e^{-n\beta\hbar v_s \lambda} \\ &= \beta\hbar v_s \lambda \frac{\partial m_{\lambda s}^{(p)i-1}}{\partial \bar{n}_{\lambda s}^{(p)}}\, \bar{n}_{\lambda s}^{(p)}(\bar{n}_{\lambda s}^{(p)}+1), \qquad i = 2, 3, 4, \ldots, \end{aligned} \tag{2.23}$$

may be written for $i = 2$ as

$$\begin{aligned} m_{\lambda s}^{(p)2} &= (\beta\hbar v_s \lambda)^2 \bar{n}_{\lambda s}^{(p)}(\bar{n}_{\lambda s}^{(p)}+1) \approx (\beta\hbar v_s \lambda)^2 [(\beta\hbar v_s \lambda)^{-2} - \tfrac{1}{4}] \\ &\approx (\beta\hbar v_s \lambda)^2 (\beta\hbar v_s \lambda)^{-2} = (\beta\hbar v_s \lambda)^2 (\bar{n}_{\lambda s}^{(p)} + \tfrac{1}{2})^2. \end{aligned} \tag{2.24}$$

Generally, we obtain

$$m_{\lambda s}^{(p)i} = (i-1)!(\beta\hbar v_s \lambda)^i (\bar{n}_{\lambda s}^{(p)} + \tfrac{1}{2})^i. \tag{2.25}$$

We see that, by virtue of (2.22), the expression (2.25) (for $i = 2, 3, 4, \ldots$) is independent of the wave vector. Thereby, the Helmholtz free energy obtained in [1] can be rederived as follows (for more details see [5])

$$
\begin{aligned}
F = {} & E_0 + \sum_\lambda \hbar v_3 \lambda \tilde{n}^{(p)}_{\lambda 3} + \sum_\lambda (L+\varepsilon_\lambda)\tilde{n}^{(m)}_\lambda \\
& + \sum_\lambda \hbar v_3 \lambda (\tilde{n}^{(p)}_{\lambda 3} + \tfrac{1}{2}) \left[a_3^{(1)} - c_3^{(1)} Y^{(m)} \left(1 - \frac{Y^{(m)}}{2S} \right) \right] - \tfrac{1}{2} N J \gamma_0 (Y^{(m)})^2 \\
& + \beta^{-1} \sum_{\lambda s} [\tilde{n}^{(p)}_{\lambda s} \ln \tilde{n}^{(p)}_{\lambda s} - (1+\tilde{n}^{(p)}_{\lambda s}) \ln (1+\tilde{n}^{(p)}_{\lambda s})] \\
& + \beta^{-1} \sum_\lambda [\tilde{n}^{(m)}_\lambda \ln \tilde{n}^{(m)}_\lambda - (1+\tilde{n}^{(m)}_\lambda) \ln (1+\tilde{n}^{(m)}_\lambda)],
\end{aligned}
\tag{2.26}
$$

with

$$Y^{(m)} = N^{-1} \sum_\lambda (1-x_\lambda) \tilde{n}^{(m)}_\lambda, \tag{2.27}$$

$$\tilde{n}^{(m)}_\lambda = \frac{1}{e^{\beta \left[L + \varepsilon_\lambda \left(1 - \frac{Y^{(p)}}{S}\right)\left(1 - \frac{Y^{(m)}}{S}\right)\right]} - 1}, \tag{2.28}$$

$$Y^{(p)} = \frac{1}{NJ\gamma_0} \sum_\lambda \hbar v_3 \lambda (\tilde{n}^{(p)}_{\lambda 3} + \tfrac{1}{2}) c_3^{(1)}, \tag{2.29}$$

$$\tilde{n}^{(p)}_{\lambda s} = \frac{1}{e^{\beta \hbar v_s \lambda \left[a_s - c_s Y^{(m)} \left(1 - \frac{Y^{(m)}}{2S}\right)\right]} - 1}, \tag{2.30}$$

$$a_s = (a_3^{(1)} + 1)\delta_{s,3} + a_s^{(2)}, \tag{2.31}$$

$$c_s = c_3^{(1)} \delta_{s,3} + c_s^{(2)}, \tag{2.32}$$

$$a_3^{(1)} = \tfrac{1}{2} N^{-1} \sum_\lambda [\exp(\lambda^2 \sum_s d_{ss}) - 1], \tag{2.33}$$

$$a_s^{(2)} = \frac{d_{s3}}{2} N^{-1} \sum_\lambda \lambda^2 \exp\left(\lambda^2 \sum_s d_{ss} \right), \tag{2.34}$$

$$c_3^{(1)} = 2 \frac{JS\gamma_0}{Mv_3^2} (a_3^{(1)} + \tfrac{1}{2}), \tag{2.35}$$

$$c_s^{(2)} = 2 \frac{JS\gamma_0}{Mv_s^2} a_s^{(2)}, \tag{2.36}$$

where

$$d_{ss'} = \frac{1}{3NM} \sum_\lambda \frac{1}{(v_s \lambda)^2} N^{-1} \sum_\mu \hbar v_{s'} \mu (\tilde{n}^{(p)}_{\mu s'} + \tfrac{1}{2}). \tag{2.37}$$

Hence, the reduced magnetization and the renormalization factors of the magnon and phonon energies are given by

$$\mu = -(SN)^{-1}\frac{\partial F}{\partial L} = 1-(SN)^{-1}\sum_{\lambda}\tilde{n}_{\lambda}^{(m)}, \tag{2.38}$$

$$\alpha^{(m)} = \left(1-\frac{Y^{(p)}}{S}\right)\left(1-\frac{Y^{(m)}}{S}\right), \tag{2.39}$$

$$\alpha_s^{(p)} = a_s - c_s Y^{(m)}\left(1-\frac{Y^{(m)}}{2S}\right), \tag{2.40}$$

where $\tilde{n}_{\lambda}^{(m)}$, $Y^{(m)}$ and $Y^{(p)}$ are now defined by (2.28), (2.27) and (2.29), respectively. From Eqs. (2.31)–(2.36) and (2.40) it follows that, in contrast to (2.19), the renormalization factor of the phonon energy is nondivergent, and our earlier claim regarding this matter is thus proved. The finiteness of the renormalized phonon frequencies enables us to expand $\tilde{n}_{\lambda s}^{(p)}$ (2.30) in the same way as $\bar{n}_{\lambda s}^{(p)}$ (2.21). Then, on assuming $\alpha_s^{(p)}$ to be relatively small, we can calculate $\tilde{n}_{\lambda s}^{(p)}$ having recourse to the approximation of the type (2.22). Consequently, Eqs. (2.29)–(2.36) may be rewritten as follows:

$$Y^{(p)} = SW_3c_3^{(1)}, \tag{2.41}$$

$$W_s = \left\{X\left[a_s - c_s Y^{(m)}\left(1-\frac{Y^{(m)}}{2S}\right)\right]\right\}^{-1}, \tag{2.42}$$

$$a_s = (a_3^{(1)}+1)\delta_{s,3} + a_s^{(2)}, \tag{2.43}$$

$$c_s = c_3^{(1)}\delta_{s,3} + c_s^{(2)}, \tag{2.44}$$

$$a_3^{(1)} = \tfrac{1}{2}N^{-1}\sum_{\lambda}[\exp(\tfrac{1}{3}\lambda^2KN^{-1}\sum_{\mu}\mu^{-2})-1], \tag{2.45}$$

$$a_s^{(2)} = \tfrac{1}{6}A_sW_3N^{-1}\sum_{\lambda}\lambda^{-2}N^{-1}\sum_{\mu}\mu^2\exp(\tfrac{1}{3}\mu^2KN^{-1}\sum_{\nu}\nu^{-2}), \tag{2.46}$$

$$c_3^{(1)} = 2A_3(a_3^{(1)}+\tfrac{1}{2}), \tag{2.47}$$

$$c_s^{(2)} = 2A_3a_s^{(2)}, \tag{2.48}$$

where

$$K = \sum_{s=1}^{3} A_sW_s, \tag{2.49}$$

$$X = \frac{JS\gamma_0}{kT}, \tag{2.50}$$

$$A_s = \frac{JS\gamma_0}{Mv_s^2}. \tag{2.51}$$

It should be pointed out that in the set of self-consistent Eqs. (2.27), (2.28) and (2.41)–(2.48), which determine the dynamic and magnetic behaviour of the strongly anharmonic ferromagnet, four parameters only occur, i. e. X and A_s's. X is the well known temperature parameter, whereas A_s's (hereafter called the dynamic parameters) are a measure of the strength of the spin-phonon coupling.

Eqs. (2.41)–(2.48), in contrast to (2.29)–(2.36), have a form more convenient for numerical analysis. By utilizing the approximate formulae given in Appendix A and by using the trial and error method, we solved numerically the set of Eqs. (2.27), (2.28) and (2.41)–(2.48) for various values of parameters X and A_s's, and afterwards we have evaluated the quantities (2.38)–(2.40). The main results concerning the reduced magnetization and the renormalization factors of the energies of the magnon and phonon excitations are presented in the subsequent section.

3. Numerical results for temperatures close to the Curie point

In discussing our results we shall consider separately three cases: the longitudinal vibrations of the highly anharmonic crystal lattice, the longitudinal lattice vibrations in the lowest order anharmonic and the harmonic approximations, and finally the anharmonic case involving both the longitudinal and transverse lattice vibrations.

3.1. The case of longitudinal vibrations of the strongly anharmonic crystal lattice

The temperature dependence of the reduced magnetization and of the renormalization factor $\alpha^{(m)}$ for $S = \frac{1}{2}$ and for various values of the dynamic parameter $A \equiv A_3$ is shown in Figs. 1 and 2. The case $A = 0$ corresponds to the rigid lattice model, i. e., for $M = \infty$ (see (2.51)), and was already investigated in [5, 6]. In Figs. 3 and 4 we illustrate the temperature dependence of the reduced magnetization and the renormalization

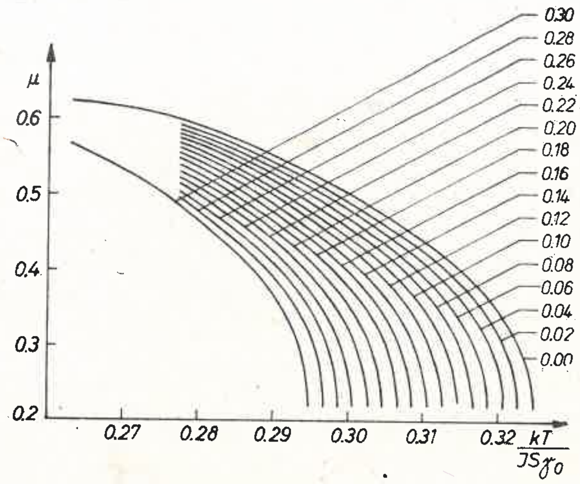

Fig. 1. Reduced magnetization for $S = \frac{1}{2}$ and for various values of A

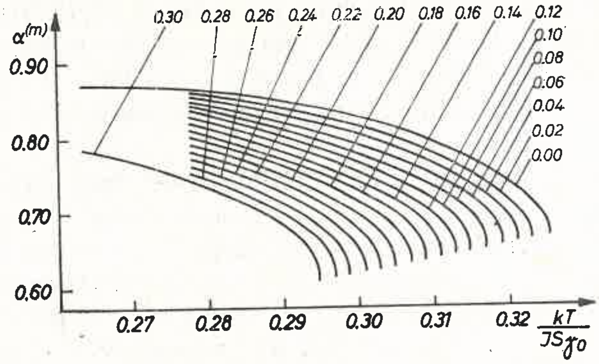

Fig. 2. Renormalization factor of the magnon energy for $S = \frac{1}{2}$ and various values of A

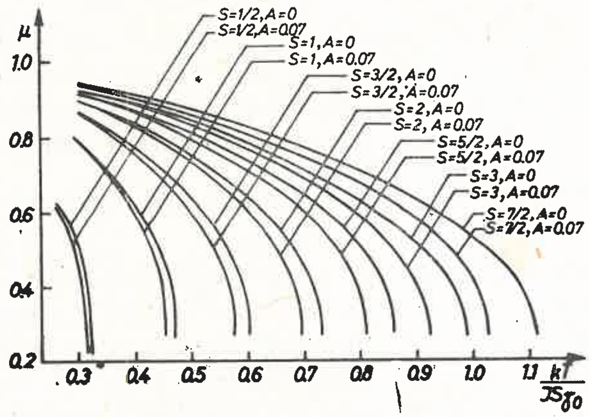

Fig. 3. Reduced magnetization for $S = \frac{1}{2}, 1, \ldots, \frac{7}{2}$ and for $A = 0;\ 0.07$

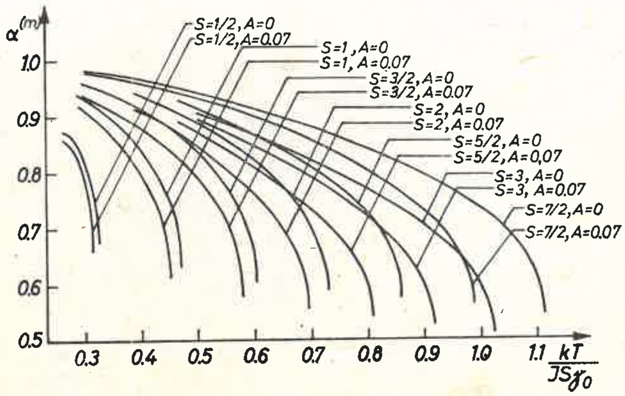

Fig. 4. Renormalization factor of the magnon energy for $S = \frac{1}{2}, 1, \ldots, \frac{7}{2}$ and for $A = 0;\ 0.07$

factor $\alpha^{(m)}$ for $S = \frac{1}{2}, 1, \ldots, \frac{7}{2}$ and for the selected value of $A = 0.07$. It can be seen from Figs. 1–4 that the set of Eqs. (2.27), (2.28) and (2.41)–(2.48) has solutions only up to a certain cut off temperature $T_m^{(A)}$ which depends on S and A. At this temperature (we call it the Curie temperature) μ has a nonzero minimal value (independent of A). Thus, the method discussed in this paper fails for temperatures from the immediate vicinity of $T_m^{(A)}$. It is to be noted that the nonzero minimum value of μ may be reduced by making better approximations, i. e., by including into the theory the kinematic interaction and by extending the calculations to the graphs which contain the energy denominators, see [7].

The shift of the Curie temperature due to the spin-phonon coupling can be described by means of the ratio $\beta_m^{(A)} \equiv (T_m^{(0)} - T_m^{(A)})/T_m^{(0)}$. It turns out that the spin-phonon interactions lead to decrease of the Curie temperature, and for values of A from the interval $0.003 \leqslant A \leqslant 0.3$ and for $S = \frac{1}{2}, 1, \ldots, \frac{7}{2}$ $\beta_m^{(A)}$ may approximately be regarded as a linear function of A. For example, in the case $S = \frac{1}{2}$ we found

$$\beta_m^{(A)} = (0.319 \pm 0.006)A. \tag{3.1}$$

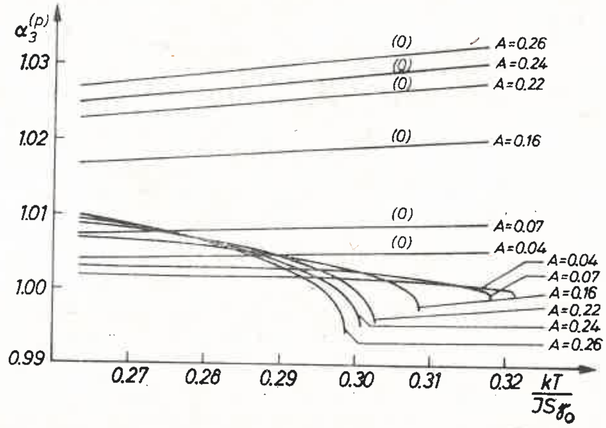

Fig. 5. Renormalization factor of the phonon energy for $S = \frac{1}{2}$ and for several values of A. (0) denotes the case of $Y^{(m)} \equiv 0$

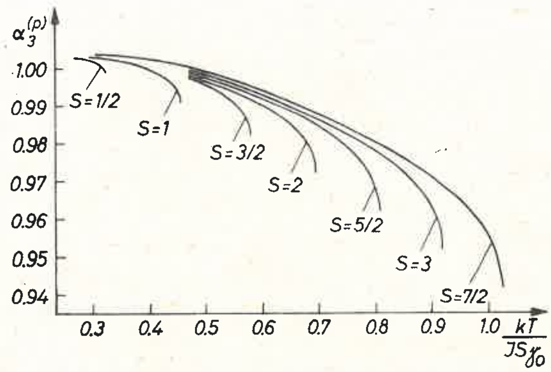

Fig. 6. Renormalization factor of the phonon energy for $S = \frac{1}{2}, 1, \ldots, \frac{7}{2}$ and for $A = 0.07$

In Fig. 5 we show the temperature dependence of the renormalization factor $\alpha_3^{(p)}$ for $S = \frac{1}{2}$ and for several values of A. In order to explain how spin-phonon interactions affect the behaviour of $\alpha_3^{(p)}$, we also considered the case of the absence of the spin-phonon coupling by putting $Y^{(m)} \equiv 0$ in Eqs. (2.41)–(2.48). The temperature dependence of $\alpha_3^{(p)}$ for $S = \frac{1}{2}, 1, \ldots, \frac{7}{2}$ and for $A = 0.07$ is represented in Fig. 6.

From Figs. 1–6 we see that the spin-phonon coupling causes a decrease (at a given temperature) of the reduced magnetization and renormalization factors $\alpha^{(m)}$ and $\alpha_3^{(p)}$. These effects become more important near the Curie point and for large values of S and A.

3.2. The case of longitudinal lattice vibrations considered in the lowest order anharmonic and the harmonic approximations

To examine how the strongly anharmonic effects influence the magnetic and dynamic properties of a ferromagnet, we have also considered the lattice vibrations in the lowest order anharmonic and the harmonic approximations (see Appendix B). Detailed numerical

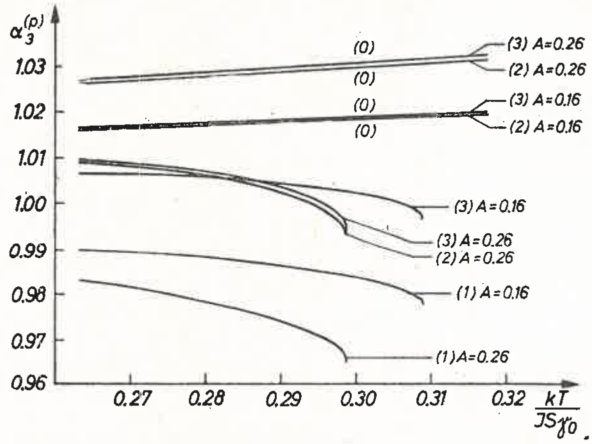

Fig. 7. Renormalization factor of the phonon energy for $S = \frac{1}{2}$ and for $A = 0.16$; 0.26. (0) denotes the case of $Y^{(m)} \equiv 0$, (1) harmonic case, (2) lowest order anharmonic case, (3) strongly anharmonic case

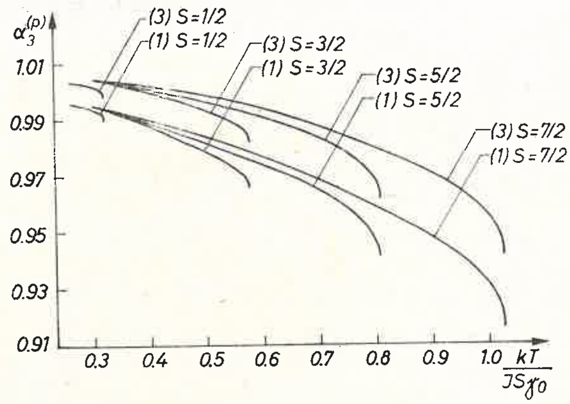

Fig. 8. Renormalization factor of the phonon energy for $S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$ and for $A = 0.07$. (1) harmonic case, (3) strongly anharmonic case

calculations prove that the reduced magnetization and the renormalization factor $\alpha^{(m)}$ derived in the highly anharmonic case differ very slightly (for various values of S and A) from those obtained in the harmonic approximation. This means that in our approach the anharmonic effects are of little consequence for investigating the magnetic properties of ferromagnets.

The anharmonicity effects are of far greater importance for describing the dynamic properties of a ferromagnet than its magnetic properties. From Figs. 7 and 8, which exhibit the temperature dependence of the renormalization factor $\alpha_3^{(p)}$, it follows that the anharmonicity of the crystal lattice is most important near the Curie temperature and for large values of S and A.

3.3. The case of the longitudinal and transverse vibrations of the anharmonic crystal lattice

Now, we consider the problem of the longitudinal and transverse lattice vibrations of the anharmonic crystal to study the applicability of our method for describing the transverse lattice vibrations. The numerical calculations were carried out for several values of S and for the parameters A_3 and A_1 (we assumed that $A_1 = A_2$) from a wide range of values. The results show that in our approach the transverse lattice vibrations, as compared to the longitudinal ones, are of far less importance.

4. *Conclusions*

The results derived in this paper were obtained by adopting the approximation of self-consistently renormalized magnons and phonons in the strongly anharmonic Heisenberg ferromagnet. In the numerical calculations concerning the reduced magnetization and the renormalization factors of the magnon and phonon energies both the longitudinal and transverse lattice vibrations have been considered, and we have shown that our method describes the longitudinal lattice vibrations much better than the transverse ones. The results have also proved that the effects of the spin-phonon coupling and the effects of the anharmonicity of ferromagnets are most important for temperatures from the vicinity of the Curie point as well as for large values of the spin quantum number and for large values of the dynamic parameters. Moreover, the anharmonicity effects turned out to be far more important for describing the dynamic than the magnetic properties of ferromagnets. It should be noted that our treatment was confined to a model in which the equilibrium lattice spacing was independent of temperature.

The breakdown of our method in the critical region means that the Hartree-Fock approximation is not adequate for investigating the behaviour of a ferromagnet in the immediate vicinity of the Curie point. To achieve better results one must improve the method by including into the theory the kinematic interaction and by extending the calculations to some classes of graphs comprising the energy denominators.

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APPENDIX A

In this appendix we derive approximate formulae which are very useful in our numerical calculations. Let us consider the quantities (2.27) and (2.28). For relatively high temperatures we can resort to the expansion of the (2.21) type. Then we have

$$\sum_{\lambda} \tilde{n}_{\lambda}^{(m)} \to \frac{1}{\pi^3} \iiint_0^{\pi} \frac{dxdydz}{e^{\varepsilon(1-x_r)}-1}$$

$$= \frac{1}{\varepsilon} c_{-1} - \tfrac{1}{2} + \tfrac{1}{12} c_1\varepsilon - \tfrac{1}{720} c_3\varepsilon^3 + \tfrac{1}{30240} c_5\varepsilon^5 + \dots, \tag{A1}$$

$$Y^{(m)} \to \frac{1}{\pi^3} \iiint_0^{\pi} (1-x_r) \frac{dxdydz}{e^{\varepsilon(1-x_r)}-1}$$

$$= \frac{1}{\varepsilon} - \tfrac{1}{2} c_1 + \tfrac{1}{12} c_2\varepsilon - \tfrac{1}{720} c_4\varepsilon^3 + \tfrac{1}{30240} c_6\varepsilon^5 + \dots, \tag{A2}$$

where

$$\varepsilon = X\left(1-\frac{Y^{(p)}}{S}\right)\left(1-\frac{Y^{(m)}}{S}\right), \tag{A3}$$

$$x_r = e^{i(x+y+z)}. \tag{A4}$$

Coefficients c_{2n-1} for the s. c. lattice are given by

$$c_{2n-1} = \frac{1}{\pi^3} \iiint_0^{\pi} [1-\tfrac{1}{3}(\cos x+\cos y+\cos z)]^{2n-1} dxdydz,$$

$$n = 0, 1, 2, \dots . \tag{A5}$$

In particular, for $n = 0$ we have [4]

$$c_{-1} = \frac{12}{\pi^2}(18+12\sqrt{2}-10\sqrt{3}-7\sqrt{6})K^2[(2-\sqrt{3})(\sqrt{3}-\sqrt{2})] = 1.516386, \tag{A6}$$

where $K(l)$ denotes the complete elliptic integral.

We now express quantities (2.45) and (2.46) in the form convenient for our computational purposes. By making use of the following formulae

$$N^{-1}\sum_{\lambda} e^{\eta^2 a^2 \lambda^2} \to \frac{1}{\pi^3} \iiint_0^{\pi} dxdydz e^{a^2(x^2+y^2+z^2)}$$

$$= e^{3\pi^2 a^2}\left[\sum_{n=0}^{\infty} \frac{(-1)^n 2^n (\pi a)^{2n}}{(2n+1)!!}\right]^3$$

$$= e^{3\pi^2 a^2}[1-2(\pi a)^2+\tfrac{32}{15}(\pi a)^4-\tfrac{1504}{945}(\pi a)^6+\dots], \tag{A7}$$

$$N^{-1}\sum_{\lambda}\eta^2\lambda^2 e^{\eta^2a^2\lambda^2} = \frac{\partial}{\partial a^2}N^{-1}\sum_{\lambda}e^{\eta^2a^2\lambda^2}$$

$$= \pi^2 e^{3\pi^2a^2}[1-\tfrac{26}{15}(\pi a)^2+\tfrac{512}{315}(\pi a)^4-\tfrac{1696}{1575}(\pi a)^6+\ldots], \tag{A8}$$

$$N^{-1}\sum_{\lambda}\lambda^{-2} \to \frac{\eta^2}{\pi^3}\iiint_0^{\pi}\frac{dxdydz}{x^2+y^2+z^2} \approx 0.119829\eta^2, \tag{A9}$$

we obtain

$$a_3^{(1)} = (0.5-0.394221K+0.165770K^2-0.048753K^3+\ldots)$$
$$\times\exp(1.182661K)-0.5, \tag{A10}$$

$$a_s^{(2)} = (0.197110-0.138022K+0.049791K^2-0.013004K^3+\ldots)$$
$$\times A_sW_3\exp(1.182661K). \tag{A11}$$

APPENDIX B

From Eqs. (2.45), (2.46) and (A10), (A11) it follows (see also [1]) that in the lowest order anharmonic approximation quantities $a_3^{(1)}$ and $a_s^{(2)}$ assume the form

$$a_3^{(1)} = 0.197110K, \tag{B1}$$

$$a_s^{(2)} = 0.197110A_sK. \tag{B2}$$

In the harmonic approximation these become

$$a_3^{(1)} = a_s^{(2)} = 0. \tag{B3}$$

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