GROUND STATE ENERGY OF A SPIN POLARIZED HARD CORE FERMI GAS

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The energy per particle, E/N, of the ground state of a spin polarized system of hard core spin 1/2 fermions is expanded in powers of $x = k_F c$ (k_F = Fermi momentum, c = hard core radius), with the result: $E/N = (\hbar^2 k_F^2/2M)\{3/5+(2/5\pi)x^3-(18/175\pi)x^5+0.01803x^6+(2728/70875\pi)x^7-0.01101x^8+\ldots\}$.

1. *Introduction*

The expansion of the ground state energy E of an infinite system of fermions interacting with a hard core potential, in powers of the gas parameter $x = k_F c$ (c = hard core radius, k_F = Fermi momentum in units of $\hbar$) has been investigated for a long time by several authors [1–15]. In general, a single particle state of a given momentum $\boldsymbol{k}$ may be occupied by ν particles, where ν is the number of spin and isospin degrees of freedom per particle. E. g., $\nu = 4$ for nuclear matter, $\nu = 2$ for neutron matter, an electron gas or liquid ^{3}He, $\nu = 1$ for a spin polarized neutron gas or electron gas or liquid ^{3}He. Usually, the expansion of E is presented for the case of a general value of ν. The x^3-approximation of this general expansion (i. e., including terms $\sim x^3$) is well established. It appears that beyond the x^3-approximation, problems arise with logarithmic terms ($\sim x^4 \ln x$) [8, 9, 12–14].

In the present paper, we consider the special case of $\nu = 1$. In this particular case it is easy to calculate all coefficients of the expansion in the x^8-approximation, and this is done in this paper. In the case of $\nu = 1$ (e. g., totally spin polarized neutron matter), all particles have spins with the same z-component (e. g., they all are spin-up particles), and their mutual interaction takes place only in states with odd relative angular momenta l. Now, the contribution to E of the interaction in the P state ($l = 1$) is at least $\sim x^3$, and the contribution of three-body diagrams to E is at least $\sim x^9$. Consequently, even if we

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restrict ourselves in the calculation of the interaction energy to the phase shift approximation, $\Delta_0 E$, and to the lowest order Pauli principle correction, $\Delta' E$ (it accounts for the effect of the Fermi sea on the two-body interaction), we get the exact energy within the x^8-approximation. This means that the x^8-approximation for $\nu = 1$ is, in principle, as simple as the x^2-approximation for $\nu = 2, 4$ (where the S state interaction gives a contribution to E, linear in x).

The results of the present paper apply to any hard core system of spin 1/2 fermions in the state of complete spin polarization (the ferromagnetic state). In particular, the model of hard core interaction has been used in discussing the possibility that neutron matter becomes ferromagnetic at a density, comparable to neutron star densities ([16, 17], and references therein). Also in the theory of the magnetism of metals, the hard core interaction model has been in use for a long time (for a review, see [18]).

Recently, the hard core fermion system has been used as a testing ground for various Jastrow type approximations [19], and the availability of our x^8-approximation in the very simple $\nu = 1$ case should be useful here. Actually, the need for the present work arose when discrepancies between variational results and the x^3-approximation were noticed in the case of $\nu = 1$ [17, 20].

We do not know any satisfactory estimate of the convergence radius of the energy expansion in powers of x. The whole expansion must blow up when the density approaches the close-packed limit $\varrho_{cp} = 2^{1/2} c^{-3}$ (for the fcc or hcp lattice), i. e., for $x = x_0 \leqslant x_{cp} = 2^{1/2}(3\pi^2/\nu)^{1/3} = 4.375/\nu^{1/3}$. The knowledge of the x^8-approximation might be helpful in bridging the gap between the low density limit (where the expansion in powers of x should work), and the high density limit where an asymptotic form $\sim (x-x_0)^{-2}$ has been suggested for the energy of a hard core gas [21].

The present paper is organized as follows. In Section 2, we derive the approximate expresion for the interaction energy, which contains the phase shift approximation $\Delta_0 E$, and the Pauli principle correction $\Delta' E$. In Section 3, we expand $\Delta_0 E$ and $\Delta' E$ in powers of x and calculate all the coefficients of the x^8-approximation. Appendix A contains formulas for integrals over momenta of two particles in the Fermi sea. In Appendix B, expressions for elements of the free reaction matrix $\mathscr{K}^0$ are derived, and expanded in powers of x. Appendix C contains a list of integrals which appear in expressions for $\Delta' E$.

The results of the present paper have been reported in [27].

2. *General expression for E*

We start with the unperturbed ground state of N spin-up fermions in a periodicity box of volume Ω. We have

$$k_F^3 = 6\pi^2 \varrho, \tag{2.1}$$

where $\varrho = N/\Omega$ is the density. The unperturbed (kinetic) energy is

$$E_0/N = (3/5)\varepsilon_F, \tag{2.2}$$

where $\varepsilon_F = \hbar^2 k_F^2/2M$ is the Fermi energy.

We express the contribution of hard core interaction to the energy, $\Delta E = E - E_0$, in terms of the Brueckner reaction matrix $\mathscr{K}$,

$$\Delta E = (1/2) \sum_{p_1 p_2}^{<k_F} (p_1 p_2 - p_2 p_1 | \mathscr{K} | p_1 p_2) + \ldots, \tag{2.3}$$

where states denoted by $|\boldsymbol{p})$ are plane waves normalized in volume Ω.

Let us introduce the relative, and center-of-mass momenta:

$$\boldsymbol{k} = (\boldsymbol{k}_1 - \boldsymbol{k}_2)/2, \quad \boldsymbol{K} = \boldsymbol{k}_1 + \boldsymbol{k}_2, \quad \boldsymbol{p} = (\boldsymbol{p}_1 - \boldsymbol{p}_2)/2, \quad \boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2, \tag{2.4}$$

$$(\boldsymbol{k}_1 \boldsymbol{k}_2 | \mathscr{K} | \boldsymbol{p}_1 \boldsymbol{p}_2) = \delta_{KP} (\boldsymbol{k} | \mathscr{K}_P | \boldsymbol{p}) = \delta_{KP} [(2\pi)^3/\Omega] \langle \boldsymbol{k} | \mathscr{K}_P | \boldsymbol{p} \rangle, \tag{2.5}$$

where $|\boldsymbol{p}\rangle = (2\pi)^{-3/2}\ |\boldsymbol{p})$.

Expression (2.3) for ΔE may be written in the form (with the help of relation (2.1)):

$$\Delta E/N = (3/8\pi k_F^3) \int_{<k_F} d\boldsymbol{p}_1 \int_{<k_F} d\boldsymbol{p}_2 [\langle \boldsymbol{p} | \mathscr{K}_P | \boldsymbol{p} \rangle - \langle -\boldsymbol{p} | \mathscr{K}_P | \boldsymbol{p} \rangle] + \ldots . \tag{2.6}$$

We use the reaction matrix $\mathscr{K}$, with pure kinetic single particle energies in the intermediate states. It is defined by the equation:

$$\langle \boldsymbol{k} | \mathscr{K}_P | \boldsymbol{p} \rangle = \langle \boldsymbol{k} | v | \boldsymbol{p} \rangle + (M/\hbar^2) \int d\boldsymbol{k}' \langle \boldsymbol{k} | v | \boldsymbol{k}' \rangle \frac{Q(\boldsymbol{P}, \boldsymbol{k}')}{p^2 - k'^2} \langle \boldsymbol{k} | \mathscr{K}_P | \boldsymbol{p} \rangle, \tag{2.7}$$

where v is the hard core two-body interaction, and Q the exclusion principle operator:

$$Q(\boldsymbol{P}, \boldsymbol{k}) = \begin{cases} 1 \text{ for } |\frac{1}{2}\boldsymbol{P} \pm \boldsymbol{k}| > k_F, \\ 0 \text{ otherwise.} \end{cases} \tag{2.8}$$

Next nonvanishing terms, denoted by dots in Eq. (2.6), which contribute to ΔE, are of the third order in $\mathscr{K}$. Strictly speaking, the higher order terms involve off-energy-shell elements of the $\mathscr{K}$ matrix, whereas Eq. (2.7) defines on-energy-shell (and half-off-energy--shell) elements, the only ones we shall actually need in our calculation.

To solve Eq. (2.7), we consider the free space reaction matrix $\mathscr{K}^0$, defined by the equation:

$$\langle \boldsymbol{k} | \mathscr{K}^0(z) | \boldsymbol{p} \rangle = \langle \boldsymbol{k} | v | \boldsymbol{p} \rangle + \frac{M}{\hbar^2} \int d\boldsymbol{k}' \langle \boldsymbol{k} | v | \boldsymbol{k}' \rangle \frac{1}{z^2 - k'^2} \langle \boldsymbol{k}' | \mathscr{K}^0(z) | \boldsymbol{p} \rangle, \tag{2.9}$$

where the principal value of the integral over k' is taken. (The principle value of all singular integrals occurring in this paper will be taken). Since we will need half-off-energy shell elements of $\mathscr{K}^0$, we indicate explicitly the dependence of $\mathscr{K}^0$ on the energy variable z. Obviously, $\mathscr{K}^0$ does not depend on the center-of-mass momentum.

From Eq. (2.9) and (2.7), we obtain

$$\langle \boldsymbol{k} | \mathscr{K}_P | \boldsymbol{p} \rangle = \langle \boldsymbol{k} | \mathscr{K}^0(p) | \boldsymbol{p} \rangle + \frac{M}{\hbar^2} \int d\boldsymbol{k}' \langle \boldsymbol{k} | \mathscr{K}^0(p) | \boldsymbol{k}' \rangle \frac{Q(\boldsymbol{P}, \boldsymbol{k}') - 1}{p^2 - k'^2} \langle \boldsymbol{k}' | \mathscr{K}_P | \boldsymbol{p} \rangle. \tag{2.10}$$

By solving Eq. (2.10) by iteration, we get

$$\langle \boldsymbol{k} | \mathscr{K}_P | \boldsymbol{p} \rangle = \langle \boldsymbol{k} | \mathscr{K}^0(p) | \boldsymbol{p} \rangle + \langle \boldsymbol{k} | \mathscr{K}'_P | \boldsymbol{p} \rangle + \ldots, \tag{2.11}$$

where

$$\langle \boldsymbol{k}|\mathscr{K}'_{\boldsymbol{P}}|\boldsymbol{p}\rangle = \frac{M}{\hbar^2}\int d\boldsymbol{k}'\langle \boldsymbol{k}|\mathscr{K}^0(p)\,|\boldsymbol{k}'\rangle \frac{Q(\boldsymbol{P},\boldsymbol{k}')-1}{p^2-k'^2}\langle \boldsymbol{k}'|\mathscr{K}^0(p)\,|\boldsymbol{p}\rangle. \tag{2.12}$$

Substituting the iteration series for $\mathscr{K}$, Eq. (2.11), into expression (2.6) for ΔE, we get

$$\Delta E = \Delta_0 E+\Delta' E+\ \dots, \tag{2.13}$$

where

$$\Delta_0 E/N = 16\pi \int_0^{k_{\mathrm{F}}} dp p^2 g(p/k_{\mathrm{F}})\,[\langle \boldsymbol{p}|\mathscr{K}^0(p)\,|\boldsymbol{p}\rangle - \langle -\boldsymbol{p}|\mathscr{K}^0(p)\,|\boldsymbol{p}\rangle], \tag{2.14}$$

where the function g is defined in Eq. (A.6), and

$$\Delta' E/N = (3/8\pi k_{\mathrm{F}}^3)\int_{<k_{\mathrm{F}}} d\boldsymbol{p}_1 \int_{<k_{\mathrm{F}}} d\boldsymbol{p}_2[\langle \boldsymbol{p}|\mathscr{K}'_{\boldsymbol{P}}|\boldsymbol{p}\rangle - \langle -\boldsymbol{p}|\mathscr{K}'_{\boldsymbol{P}}|\boldsymbol{p}\rangle]. \tag{2.15}$$

The terms indicated by dots in Eq. (2.13) are those resulting from higher iterations of Eq. (2.10) for $\mathscr{K}$, as well as terms of third and higher orders in $\mathscr{K}$ in Eq. (2.6).

If we introduce the partial wave expansion,

$$\langle \boldsymbol{k}|\mathscr{K}^0(z)\,|\boldsymbol{p}\rangle = (2\pi)^{-3}\sum_l (2l+1)\mathscr{K}_l^0(k,p;z)P_l(\hat{\boldsymbol{p}}\hat{\boldsymbol{k}}), \tag{2.16}$$

we may write expression (2.14) in the form

$$\Delta_0 E/N = (4/\pi^2)\sum_l{}^{\mathrm{o}}\,(2l+1)\int_0^{k_{\mathrm{F}}} dp p^2 g(p/k_{\mathrm{F}})\mathscr{K}_l^0(p,p;p), \tag{2.17}$$

where $\sum_l^{\mathrm{o}}$ indicates summations over odd values of l.

A partial wave expansion for $\mathscr{K}'_{\boldsymbol{P}}$ is complicated by the presence in Eq. (2.12) of the operator Q which we expand in Legendre polynomials [22][1],

$$Q(\boldsymbol{P},\boldsymbol{k}) = \sum_L{}^{e}\,Q_L(P,k)P_L(\hat{\boldsymbol{P}}\hat{\boldsymbol{k}}), \tag{2.18}$$

where $\sum_L^{e}$ indicates summation over even values of L,

$$Q_0(P,k) = \begin{cases} 1 \text{ for } k > \frac{1}{2}P+k_{\mathrm{F}}, \\ \gamma \text{ for } \frac{1}{2}P+k_{\mathrm{F}} > k > \sqrt{k_{\mathrm{F}}^2-P^2/4}\,, \\ 0 \text{ otherwise,} \end{cases} \tag{2.19}$$

$$\underset{(L\neq 0)}{Q_L(P,k)} = \begin{cases} P_{L+1}(\gamma)-P_{L-1}(\gamma) \text{ for } \frac{1}{2}P+k_{\mathrm{F}} > k > \sqrt{k_{\mathrm{F}}^2-P^2/4}, \\ 0 \text{ otherwise,} \end{cases} \tag{2.20}$$

where

$$\gamma = \gamma(P,k) = (k^2+\tfrac{1}{4}P^2-k_{\mathrm{F}}^2)/kP. \tag{2.21}$$

[1] We thank Dr P. Haensel for calling our attention to Ref. [22].

If we insert expansions (2.16) and (2.18) into Eq. (2.12) for $\langle \pm \boldsymbol{p}|\mathscr{K}'_P|\boldsymbol{p}\rangle$, we get

$$\langle \pm \boldsymbol{p}|\mathscr{K}'_P|\boldsymbol{p}\rangle = (2\pi)^{-3}\sum_l (\pm)^l(2l+1)\mathscr{K}'_l(\boldsymbol{P}, \boldsymbol{p}), \tag{2.22}$$

where

$$\mathscr{K}'_l(P, p) = \sum_L{}^e \mathscr{K}'_{l,L}(P, p)P_L(\hat{\boldsymbol{P}}\hat{\boldsymbol{p}}), \tag{2.23}$$

where

$$\mathscr{K}'_{l,0}(P, p) = \frac{M}{\hbar^2}4\pi\int\frac{dkk^2}{(2\pi)^3}\frac{Q_0(P, k)-1}{p^2-k^2}\mathscr{K}^0_l(k, p)^2, \tag{2.24}$$

$$\mathscr{K}'_{\substack{l,L\\(L\neq 0)}}(P, p) = \frac{M}{\hbar^2}\sum_{l'}(lL00|l'0)^2 4\pi\int\frac{dkk^2}{(2\pi)^3}\frac{Q_L(P, k)}{p^2-k^2}\mathscr{K}^0_l(k, p)\mathscr{K}^0_{l'}(k, p). \tag{2.25}$$

In the last two equations, we apply the notation

$$\mathscr{K}^0_l(k, p) = \mathscr{K}^0_l(k, p; p) = \mathscr{K}^0_l(p, k; p), \tag{2.26}$$

in which the symmetry property of $\mathscr{K}^0_l$ is used. The definition of $Q_L(P, k)$ implies finite limits for the k-integrals in Eqs (2.24) and (2.25). The upper limit is $P/2+k_F$ in both equations. The lower limit in Eq. (2.25) and in the Q_0 part of Eq. (2.24) is $(k_F^2-P^2/4)^{1/2}$, and in the -1 part of Eq. (2.24) is zero.

With the help of Eqs (2.22), (2.23), we may write expression (2.15) in the form (see Appendix A):

$$\Delta' E/N = (3/2\pi^2k_F^3)\sum_l{}^0 (2l+1)\int_0^{k_F} dpp^2\int dPP^2\{\alpha\mathscr{K}'_{l,0}(P, p)$$

$$+\sum_{L>0}{}^e (2L+1)^{-1}[P_{L+1}(\alpha)-P_{L-1}(\alpha)]\mathscr{K}'_{l,L}(P, p)\}, \tag{2.27}$$

where the function $\alpha = \alpha(P, p)$ is defined in Eq. (A.4).

Expressions for $\mathscr{K}^0_l(k, p)$ are given in Appendix B.

3. *The x^8-approximation*

The expansion of $\mathscr{K}^0_l(k, p)$ in powers of $x = k_Fc$ starts with the leading term $\sim x^{2l+1}$ (see Eq. (B.12)). For the lowest partial wave, the P wave ($l = 1$), we have $\mathscr{K}^0_1(k, p) \sim x^3 + O(x^5)$. This means (see Eq. (2.11)) that also the expansion of $\mathscr{K}_P$ in powers of x starts with x^3. Consequently, the contribution of third (and higher) order terms in $\mathscr{K}_P$ is at least $\sim x^9$. We conclude then that it is sufficient to keep in Eq. (2.6) terms linear in $\mathscr{K}_P$, if we want to calculate in the expansion of E terms of lower order than x^9.

Similarly, in our iterative procedure of determining $\mathscr{K}_P$ in terms of $\mathscr{K}^0$, Eq. (2.11), we may stop at $\mathscr{K}'_P$, if we want to calculate in E terms of lower order than x^9. Namely, the next iteration (beyond $\mathscr{K}'_P$) is of third order in $\mathscr{K}^0$, and thus would lead to terms at least $\sim x^9$. Furthermore, it is sufficient to consider only P wave contribution to $\mathscr{K}'_P$. Namely, for the next partial wave, the F wave ($l = 3$), we have $\mathscr{K}^0_3(k, p) \sim x^7 + O(x^9)$. Now, looking at Eq. (2.25), we see that the lowest power of x, in which the F wave participates, arises by combining in Eq. (2.25) P and F waves ($l = 1$, $l' = 3$ or $l = 3$, $l' = 1$). This, however, would lead to terms $\sim x^{3+7}$ in $\mathscr{K}'_P$.

On the other hand, we have to include the F wave in calculating $\Delta_0 E$ in the x^8-approximation, as it leads to a contribution $\sim x^7$.

First, we calculate $\Delta_0 E$. With the help of Eq. (B.8), we may write expression (2.17) for $\Delta_0 E$ in the form:

$$(\Delta_0 E/N)/\varepsilon_F = -(32/\pi) \sum_l^0 (2l+1) \int_0^1 d\tilde{p}\tilde{p}g(\tilde{p}) \operatorname{tg} \delta_l(p), \tag{3.1}$$

where $\tilde{p} = p/k_F$. After substituting for $\operatorname{tg} \delta_l(p)$ the expressions given in Eq. (B.13), and performing the simple integrations over $\tilde{p}$, we get

$$\begin{aligned}(\Delta_0 E/N)/\varepsilon_F &= (2/5\pi)x^3 - (18/175\pi)x^5 + \{(4/105\pi) + [28/70875\pi]\}x^7 + O(x^9) \\ &= 0.127324x^3 - 0.032740x^5 + \{0.012126 + [0.000126]\}x^7 + O(x^9),\end{aligned} \tag{3.2}$$

where all terms are due to the P state interaction, except for the very small term in the square brackets, which is due to the F state interaction. The total coefficient at x^7 is $\{2728/70875\,\pi\} = 0.012252$.

To calculate $\Delta' E$ in the x^8-approximation, we consider only the $l = 1$ term in Eq. (2.27), and only the $l' = 1$ term in Eq. (2.25), in which then the only possible value of L is 2. Consequently, we have

$$\Delta' E/N = [\Delta' E/N]_{L=0} + [\Delta' E/N]_{L=2} + O(x^{10}), \tag{3.3}$$

where

$$[\Delta' E/N]_{L=0} = (9/2\pi^2 k_F^3) \int_0^{k_F} dp p^2 \int dP P^2 \alpha \mathscr{K}'_{1,0}, \tag{3.4}$$

$$[\Delta' E/N]_{L=2} = (9/4\pi^2 k_F^3) \int_0^{k_F} dp p^2 \int dP P^2 \alpha(\alpha^2 - 1) \mathscr{K}'_{1,2}. \tag{3.5}$$

When we insert for $\mathscr{K}_{1,L}$, expressions (2.25), (2.26), and apply notation (B.10) we get

$$[(\Delta' E/N)/\varepsilon_F]_{L=0} = (72/\pi^2) \int_0^1 d\tilde{p}\tilde{p}^2 \int d\tilde{P}\tilde{P}^2 \tilde{\alpha} \int d\tilde{k}\tilde{k}^2 \frac{\tilde{Q}_0(\tilde{P}, \tilde{k}) - 1}{\tilde{p}^2 - \tilde{k}^2} k_1^0(\tilde{k}, \tilde{p})^2, \tag{3.6}$$

$$[(\Delta' E/N)/\varepsilon_F]_{L=2} = (72/5\pi^2) \int_0^1 d\tilde{p}\tilde{p}^2 \int d\tilde{P}\tilde{P}^2 \tilde{\alpha}(\tilde{\alpha}^2 - 1) \int d\tilde{k}\tilde{k}^2 \frac{\tilde{Q}_2(\tilde{P}, \tilde{k})}{\tilde{p}^2 - \tilde{k}^2} k_1^0(\tilde{k}, \tilde{p})^2, \tag{3.7}$$

where $\tilde{P} = P/k_F$, $\tilde{k} = k/k_F$, $\tilde{\alpha} = \tilde{\alpha}(\tilde{P}, \tilde{p}) \equiv \alpha(P, p)$ and $\tilde{Q}_L(\tilde{P}, \tilde{k}) \equiv Q_L(P, k)$.

If we substitute for $k_1^0(\tilde{k}, \tilde{p})^2$ the expression given in Eq. (B. 14), we get both for $L = 0$ and $L = 2$:

$$[(\Delta E/N)/\varepsilon_F]_L = X_{6,L}x^6 + X_{8,L}x^8 + O(x^{10}), \tag{3.8}$$

where

$$X_{6,L} = (8/\pi^2) \int_0^1 d\tilde{p}\tilde{p}^4 \int d\tilde{P}\tilde{P}^2\tilde{\alpha} \begin{cases} I_0(\tilde{P}, \tilde{p}) & \text{for } L = 0, \\ \frac{1}{5}(\tilde{\alpha}^2 - 1)I_2(\tilde{P}, \tilde{p}) & \text{for } L = 2, \end{cases} \tag{3.9}$$

$$X_{8,L} = -(6/5)X'_{6,L} + (8/5\pi^2) \int d\tilde{p}\tilde{p}^4 \int d\tilde{P}\tilde{P}^2\tilde{\alpha} \begin{cases} I'_0(\tilde{P}, \tilde{p}) & \text{for } L = 0, \\ \frac{1}{5}(\alpha^2 - 1)I'_2(\tilde{P}, \tilde{p}) & \text{for } L = 2, \end{cases} \tag{3.10}$$

where

$$I_0(\tilde{P}, \tilde{p}) = \int d\tilde{k}\tilde{k}^4(\tilde{Q}_0 - 1)/(\tilde{p}^2 - k^2), \tag{3.11}$$

$$I_2(\tilde{P}, \tilde{p}) = \int d\tilde{k}\tilde{k}^4\tilde{Q}_2/(\tilde{p}^2 - \tilde{k}^2), \tag{3.12}$$

$$I'_0(\tilde{P}, \tilde{p}) = \int d\tilde{k}\tilde{k}^4(\tilde{Q}_0 - 1), \tag{3.13}$$

$$I'_2(\tilde{P}, \tilde{p}) = \int d\tilde{k}\tilde{k}^4\tilde{Q}_2. \tag{3.14}$$

Formulas for I_L and I'_L are given in Appendix C. The expressions for $X'_{6,L}$ are the same as those for $X_{6,L}$, with the only difference that $\tilde{p}^4$ should be replaced by $\tilde{p}^6$.

Although the integrals over $\tilde{p}$ and $\tilde{P}$ can be performed analytically, we chose the much easier and safer way of numerical integration. We have applied an improved Gauss–Kronrod method [23, 24] with an expected relative error $< 10^{-5}$. Our results are:

$$X_{6,0} = 0.020573, \quad X_{6,2} = -0.002544,$$

$$X_{8,0} = -0.012944, \quad X_{8,2} = 0.001933. \tag{3.15}$$

Notice that if we neglected Q_2, i. e., approximated Q by its angle average Q_0, then $X_{6,2}$ and $X_{8,2}$ would vanish. Our results (3.15) show that the approximation $Q \cong Q_0$ would introduce an error of about 10% in calculating X_6 and X_8.

Let us collect Eqs (3.15), (3.2), and (2.2), and write our final result:

$$(E/N)/\varepsilon_F = \sum_{n=0}^{8} X_n x^n + O(x^9), \tag{3.16}$$

$$X_0 = 0.6, \quad X_1 = X_2 = X_4 = 0, \quad X_3 = 2/5\pi = 0.12732,$$

$$X_5 = -18/175\pi = -0.03274, \quad X_6 = 0.01803,$$

$$X_7 = 2728/70875\pi = 0.012225, \quad X_8 = -0.01101. \tag{3.17}$$

APPENDIX A

Formulas for integrals

Let us consider functions $F(P, p, \theta)$, $G(P, p)$, and $H(p)$, where $\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2$, $\boldsymbol{p} = (\boldsymbol{p}_1 - \boldsymbol{p}_2)/2$, and θ is the angle between $\boldsymbol{P}$ and $\boldsymbol{p}$. We have:

$$\int_{<k_F} d\boldsymbol{p}_1 \int_{<k_F} d\boldsymbol{p}_2 F(P, p, \theta) = (4\pi)^2 \int_0^{k_F} dp p^2 \left\{ \int_0^{2(k_F-p)} dP P^2 \int_{-1}^{1} \frac{d\cos\theta}{2} \right.$$

$$\left. + \int_{2(k_F-p)}^{2\sqrt{k_F{}^2-p^2}} dP P^2 \int_{-\bar{\alpha}}^{\bar{\alpha}} \frac{d\cos\theta}{2} \right\} F(P, p, \theta), \tag{A.1}$$

where

$$\bar{\alpha} = -\gamma(P, p) = (k_F^2 - P^2/4 - p^2)/Pp; \tag{A.2}$$

$$\int_{<k_F} d\boldsymbol{p}_1 \int_{<k_F} d\boldsymbol{p}_2 G(P, p) = (4\pi)^2 \int_0^{k_F} dp p^2 \left\{ \int_0^{2(k_F-p)} dP P^2 + \int_{2(k_F-p)}^{2\sqrt{k_F{}^2-p^2}} dP P^2 \bar{\alpha} \right\} G(P, p)$$

$$= (4\pi)^2 \int_0^{k_F} dp p^2 \int_0^{\infty} dP P^2 \alpha(P, p) G(P, p), \tag{A.3}$$

where

$$\alpha(P, p) = \begin{cases} 0 \text{ for } P > 2\sqrt{k_F^2 - p^2}\,, \\ \bar{\alpha} \text{ for } 2\sqrt{k_F^2-p^2} > P > 2(k_F - p), \\ 1 \text{ for } 2(k_F - p) > P; \end{cases} \tag{A.4}$$

$$\int_{<k_F} d\boldsymbol{p}_1 \int_{<k_F} d\boldsymbol{p}_2 H(p) = (8/3) k_F^3 (4\pi)^2 \int dp p^2 g(p/k_F) H(p), \tag{A.5}$$

where

$$g(\xi) = 1 - \tfrac{3}{2}\xi + \tfrac{1}{2}\xi^3. \tag{A.6}$$

APPENDIX B

Matrix elements of $\mathscr{K}^0$

We present here a very simple derivation of expressions for the half-off-energy-shell elements of $\mathscr{K}^0$.[2] We introduce partial wave radial functions u_l:

$$\mathscr{K}_l^0(k, p) = \mathscr{K}_l^0(k, p; p) = 4\pi \int dr r^2 j_l(kr) v(r) u_l(p, r). \tag{B.1}$$

Eq. (2.9) implies the following equation for u_l:

$$u_l(p, r) = j_l(pr) + 4\pi \int dr' r'^2 G_l(r, r'; p) v(r') u_l(p, r'), \tag{B.2}$$

[2] Expressions for completely off-energy-shell matrix elements of $\mathscr{K}^0$ are given, e. g., in [14], where they are derived from expressions for free scattering matrix t, given in [26].

where the Green function

$$G_l(r, r'; p) = \frac{1}{2\pi^2}\frac{M}{\hbar^2}\int dkk^2 j_l(kr)j_l(kr')/(p^2-k^2) = \frac{1}{4\pi}\frac{M}{\hbar^2}\,pj_l(pr_<)n_l(pr_>), \quad \text{(B.3)}$$

where $r_< = \min(r, r')$, and $r_> = \max(r, r')$.

Since the half-off-energy-shell elements of $\mathscr{K}^0$ are the same for hard core interaction, as for hard shell interaction [25], we insert for v in Eq. (B.2) the hard shell potential

$$v(r) = \lim_{\Lambda\to\infty} \Lambda\delta(r-c), \quad \text{(B.4)}$$

and get for the product vu_l:

$$v(r)u_l(p, r) = \lambda_l(p)\delta(r-c), \quad \text{(B.5)}$$

where

$$\lambda_l(p) = -j_l(pc)/4\pi c^2 G_l(c, c; p), \quad \text{(B.6)}$$

which follows from Eq. (B.2) and the requirement that $u_l(p, c) = 0$.

Inserting expression (B.5) into Eq. (B.1), and using expression (B.3) for G_l, we get

$$\mathscr{K}_l^0(k, p) = -4\pi(\hbar^2/M)p^{-1}j_l(kc)/n_l(pc). \quad \text{(B.7)}$$

In particular, we have

$$\mathscr{K}_l^0(p, p) = -4\pi(\hbar^2/M)p^{-1}\,\mathrm{tg}\,\delta_l(p), \quad \text{(B.8)}$$

where

$$\mathrm{tg}\,\delta_l(p) = j_l(pc)/n_l(pc), \quad \text{(B.9)}$$

where $\delta_l(p)$ is the phase shift for the hard core potential.

Eq. (B.7) may be written as

$$\mathscr{K}_l^0(k, p) = 4\pi(\hbar^2/M)p^{-1}k_l^0(\tilde{k}, \tilde{p}), \quad \text{(B.10)}$$

where $\tilde{k} = k/k_F$, $\tilde{p} = p/k_F$, and

$$k_l^0(\tilde{k}, \tilde{p}) = -j_l(\tilde{k}x)/n_l(\tilde{p}x), \quad \text{(B.11)}$$

where $x = k_F c$. Expanding k_l^0 in powers of x, we get

$$k_1^0(\tilde{k}, \tilde{p}) = \tfrac{1}{3}\,\tilde{p}^2\tilde{k}x^3 - (\tfrac{1}{30}\,\tilde{p}^2\tilde{k}^3 + \tfrac{1}{6}\,\tilde{p}^4\tilde{k})x^5$$

$$+(\tfrac{1}{840}\,\tilde{p}^2\tilde{k}^5 + \tfrac{1}{60}\,\tilde{p}^4\tilde{k}^3 + \tfrac{1}{60}\,\tilde{p}^6\tilde{k})x^7 + O(x^9),$$

$$k_3^0(\tilde{k}, \tilde{p}) = \tilde{p}^4\tilde{k}^3x^7/1575 + O(x^9),$$

$$k_l^0(\tilde{k}, \tilde{p}) = (2l+1)\tilde{p}^{l+1}\tilde{k}^l x^{2l+1}/[1\times 3\times \ldots (2l+1)]^2 + O(x^{2l+3}), \quad \text{(B.12)}$$

and for tg $\delta_l(p) = -k_l^0(\tilde{p}, \tilde{p})$, we get

$$\operatorname{tg} \delta_1(p) = -\tfrac{1}{3}\tilde{p}^3x^3 + \tfrac{1}{5}\tilde{p}^5x^5 - \tfrac{1}{7}\tilde{p}^7x^7 + O(x^9),$$

$$\operatorname{tg} \delta_3(p) = -\tilde{p}^7x^7/1575 + O(x^9),$$

$$\operatorname{tg} \delta_l(p) = -(2l+1)\tilde{p}^{2l+1}x^{2l+1}/[1\times 3\times \ldots (2l+1)]^2 + O(x^{2l+3}). \tag{B.13}$$

Finally, we have

$$k_1^0(\tilde{k}, \tilde{p})^2 = \tfrac{1}{9}x^6\tilde{p}^4k^2 - \tfrac{1}{9}x^8\tilde{p}^4\tilde{k}^2(\tfrac{1}{5}\tilde{k}^2 + \tilde{p}^2) + O(x^{10}). \tag{B.14}$$

APPENDIX C

Formulas for I_L and I'_L

After performing elementary integrations over $\tilde{k}$ in Eqs (3.11)–(3.14), we get:

$$I_0(\tilde{P}, \tilde{p}) = \tfrac{1}{3}(1+\tfrac{1}{2}\tilde{P})^2(1-\tfrac{1}{4}\tilde{P}) + \tilde{p}^2\{\tfrac{1}{2}(1+\tfrac{1}{2}\tilde{P})$$
$$+[(1-\tfrac{1}{4}\tilde{P}^2-\tilde{p}^2)/2\tilde{P}]\ln([(1+\tfrac{1}{2}\tilde{P})^2-\tilde{p}^2]/[1-\tfrac{1}{4}\tilde{P}^2-p^2])$$
$$-\tfrac{1}{2}\tilde{p}\ln([1+\tfrac{1}{2}\tilde{P}+\tilde{p}]/[1+\tfrac{1}{2}\tilde{P}-\tilde{p}])\}, \tag{C.1}$$

$$I_2(\tilde{P}, \tilde{p}) = (5/4\tilde{P}^3)\{A(\tilde{P}, \tilde{p}) + B(\tilde{P}, \tilde{p})\ln([(1+\tfrac{1}{2}\tilde{P})^2-\tilde{p}^2]/[1-\tfrac{1}{4}\tilde{P}^2-\tilde{p}^2])\}, \tag{C.2}$$

$$I'_0(\tilde{P}, \tilde{p}) = -\tfrac{1}{5}\{1+\tfrac{5}{4}\tilde{P}+\tfrac{5}{12}\tilde{P}^2+\tfrac{1}{192}\tilde{P}^5\}, \tag{C.3}$$

$$I'_2(\tilde{P}, \tilde{p}) = \tfrac{5}{16}\tilde{P}\{-1-\tfrac{4}{3}\tilde{P}-\tfrac{1}{2}\tilde{P}^2+\tfrac{1}{48}\tilde{P}^4\}, \tag{C.4}$$

where

$$A(\tilde{P}, \tilde{p}) = \tilde{P}\{-1+\tfrac{2}{3}\tilde{P}^2+\tfrac{1}{4}\tilde{P}^3+\tfrac{1}{16}\tilde{P}^4+\tfrac{1}{48}\tilde{P}^5$$
$$+\tilde{p}^2(2+\tfrac{1}{2}\tilde{P}+\tfrac{1}{8}\tilde{P}^3)-\tilde{p}^4(1+\tfrac{1}{2}\tilde{P})\}, \tag{C.5}$$

$$B(\tilde{P}, \tilde{p}) = (1-\tfrac{1}{4}\tilde{P}^2)^3-\tilde{p}^2(3-\tfrac{1}{2}\tilde{P}^2-\tfrac{1}{16}\tilde{P}^4)+\tilde{p}^4(3+\tfrac{1}{4}\tilde{P}^2)-\tilde{p}^6. \tag{C.6}$$

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