CRITICAL COMPARISON OF QUANTUM-MECHANICAL AND CLASSICAL DESCRIPTIONS OF NONLINEAR OPTICAL PROCESSES

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In this paper a comparison of quantum-mechanical description of three mode nonlinear process and classical description of nonlinear interaction of three travelling waves is carried out. It is shown here that both these descriptions correspond only if the spatial and frequentional dispersions are neglected. For the dispersional case the average index of refraction and corresponding normalization relations have been introduced here that enable the transition from one description to the other one. These relations were used to demonstrate the parametric generation from quantum noise in terms of measurable quantities and spatial parameters.

1. *Introduction*

Recently the quantum dynamics of nonlinear proccsses with three mode Hamiltonian has been developed [1–18]. This theory enables the description of great number of qualitatively new effects which are mainly connected with statistical properties of light, as well as explanation of such effects, as e.g. parametric generation from quantum noise, that cannot be explained by means of the classical theory.

However, the simple quantum-mechanical description of nonlinear processes by means of three mode Hamiltonian does not comprehend some physical parameters that are included in the classical theory [19–21]. Physical meaning of the coupling constant in the interaction Hamiltonian has to be determined by comparing the quantum-mechanical and classical models (see e.g. [18]).

If we are concerned with the description of travelling waves nonlinear interaction by means of quantum-mechanical model two questions arise here, namely what is the relation between the time coordinate t and spatial coordinate z (z being the normal distance from the boundary of nonlinear medium) and how the normalization volume V has to be chosen.

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As for the first question some authors (see e.g. [8, 12]) reduce the problem of nonlinear interaction to the vacuum so that they use the simple relation $z = ct$; c being velocity of light in vacuum. The other authors (see e.g. [2]) make use of phase velocity: $z = vt$; v being the phase velocity that is the same for all waves. Colinear propagation of all considered waves and phase matching are assumed in all cases.

Also normalization volume V is considered in different ways. For instance in [8, 12] the normalization volume is related to the counting time T: $V = ScT$ where S is transverse section of the beam, whilst in other studies (see e.g. [17, 22]) the volume V is considered as the effective volume of the nonlinear crystal.

In this paper we shall compare both mentioned descriptions with the aim of demonstrating the parametric generation from quantum noise in terms of measurable quantities and spatial parameters.

For simplification we shall assume nonlinear interaction of three monochromatic plane waves at frequencies ω_1, ω_2, ω_3 only:

$$\omega_3 = \omega_1 + \omega_2 \tag{1}$$

and phase matching condition for wave vectors $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, $\boldsymbol{k}_3$ at ω_1, ω_2, ω_3, respectively, will be assumed as well:

$$\boldsymbol{k}_3 = \boldsymbol{k}_1 + \boldsymbol{k}_2. \tag{2}$$

The nonlinear medium will be considered to be optically transparent for all three frequencies ω_1, ω_2, ω_3.

2. *Classical description of nonlinear interaction of three travelling waves in z-domain*

The mutual interaction of three monochromatic plane waves

$$\boldsymbol{E}_i = \boldsymbol{e}_i A_i(z) \exp[i(\boldsymbol{k}_i \cdot \boldsymbol{r} - \omega_i t)], \; i = 1, 2, 3, \tag{3}$$

that satisfy the relations (1) and (2), in nonlinear quadratic medium can be described from the point of view of Maxwell electromagnetic theory by means of three coupled first order differential equations for complex amplitudes $A_1(z)$, $A_2(z)$, $A_3(z)$ (see e.g. [20]):

$$\frac{dA_1(z)}{dz} = -i\sigma_1 A_3(z) A_2^*(z), \tag{4a}$$

$$\frac{dA_2(z)}{dz} = -i\sigma_2 A_3(z) A_1^*(z), \tag{4b}$$

$$\frac{dA_3(z)}{dz} = -i\sigma_3 A_1(z) A_2(z). \tag{4c}$$

The notation is as follows: z is the normal distance from the plane boundary of the nonlinear medium, $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are unit polarization vectors, σ_1, σ_2, σ_3 are constants of nonlinear coupling:

$$\sigma_i = \left(\frac{\mu_0}{\varepsilon_0}\right)^{1/2} \frac{\omega_i K}{v_i(\omega_i, \boldsymbol{s}_i) \cos\delta_i \cos\beta_i}, \quad i = 1, 2, 3, \tag{5}$$

with

$$K = \tfrac{1}{2}\, e_1 \cdot \chi(\omega_1 = \omega_3-\omega_2) : e_3 e_2 = \tfrac{1}{2}\, e_2 \cdot \chi(\omega_2 = \omega_3-\omega_1) : e_3 e_1$$
$$= \tfrac{1}{2}\, e_3 \cdot \chi(\omega_3 = \omega_1+\omega_2) : e_1 e_2, \tag{6}$$

ε_0 is electric permittivity and μ_0 magnetic permeability of vacuum in SI units, $\nu_1(\omega_1, s_1)$, $\nu_2(\omega_2, s_2)$, $\nu_3(\omega_3, s_3)$ are linear indices of refraction in anisotropic medium, $\beta_1, \beta_2, \beta_3$ are angles of refraction for ray directions $f_1(s_1), f_2(s_2), f_3(s_3)$ in the individual waves and $\delta_1, \delta_2, \delta_3$ are angles of anisotropy, i.e. the angles between the ray directions f_1, f_2, f_3 and the normal directions s_1, s_2, s_3, respectively. χ represents third rank tensors of the nonlinear quadratic susceptibilities.

The time average values of z-components of Poynting vectors in the individual waves are given as follows [23, 29]:

$$G_{i,z} = \tfrac{1}{2}\left(\frac{\varepsilon_0}{\mu_0}\right)^{1/2} |A_i|^2 \nu_i(\omega_i, s_i) \cos\delta_i \cos\beta_i, \qquad i = 1, 2, 3. \tag{7}$$

If we denote $N_{1,z}(z)$, $N_{2,z}(z)$, $N_{3,z}(z)$ the average numbers of photons at frequencies $\omega_1, \omega_2, \omega_3$, respectively, which flow through the unit area oriented in the direction of the normal to the boundary (z-axis direction) per unit time (the photon fluxes), then it holds:

$$\overline{G}_{i,z}(z) = \hbar\omega_i N_{i,z}(z), \qquad i = 1, 2, 3. \tag{8}$$

From the equations (4) and using (7) and (8) we can simply derive the following conservation laws for the photon fluxes in z-domain [24]:

$$\frac{dN_{1,z}(z)}{dz} - \frac{dN_{2,z}(z)}{dz} = 0, \tag{9a}$$

$$\frac{dN_{1,z}(z)}{dz} + \frac{dN_{3,z}(z)}{dz} = 0, \tag{9b}$$

$$\frac{dN_{2,z}(z)}{dz} + \frac{dN_{3,z}(z)}{dz} = 0. \tag{9c}$$

The solution of the equations (4) for real amplitudes leads usually to the elliptical functions which can be reduced to the hyperbolical functions in some asymptotic cases [18–21].

For our needs we shall introduce the solution for sum frequency generation $\omega_3 = \omega_1+\omega_2$. When considering the boundary conditions

$$\frac{|A_1(0)|^2}{\sigma_1} = \frac{|A_2(0)|^2}{\sigma_2} = \frac{2\hbar}{K} N_{1,z}(0) = \frac{2\hbar}{K} N_{2,z}(0) = \frac{2\hbar}{K} N_0, \tag{10a}$$

$$A_3(0) = 0, \quad \text{or} \quad N_{3,z}(0) = 0, \tag{10b}$$

then we can obtain the solution for photon fluxes in the following form (see [19, 20]):

$$N_{1,z}(z) = N_{2,z}(z) = N_0 \operatorname{sech}^2 (N_0^{1/2}\mu z), \tag{11a}$$

$$N_{3,z}(z) = N_0 \tanh^2 (N_0^{1/2}\mu z), \tag{11b}$$

where

$$\mu = K\left[\left(\frac{\mu_0}{\varepsilon_0}\right)^{3/2} \frac{2\hbar\omega_1\omega_2\omega_3}{v_1 v_2 v_3 \cos\delta_1 \cos\delta_2 \cos\delta_3 \cos\beta_1 \cos\beta_2 \cos\beta_3}\right]^{1/2}. \tag{12}$$

The solution for phases is not introduced here because it is not important for our considerations.

3. *Quantum-mechanical description of three mode nonlinear interaction in t-domain*

The three mode nonlinear quadratic process can be described from quantum-mechanical point of view by means of the trilinear time-dependent Hamiltonian (see e.g. [1, 2, 5]). When using the Heisenberg equation of motion this description leads to three coupled first order differential equations for the annihilation and creation operators $\hat{a}_i(t)$ and $\hat{a}_i^+(t)$ relative to the i-th mode ($i = 1, 2, 3$) as follows (see e.g. [9, 11, 26]):

$$i\frac{d\hat{a}_1(t)}{dt} = \omega_1\hat{a}_1(t) + g\hat{a}_2^+(t)\hat{a}_3(t), \tag{13a}$$

$$i\frac{d\hat{a}_2(t)}{dt} = \omega_2\hat{a}_2(t) + g\hat{a}_1^+(t)\hat{a}_3(t), \tag{13b}$$

$$i\frac{d\hat{a}_3(t)}{dt} = \omega_3\hat{a}_3(t) + g\hat{a}_1(t)\hat{a}_2(t), \tag{13c}$$

where g labels the real coupling constant.

The total photon number in the i-th mode is given by the expectation value $\langle n_i(t)\rangle = \langle\psi|\hat{n}_i(t)|\psi\rangle$; $\hat{n}_i(t) = \hat{a}_i^+(t)\hat{a}_i(t)$ is the photon number operator and $|\psi\rangle$ representing a state of the system.

Making use of the equations (13) we obtain three photon number conservation laws in the standard way:

$$\frac{d}{dt}\langle n_1(t)\rangle - \frac{d}{dt}\langle n_2(t)\rangle = 0, \tag{14a}$$

$$\frac{d}{dt}\langle n_1(t)\rangle + \frac{d}{dt}\langle n_3(t)\rangle = 0, \tag{14b}$$

$$\frac{d}{dt}\langle n_2(t)\rangle + \frac{d}{dt}\langle n_3(t)\rangle = 0. \tag{14c}$$

Supposing that the photon statistics of the generating modes are conserved in the course of the nonlinear process, we can find closed solutions for mean photon numbers $\langle n_i(t)\rangle$ (see e.g. [12, 25, 26]).

The sum frequency generation can be described in the similar way as in [12]. For the initial conditions

$$\langle n_1(0)\rangle = \langle n_2(0)\rangle = n_0, \tag{15a}$$

$$\langle n_3(0)\rangle = 0, \tag{15b}$$

and coherent generating radiations with the Poisson photon number distributions [28], when neglecting the effects of quantum fluctuations, the following expressions were found [27]:

$$\langle n_1(t)\rangle = \langle n_2(t)\rangle = n_0 \operatorname{sech}^2 (n_0^{1/2} gt), \tag{16a}$$

$$\langle n_3(t)\rangle = n_0 \tanh^2 (n_0^{1/2} gt). \tag{16b}$$

4. *Comparison of quantum-mechanical and classical descriptions*

If we compare the conservation laws (9) and (14), and the courses of photon fluxes (11) in z-domain and the courses of mean photon numbers (16) in t-domain, certain analogy appears at first sight.

In order to compare both descriptions it is necessary to express the expectation value of the Poynting vector operator $\langle\psi|\hat{\boldsymbol{G}}_i|\psi\rangle$ in terms of $\langle n_i\rangle$ for each mode. It must be also considered that the most nonlinear processes take place in anisotropic crystals. By the similar way as in [1, 2, 13] and using [29] we found for $\langle\psi|\hat{\boldsymbol{G}}_i|\psi\rangle$ the following expression in linear anisotropic medium:

$$\langle\psi|\hat{\boldsymbol{G}}_i|\psi\rangle = \langle\psi|\hat{\boldsymbol{E}}_i\times\hat{\boldsymbol{H}}_i|\psi\rangle = \frac{\hbar\omega_i c}{V v_i \cos\delta_i}\left(\langle n_i\rangle+\tfrac{1}{2}\right)\boldsymbol{f}_i. \tag{17}$$

V labels the volume in which the electromagnetic field is considered (normalization or quantization volume) (see e.g. [1, 2, 13]); $\dfrac{c}{v_i\cos\delta_i}$ represents the ray velocity in $\boldsymbol{f}_i$ direction.

Making use of (7) and (17) and neglecting $\frac{1}{2}$ we obtain the following equivalence for the z-component of time average of the Poynting vector:

$$\overline{G}_{i,z} = N_{i,z}\hbar\omega_i = \frac{\hbar\omega_i c\cos\beta_i}{V v_i\cos\delta_i}\langle n_i\rangle = \frac{\hbar\omega_i}{S\tau_i}\langle n_i\rangle, \tag{18}$$

where the volume $V = Sl$ was chosen in such a way that the area S is perpendicular to the z-axis and l lies in z-direction. The time τ_i is given by

$$\tau_i = \frac{l v_i\cos\delta_i}{c\cos\beta_i}, \tag{19}$$

i.e. the time that a travelling wave at ω_i needs for passing the distance l in z-axis. The quantity $\dfrac{c\cos\beta_i}{v_i\cos\delta_i}$ represents the z-component of ray velocity of the monochromatic wave at frequency ω_i.

Now let us consider a nonlinear plane-parallel plate of the thickness L. The Cartesian coordinate system x, y, z is oriented so that the z-axis is identical with the normal to the both boundaries of the plate. An aperture having the area S is assumed to be placed in the first boundary. The real amplitude (intensity) of a wave at ω_i is changed at the passage through the nonlinear plate in dependence of the normal distance from the first boundary z. Thus the amplitude (intensity) in the second boundary differs from the amplitude in the first boundary. We shall assume continual flowing of electromagnetic waves through the nonlinear plate.

The matter of our interest is to calculate the total photon number in each mode in quantum mechanical description which corresponds to the classical description of nonlinear interaction of travelling waves.

Let us assume that, in principle, we are able to count the photons that pass through the first and second boundaries, respectively. When the counting is made in a time T, then the number of photons at ω_i which have passed through the first boundary in the time T is

$$\langle n_i(0)\rangle = \int_0^T N_{i,z}(0)S\,dt = N_{i,z}(0)ST = N_{i,z}(0)Sl_{i,T}\,\frac{v_i\cos\delta_i}{c\cos\beta_i} \tag{20}$$

and the number of photons at ω_i which have passed through the second boundary in the same time T is

$$\langle n_i(\tau_{i,L})\rangle = \int_0^T N_{i,z}(L)S\,dt = N_{i,z}(L)ST = N_{i,z}(L)Sl_{i,T}\,\frac{v_i\cos\delta_i}{c\cos\beta_i}, \tag{21}$$

where $l_{i,T} = T\dfrac{c\cos\beta_i}{v_i\cos\delta_i}$ is the distance which was passed by a travelling wave at ω_i in the counting time T and

$$\tau_{i,L} = \frac{Lv_i\cos\delta_i}{c\cos\beta_i} \tag{22}$$

is the time of passage of the travelling wave at ω_i through the nonlinear plate.

If we want to determine the ensemble of photons in each i-th mode of quantum-mechanical model which is analogic to the passage of the travelling wave at ω_i through the nonlinear plate, that is clear that the time of interaction T_{inter}, which has to be substituted for T in (20) and (21), must be equal to the time of passage of the wave at ω_i through the plate[1]:

$$T_{\text{inter}} = \tau_{i,L}. \tag{23}$$

[1] In most cases the comparison of classical and quantum-mechanical results is independent from the choice of T_{inter}. However, there are effects there, such as parametric generation from quantum noise, for which the magnitude of T_{inter} is important for deriving some formulae in terms of measurable quantities.

The time of interaction is connected with the normalization volume $V = ST_{\text{inter}} \times \frac{c \cos \beta_i}{v_i \cos \delta_i}$ and it occurs in the mutual relation between the coupling constant μ in the classical description and the coupling constant g in the quantum-mechanical description.

Considering the interaction time greater than the time of passing the wave through the plate: $T_{\text{inter}} > \tau_{i,L}$, it would mean that even such photons were included in the ensemble of photons in quantum-mechanical model which correspond to such waves that are not present in the nonlinear medium simultaneously and, consequently, their mutual interaction is impossible.

In the opposite case, considering $T_{\text{inter}} < \tau_{i,L}$, it would mean that a smaller photon number was considered than that one which would correspond to the limit case of simultaneous presence of two waves in the first and second boundary, that follow in the time interval $\tau_{i,L}$.

An unpleasant situation takes place when the passage of three waves at $\omega_1, \omega_2, \omega_3$ through the nonlinear plate is considered simultaneously. Namely the z-components of individual ray velocities $\frac{c \cos \beta_i}{v_i \cos \delta_i}$ $(i = 1, 2, 3)$ are different because of frequentional and spatial dispersions, thus also the times of passing the waves through the plate $\tau_{1,L}, \tau_{2,L}, \tau_{3,L}$ must be different.

When using the relations (20), (21), (22) and (23) for comparison of quantum-mechanical and classical descriptions, we can see that for a dispersive medium both the conservation laws (9) and (14) and the expressions for photon fluxes (11) and mean photon numbers (15), respectively, cannot be fulfilled simultaneously.

The mutual non-correspondence of the both descriptions can be, perhaps, demonstrated more clearly when comparing the classical (4) and quantum-mechanical (13) equations in t-domain.

From (13) we can obtain in a standard way the following differential equations for quantum-mechanical operators in t-domain:

$$i \frac{d\hat{n}_1(t)}{dt} = i \frac{d\hat{n}_2(t)}{dt} = -i \frac{d\hat{n}_3(t)}{dt}$$

$$= i \frac{d}{dt}(\hat{a}_1^+(t)\hat{a}_1(t)) = i \frac{d}{dt}(\hat{a}_2^+(t)\hat{a}_2(t)) = -i \frac{d}{dt}(\hat{a}_3^+(t)\hat{a}_3(t))$$

$$= g\{\hat{a}_1^+(t)\hat{a}_2^+(t)\hat{a}_3(t) - \hat{a}_1(t)\hat{a}_2(t)\hat{a}_3^+(t)\}. \tag{24}$$

The quantum-mechanical operators $\hat{a}_i$ and $\hat{a}_i^+$ $(\hat{n}_i = \hat{a}_i^+ \hat{a}_i)$ correspond to normalized amplitudes a_i, a_i^* satisfying the relation $\langle n_i \rangle = a_i^* a_i = |a_i|^2$ (see e.g. [1, 2, 18]). Using (18), (7) and (8) we can find the normalization relation:

$$|A_i| = \left[\left(\frac{\mu_0}{\varepsilon_0}\right)^{1/2} \frac{2\hbar\omega_i c}{V v_i^2 \cos^2 \delta_i}\right]^{1/2} |a_i|, \quad i = 1, 2, 3. \tag{25}$$

In order to obtain the classical equations for the normalized amplitudes a_1, a_2, a_3 in t-domain it is necessary to use a suitable relation between the space z-parameter and the time t-parameter.

Since in the classical model of continuous flowing electromagnetic waves there appear ray velocities or their components only, that is clear that each amplitude a_i is connected with t-coordinate by means of corresponding z-component of ray velocity. Thus it holds:

$$z = \frac{c \cos \beta_i}{v_i \cos \delta_i} t_i \text{ for the amplitude } a_i. \tag{26}$$

Using the last relations (25) and (26) we can obtain from (4) the classical analogue to (24) as follows:

$$i\left[\frac{d}{dt}\langle n_1(t)\rangle\right]_{t=t_1} = i\left[\frac{d}{dt}\langle n_2(t)\rangle\right]_{t=t_2} = -i\left[\frac{d}{dt}\langle n_3(t)\rangle\right]_{t=t_3}$$

$$= i\left[\frac{d}{dt}(a_1^*(t)a_1(t))\right]_{t=t_1} = i\left[\frac{d}{dt}(a_2^*(t)a_2(t))\right]_{t=t_2} = -i\left[\frac{d}{dt}(a_3^*(t)a_3(t))\right]_{t=t_3}$$

$$= \kappa\{a_1^*(t_1)a_2^*(t_2)a_3(t_3) - a_1(t_1)a_2(t_2)a_3^*(t_3)\}, \tag{27}$$

where

$$\kappa = \left[\frac{c^3 \cos \beta_1 \cos \beta_2 \cos \beta_3}{V v_1 v_2 v_3 \cos \delta_1 \cos \delta_2 \cos \delta_3}\right]^{1/2} \mu \tag{28}$$

and μ is given by (12). The times t_1, t_2, t_3 are given by (26).

When comparing the equations (24) and (27) we can see that both equations do not correspond because of the dispersion of z-components of ray velocities of the individual waves. Namely each normalized amplitude a_i is considered in the time t_i and the times t_1, t_2, t_3 are different. Thus the equation (27) is incorrect because three different times occur as parameters there.

Moreover when quantizing the normalized amplitudes $a_i(t_i)$ we are not able to guarantee the validity of corresponding commutation relations.

Mutual disagreement of quantum-mechanical and classical descriptions is, apparently, the consequence of the fact that both the spatial and frequentional dispersions are not included in the quantum-mechanical model of nonlinear interaction.

Only in the degenerate case, when $\omega_1 = \omega_2$, $\boldsymbol{e}_1 A_1 = \boldsymbol{e}_2 A_2$ and $\boldsymbol{k}_1 = \boldsymbol{k}_2$, i.e. for second harmonic generation or for parametric generation of half frequency, there is a perfect agreement between the quantum-mechanical and classical descriptions there.

The greater the dispersion of z-components of ray velocities is, the greater disagreement of both models appears; especially this disagreement is extremely great if any of considered frequencies lies near the absorption region.

To the similar inconveniences leads also a quantum-mechanical description that uses the localized momentum operator and the Heisenberg equations in spatial coordinates [30], whose application for studying nonlinear problems was proposed in [8].

The complete quantum-mechanical treatment of nonlinear optical interactions including dispersional effects is, perhaps, possible using the perturbation theory only.

However, the simple quantum-mechanical description of nonlinear optical processes with the trilinear Hamiltonian enables the treatment of many new effects, which do not cohere essentially with the dispersional effects. The description of such effects by means of more complicated quantum-mechanical model would inadequately aggravate their mathematical treatment.

We are of the opinion that, in rather good approximation, the results following from the quantum-mechanical description can be expressed in terms of classical quantities and spatial parameters, when reduced quantities are introduced which neglect the dispersional effects.

Generally the requirement of mutual correlation between the quantum-mechanical and classical descriptions leads to the following relations:
(*i*) The relation between z and t must be the same for all frequentional components that are considered:

$$z = \frac{c}{\nu_0} t, \tag{29}$$

where ν_0 is a reduced index of refraction.
(*ii*) The normalization relation (25) is to be replaced by the following one:

$$|A_i| = \left[\left(\frac{\mu_0}{\varepsilon_0}\right)^{1/2} \frac{2\hbar\omega_i c}{V\nu_0\nu_i \cos\delta_i \cos\beta_i}\right]^{1/2} |a_i|. \tag{30}$$

The relation between the photon fluxes (in classical description) and the mean photon numbers (in quantum-mechanical description) is then given by:

$$N_{i,z}(z) = \frac{c}{V\nu_0}\left\langle n_i\left(t = \frac{z\nu_0}{c}\right)\right\rangle, \tag{31}$$

and the constant of nonlinear coupling g in the quantum-mechanical description is given by

$$g = \left(\frac{c^3}{V\nu_0^3}\right)^{1/2} \mu$$
$$= K\left[\left(\frac{\mu_0}{\varepsilon_0}\right)^{3/2} \frac{2\hbar c^3\omega_1\omega_2\omega_3}{V\nu_0^3\nu_1\nu_2\nu_3 \cos\delta_1 \cos\delta_2 \cos\delta_3 \cos\beta_1 \cos\beta_2 \cos\beta_3}\right]^{1/2}. \tag{32}$$

It is very simple to make sure that using (29)–(32) the excellent agreement between all results of classical and quantum-mechanical descriptions appears for any reduced index of refraction ν_0 and any normalization volume V.

However, there are effects, such as parametric generation from quantum noise, for which the resulting formulae in terms of classical quantities depend on both the reduced index of refraction ν_0 and the normalization volume V.

With respect to above comparison we assume that the most proper way is to choose ν_0 as the geometrical average of "z-projections" of all three indices of refraction:

$$\nu_0 = \left(\frac{\nu_1\nu_2\nu_3 \cos\delta_1 \cos\delta_2 \cos\delta_3}{\cos\beta_1 \cos\beta_2 \cos\beta_3}\right)^{1/3} \tag{33}$$

and the normalization volume V is to be chosen as the effective volume of the nonlinear medium

$$V = S_{\text{effect}} L, \tag{34}$$

where S_{effect} labels the effective area on the surface of the nonlinear plate and L is the thickness of the plate.

However, the choice of the relations (33) and (34) could be a matter of discussion.

When introducing the relations for reduced quantities, one can see that the correlation relations for photon number operators of the type $\langle \mathcal{N} \hat{n}_i^k(t)\, \hat{n}_i^m(t+\tau)\rangle$ in t-domain correspond well to the correlation relations for intensities $\langle I_i^k(z)\, I_i^m(z+z_0)\rangle$ in z-domain, whilst the use of correlations of the type $\langle \mathcal{N} \hat{n}_i^k(t)\, \hat{n}_j^m(t+\tau)\rangle$ $(i \neq j)$ is irrelevant to the use of the correlations $\langle I_i^k(z)\, I_j^m(z+z_0)\rangle$ $(i \neq j)$.

In this paper we did not deal with the conditions that are put for the volume V in which the electromagnetic field is quantized. These conditions were treated in many studies (see e. g. [1, 2, 13, 18, 30]) and they have no special meaning for our purpose.

5. *Description of parametric generation from quantum noise in measurable quantities*

In this chapter we shall use the above introduced comparison to demonstrate the parametric generation from quantum noise in terms of measurable quantities in z-domain.

Parametric generation from quantum noise is a spontaneous decay of a pumping photon ω_3 into two photons ω_1 and ω_2 when both amplitudes of subfrequencies ω_1 and ω_2 equal zero at the beginning of the process. The classical theory is not able to explain this effect because the nonlinear polarizations at all considered frequencies equal zero at the beginning of the process [19–21]. The effect of parametric generation from quantum noise can be explained by means of quantum fluctuations when using the quantum-mechanical description [1, 2, 7, 17, 18, 26]. Thus the results of quantum-mechanical treatment have no analogy in the classical description.

In [26] it was shown that the parametric generation can begin from quantum noise with coherent pumping light having the Poisson photon number distribution [28] and the complete depletion of pumping light is possible.

The following expressions for the average photon numbers in the individual modes were found in t-domain in [26]:

$$\langle n_3(t)\rangle = n_{3,0}\, \frac{(n_{3,0}+1)\,\text{sech}^2\,(n_{3,0}^{1/2} g t)}{[1+n_{3,0}\,\text{sech}^2\,(n_{3,0}^{1/2} g t)]}\,, \tag{35}$$

$$\langle n_1(t)\rangle = \langle n_2(t)\rangle = n_{3,0}\, \frac{\tanh^2\,(n_{3,0}^{1/2} g t)}{[1+n_{3,0}\,\text{sech}^2\,(n_{3,0}^{1/2} g t)]}\,, \tag{36}$$

where $n_{3,0}$ labels the average photon number in the pumping mode at the frequency ω_3 at the beginning of the process ($t = 0$) and g is the constant of nonlinear coupling given by (32).

The average time of the first photon decay was computed as [26]:

$$\langle \tau_{\text{phot}} \rangle \doteq \frac{0.88}{n_{3,0}^{1/2} g} \,. \tag{37}$$

Now we shall use the relations (29)-(34) to express the formulae (35), (36), (37) in terms of classical quantities in z-domain.

It is convenient to express all quantities as functions of the thickness of nonlinear plate L.

The following expressions for the real amplitudes $A_1(L)$, $A_2(L)$, $A_3(L)$ at frequencies $\omega_1, \omega_2, \omega_3$, respectively, were found in the second boundary of the nonlinear plate:

$$|A_3(L)|^2 = |A_{3,0}|^2 \frac{\left(1 + \dfrac{S v_0 K}{2\hbar c \sigma_3} |A_{3,0}|^2 L\right) \operatorname{sech}^2 \left(|A_{3,0}| \sigma_1^{1/2} \sigma_2^{1/2} L\right)}{\left[1 + \dfrac{S v_0 K}{2\hbar c \sigma_3} |A_{3,0}|^2 L \operatorname{sech}^2 \left(|A_{3,0}| \sigma_1^{1/2} \sigma_2^{1/2} L\right)\right]}, \tag{38}$$

$$|A_{1,2}(L)|^2 = \frac{\sigma_{1,2}}{\sigma_3} |A_{3,0}|^2 \frac{\tanh^2 \left(|A_{3,0}| \sigma_1^{1/2} \sigma_2^{1/2} L\right)}{\left[1 + \dfrac{S v_0 K}{2\hbar c \sigma_3} |A_{3,0}|^2 L \operatorname{sech}^2 \left(|A_{3,0}| \sigma_1^{1/2} \sigma_2^{1/2} L\right)\right]}, \tag{39}$$

where $|A_{3,0}|$ is the absolute value of the amplitude of pumping radiation at frequency ω_3 in the first boundary of the nonlinear plate.

The total powers of the three considered waves at $\omega_1, \omega_2, \omega_3$ that emerge from the second boundary of the nonlinear plate, related to the vacuum, are given as follows:

The emerging power of pumping radiation at ω_3 behind the plate is given by

$$\mathscr{P}_3(L) = \mathscr{P}_{3,0} \frac{\left[1 + \left(\dfrac{\mu_0}{\varepsilon_0}\right)^{1/2} \dfrac{v_0 K \mathscr{P}_{3,0} L}{\hbar c \sigma_3 \cos \alpha_3}\right] \operatorname{sech}^2 \left(\left[2 \left(\dfrac{\mu_0}{\varepsilon_0}\right)^{1/2} \dfrac{\mathscr{P}_{3,0}}{S \cos \alpha_3} \sigma_1 \sigma_2\right]^{1/2} L\right)}{\left\{1 + \left(\dfrac{\mu_0}{\varepsilon_0}\right)^{1/2} \dfrac{v_0 K \mathscr{P}_{3,0} L}{\hbar c \sigma_3 \cos \alpha_3} \operatorname{sech}^2 \left(\left[2 \left(\dfrac{\mu_0}{\varepsilon_0}\right)^{1/2} \dfrac{\mathscr{P}_{3,0}}{S \cos \alpha_3} \sigma_1 \sigma_2\right]^{1/2} L\right)\right\}} \tag{40}$$

and the emerging powers of subfrequencies at ω_1 and ω_2 behind the plate are given by

$$\mathscr{P}_{1,2}(L) = \mathscr{P}_{3,0} \frac{\sigma_{1,2} \cos \alpha_{1,2}}{\sigma_3 \cos \alpha_3}$$

$$\times \frac{\tanh^2 \left(\left[2 \left(\dfrac{\mu_0}{\varepsilon_0}\right)^{1/2} \dfrac{\mathscr{P}_{3,0}}{S \cos \alpha_3} \sigma_1 \sigma_2\right]^{1/2} L\right)}{\left\{1 + \left(\dfrac{\mu_0}{\varepsilon_0}\right)^{1/2} \dfrac{v_0 K \mathscr{P}_{3,0} L}{\hbar c \sigma_3 \cos \alpha_3} \operatorname{sech}^2 \left(\left[2 \left(\dfrac{\mu_0}{\varepsilon_0}\right)^{1/2} \dfrac{\mathscr{P}_{3,0}}{S \cos \alpha_3} \sigma_1 \sigma_2\right]^{1/2} L\right)\right\}} \tag{41}$$

where $\sigma_1, \sigma_2, \sigma_3$ are given by (6), v_0 is to be taken according to (33) and S represents the effective area in the second boundary of the plate which covers all the emerging beams at frequencies $\omega_1, \omega_2, \omega_3$; $\alpha_1, \alpha_2, \alpha_3$ represent the angles of refraction into the medium behind the plate (vacuum or air), especially α_3 represents also the angle of incidence of pumping radiation at the first boundary.

For "one photon plate thickness", i. e. the thickness of the nonlinear plate in which the first pumping photon is decayed, we found with respect to (37)

$$\bar{L}_{\text{phot}} \doteq \frac{0.88}{\left[2\left(\frac{\mu_0}{\varepsilon_0}\right)^{1/2} \frac{\mathscr{P}_{3,0}}{S \cos \alpha_3} \sigma_1 \sigma_2\right]^{1/2}} . \tag{42}$$

The thickness of nonlinear plate $\bar{L}_{\text{phot}}$ can be considered as the minimum thickness of the nonlinear medium in which the process of parametric generation from quantum noise can start.

The boundary effects were neglected here.

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