EFFECT OF PHOTON STATISTICS ON OPTICAL PARAMETRIC GENERATION FROM QUANTUM NOISE

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In this paper the process of parametric generation from quantum noise with both coherent pumping light, having the Poisson photon number distribution, and chaotic pumping light, having the Bose-Einstein photon number distribution, is studied by means of simple quantum-mechanical model. It is shown that in the case of coherent pumping light the complete depletion of the pumping wave is possible, whilst in the case of chaotic pumping light the parametric process does not begin what is, probably, due to the bunching effect in the chaotic light. Photon statistics of generated subfrequency waves are treated as well.

1. Introduction

Parametric generation of light in optically transparent nonlinear crystals is an effective method for construction of continuously tunable lasers [1, 2]. A spontaneous decay of a pumping photon ω_3 into two photons ω_1 and ω_2

$$\omega_3 = \omega_1 + \omega_2 \tag{1}$$

will be a matter of interest in this paper.

The greatest emission occurs when phase matching condition for wave vectors $\vec{k}_1, \vec{k}_2, \vec{k}_3$ at $\omega_1, \omega_2, \omega_3$, respectively, is also satisfied

$$\vec{k}_3 = \vec{k}_1 + \vec{k}_2. \tag{2}$$

Furthermore, this process can be controlled by a resonator which considerably narrows the spectral bandwidth of generated subfrequency radiations.

In this paper a three mode process will be considered only and the contributions of other frequency components which do not strictly satisfy both the phase matching condition (2) and corresponding resonance conditions will be neglected [1-4].

Assuming both amplitudes of subfrequencies ω_1, ω_2 to be zero at the beginning of the process, the parametric amplification can take place from quantum noise only and the process is told to start from "zero-point".

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From the point of view of phenomenological electrodynamics the parametric amplification of the original "zero-point" is impossible because nonlinear quadratic polarizations at ω_1 and ω_2 are equal to zero at the beginning of the process (see e.g. [4, 5]).

Parametric amplification of "zero-point" was studied by means of quantum-mechanical perturbation theory in [6, 7]. The satisfactory results were obtained there when the noise waves were assumed to possess the zero-point energy $\frac{1}{2}\hbar\omega_i$ $(i = 1, 2)$ per mode. However, the result in [6] providing that the parametric amplification of quantum noise is independent of the coherence properties of pump radiation is incorrect because the authors [6] distook the incoherent state for the Fock state.

Recently the dynamics of parametric processes with three mode Hamiltonian has been developed [8–12]. In some studies [8–10] the problem of three mode interaction is solved in such a way that the pump mode is assumed to be quite intense and it can be described classically. Such a method does not provide any results for the problem we are dealing with. More detailed solution was obtained in [11, 12] using the short time approximation. However, also this approximation seems not to be suitable for the description of parametric amplification of quantum noise because the average time necessary for decay of the first pump photon is quite behind the above approximation.

In this paper we shall use the method proposed by Crosignani et al. [13] for treatment of second harmonic generation which makes it possbile to find a closed solution for average photon numbers. The parametric generation process from quantum noise will be solved for two important cases, namely for coherent pumping light having the Poisson photon number distribution and for chaotic pumping light having the Bose–Einstein photon number distribution (Gaussian light). In both cases it will be necessary to admit the assumption that the statistics of pumping radiation is not changed in the course of nonlinear process. Such assumption seems to be reasonable with respect to the approximate results in [11, 12] and the results can be guaranteed for the beginning of the nonlinear process at least provided the pump field to be quite intense.

2. *The model Hamiltonian*

We shall assume transparent nonlinear medium in which the phase matching condition (2) is satisfied. The Hamiltonian describing three mode nonlinear process can be written in the form [8, 9]

$$H = \hbar\,(\omega_1 a_1^+ a_1 + \omega_2 a_2^+ a_2 + \omega_3 a_3^+ a_3) + \hbar g (a_1 a_2 a_3^+ + a_1^+ a_2^+ a_3), \tag{3}$$

where a_i and a_i^+ label the annihilation and creation operators relative to the i-th mode, and g is the real coupling constant. The annihilation and creation operators fulfil the commutation relations

$$[a_i, a_j] = [a_i^+, a_j^+] = 0; \quad [a_i, a_j^+] = \delta_{ij}. \tag{4}$$

The equations of motion for the number operators $n_i(t) = a_i^+(t) a_i(t)$ can be obtained using the Heisenberg equation of motion for any operator a_i

$$i\hbar \frac{da_i}{dt} = [a_i, H]. \tag{5}$$

Using the Heisenberg equations we obtain three photon number conservation laws in the standard way

$$\frac{dn_1}{dt} - \frac{dn_2}{dt} = 0, \tag{6a}$$

$$\frac{dn_1}{dt} + \frac{dn_3}{dt} = 0, \tag{6b}$$

$$\frac{dn_2}{dt} + \frac{dn_3}{dt} = 0. \tag{6c}$$

The second order differential equations of motion for the number operators were found in the form

$$\frac{d^2n_1}{dt^2} = \frac{d^2n_2}{dt^2} = -\frac{d^2n_3}{dt^2} = 2g^2(n_3n_1+n_3n_2-n_1n_2+n_3). \tag{7}$$

The Eq. (7) has to be supplemented by initial conditions

$$n_1(0)|\psi\rangle = n_2(0)|\psi\rangle = 0; \tag{8}$$

$$\langle\psi|n_3(0)\,|\psi\rangle = n_{3,0}; \quad \left\langle\psi\left|\frac{dn_3}{dt}\right|_{t=0}\psi\right\rangle = 0, \tag{9}$$

which correspond to the initial absence of quanta at frequencies ω_1 and ω_2; $|\psi\rangle$ representing a state of the system.

The equation of motion for expectation value $\langle n_3(t)\rangle = \langle\psi|n_3(t)|\psi\rangle$ is immediately obtained by means of equations (6)—(9) in the form

$$\frac{d^2}{dt^2}\langle n_3(t)\rangle = 2g^2[3\langle n_3^2(t)\rangle-4\langle n_3(0)n_3(t)\rangle-\langle n_3(t)\rangle+\langle n_3^2(0)\rangle]. \tag{10}$$

Solving the equation (10), without the help of a hierarchy of equations for higher order expressions of the kind $\langle n_3^k(t)n_3^l(0)\rangle$, requires a suitable factorization assumption on $\langle n_3(t)n_3(0)\rangle$ and $\langle n_3^2(t)\rangle$ [13].

Two important cases will be treated in the following sections.

3. Parametric generation with coherent pumping light

First we shall consider the coherent pumping light with Poisson photon number distribution. The factorization condition is to be taken in the form [14]

$$\langle n_3^2(t)\rangle = \langle n_3(t)\rangle^2+\langle n_3(t)\rangle, \tag{11a}$$

$$\langle n_3^2(0)\rangle = n_{3,0}^2+n_{3,0}, \tag{11b}$$

$$\langle n_3(t)n_3(0)\rangle = \langle n_3(t)\rangle n_{3,0}+\langle n_3(t)\rangle. \tag{11c}$$

Supposing that the pump mode statistics is conserved in the course of the nonlinear process we obtain the following second order non-linear differential equation for $\langle n_3(t)\rangle$

$$\frac{d^2}{dt^2}\langle n_3(t)\rangle = 2g^2[3\langle n_3(t)\rangle^2-2(2n_{3,0}+1)\langle n_3(t)\rangle+(n_{3,0}+1)n_{3,0}]. \tag{12}$$

Taking into consideration the initial conditions (9) we obtain the solution of the equation (12) in the form (see e.g. [15])

$$t=\frac{1}{2g}\int_{\langle n_3(t)\rangle}^{n_{3,0}}[(n_{3,0}+1-x)(n_{3,0}-x)x]^{-1/2}dx, \tag{13}$$

or it can be written in the form of the elliptical integral [16]

$$t=\frac{1}{g(n_{3,0}+1)^{1/2}}\int_0^{\left[\frac{(n_{3,0}+1)(n_{3,0}-\langle n_3(t)\rangle)}{n_{3,0}(n_{3,0}+1-\langle n_3(t)\rangle)}\right]^{1/2}}\left\{(1-x^2)\left[1-\left(\frac{n_{3,0}}{n_{3,0}+1}\right)x^2\right]\right\}^{-1/2}dx. \tag{14}$$

Supposing that the pumping radiation is quite intense and it holds $n_{3,0}\gg 1$, equation (14) can be put in explicit form in a very good approximation as follows

$$\langle n_3(t)\rangle_{coh}=n_{3,0}\frac{(n_{3,0}+1)\,\mathrm{sech}^2\,(n_{3,0}^{1/2}gt)}{[1+n_{3,0}\,\mathrm{sech}^2\,(n_{3,0}^{1/2}gt)]}. \tag{15}$$

The last form describes the decrease of average pump photon number at ω_3 with the time.

The expressions for increase of average photon numbers of generated subfrequencies ω_1 and ω_2 can be found by means of conservation laws (6)

$$\langle n_1(t)\rangle_{coh}=\langle n_2(t)\rangle_{coh}=n_{3,0}\frac{\tanh^2\,(n_{3,0}^{1/2}\,gt)}{[1+n_{3,0}\,\mathrm{sech}^2\,(n_{3,0}^{1/2}gt)]}. \tag{16}$$

It is obvious from (15) and (16) that the total conversion of pump radiation energy at ω_3 into energy of two subfrequencies ω_1 and ω_2 can take place as t goes to infinity provided that $n_{3,0}\gg 1$.

From (15) or (16) we can compute the average time of the first photon decay

$$\langle\tau_{phot}\rangle_{coh}=\frac{1}{n_{3,0}^{1/2}g}\,\mathrm{Arg\,tanh}\left[\tfrac{1}{2}\left(1+\frac{1}{n_{3,0}}\right)\right]^{1/2}\doteq\frac{0.88}{n_{3,0}^{1/2}g}. \tag{17}$$

The process of parametric generation gains its maximum efficiency in a time τ_{eff} when $\frac{d^2}{dt^2}\langle n_3(\tau_{eff})\rangle=-\frac{d^2}{dt^2}\langle n_{1,2}(\tau_{eff})\rangle=0$. It was found

$$\tau_{eff}\doteq\frac{\mathrm{Arg\,cosh}\,(n_{3,0}^{1/2})}{n_{3,0}^{1/2}g}. \tag{18}$$

One can see from (17) and (18) that the effective parametric generation needs relatively long times in comparison to other nonlinear processes. For instance the average time of the first pump photon decay $\langle\tau_{phot}\rangle$ given by (17) is comparable with the time in which the majority of fundamental frequency wave photons can be converted in second harmonic wave photons [13]. All the times describing the real processes of parametric amplification of quantum noise fulfil the appropximate condition

$$\tau n_{3,0}^{1/2} g \gtrsim 0.9, \tag{19}$$

which is in contradiction with the approximation of short times used in [11] and [12].

5. *Parametric generation with chaotic pumping light*

Now we shall consider the chaotic pumping light having the Bose–Einstein photon number distribution with the following factorization properties [14, 19]

$$\langle n_3^2(t)\rangle = 2\langle n_3(t)\rangle^2 + \langle n_3(t)\rangle, \tag{20a}$$

$$\langle n_3^2(0)\rangle = 2n_{3,0}^2 + n_{3,0}, \tag{20b}$$

$$\langle n_3(t)n_3(0)\rangle = \langle n_3(t)\rangle n_{3,0}[1+|\gamma_{11}(t)|^2] + \langle n_3(t)\rangle, \tag{20c}$$

where $\gamma_{11}(t)$ is the second-order degree of coherence. We suppose the light to have the Lorentzian shape of spectral line so that [14]

$$|\gamma_{11}(t)|^2 = \exp(-2\Gamma|t|), \tag{21}$$

where Γ denotes the spectral half width.

Inserting the relations (20) in the differential equation (10) we get following second--order non-linear differential equation for $\langle n_3(t)\rangle$

$$\frac{d^2}{dt^2}\langle n_3(t)\rangle = 2g^2\{6\langle n_3(t)\rangle^2 - 2[2n_{3,0}(1+|\gamma_{11}(t)|^2)+1]\langle n_3(t)\rangle + (2n_{3,0}+1)n_{3,0}\}. \tag{22}$$

By rather simple procedure (see e.g. [15]) regarding the initial conditions (9) we obtain from (22) the following non-linear integro-differential equation which is of the first order with respect to the time derivative of $\langle n_3(t)\rangle$

$$\left(\frac{d}{dt}\langle n_3(t)\rangle\right)^2 = 4g^2\Big\{2\langle n_3(t)\rangle^3 - (2n_{3,0}+1)\langle n_3(t)\rangle^2 + (2n_{3,0}+1)n_{3,0}\langle n_3(t)\rangle$$

$$-2n_{3,0}^3 - 4n_{3,0}\int_0^t \langle n_3(t')\rangle \frac{d\langle n_3(t')\rangle}{dt'}|\gamma_{11}(t')|^2 dt'\Big\}. \tag{23}$$

For the real process of parametric noise amplification the first derivative $\frac{d}{dt}\langle n_3(t)\rangle$

must be negative in certain time interval at the beginning of the non-linear process. Thus we can suppose that the following relation holds in a time interval $(0, t)$:

$$-\langle n_3(t')\rangle \frac{d\langle n_3(t')\rangle}{dt'} \geqslant 0. \tag{24}$$

Making use of the well known mean value theorem for the definite integral we can express the integral on the right-hand side of (23) as follows

$$\int_0^t n_3(t') \frac{d\langle n_3(t')\rangle}{dt'} |\gamma_{11}(t')|^2 dt' = \frac{a}{2}(\langle n_3(t)\rangle^2 - n_{3,0}^2), \tag{25}$$

where $0 \leqslant a \leqslant 1$.

Then the equation (23) takes the form

$$\left(\frac{d}{dt}\langle n_3(t)\rangle\right)^2 = 4g^2[\langle n_3(t)\rangle - n_{3,0}]$$

$$\times[2\langle n_3(t)\rangle^2 - (2n_{3,0}a+1)\langle n_3(t)\rangle + 2n_{3,0}^2(1-a)]. \tag{26}$$

First we shall assume that the process of parametric amplification of quantum noise does start and that the pumping radiation has extremely high degree of coherence so that it holds $|\gamma_{11}(t)|^2 = 1$ at the beginning of the process.

The solution of (26) for this case can be obtained laying $a = 1$ in the following form (see e.g. [15] and [16])

$$\langle n_3(t)\rangle_{\text{chaot}} = n_{3,0} \frac{(n_{3,0}+\frac{1}{2})\,\text{sech}^2\,(2^{1/2}n_{3,0}^{1/2}gt)}{[\frac{1}{2}+n_{3,0}\,\text{sech}^2\,(2^{1/2}n_{3,0}^{1/2}gt)]}. \tag{27}$$

From the last expression the average time of the first pump photon decay for chaotic pumping light can be computed

$$\langle\tau_{\text{phot}}\rangle_{\text{chaot}} = \frac{1}{2^{1/2}n_{3,0}^{1/2}g}\,\text{Arg}\tanh\left[\frac{2(n_{3,0}+\frac{1}{2})}{3n_{3,0}}\right]^{1/2} \doteq \frac{0.8}{n_{3,0}^{1/2}g}, \tag{28}$$

which is approximately equal to the $\langle\tau_{\text{phot}}\rangle_{\text{coh}}$ given by (17).

For the equation (26) to have a real solution the second term on the right-hand side of (26) must be negative or equal to zero, i.e.

$$[2\langle n_3(t)\rangle^2 - (2n_{3,0}a+1)\langle n_3(t)\rangle + 2n_{3,0}^2(1-a)] \leqslant 0. \tag{29}$$

Assuming that the last condition must be fulfilled at the beginning of the process at least, though in the time before the first pump photon will be decayed, we obtain the existence condition for realization of parametric generation process

$$1 - \frac{1}{4n_{3,0}} \leqslant a \leqslant 1. \tag{30}$$

Now we shall introduce the following average quantities for the decay of the first photon

$$\overline{\langle n_3(t)\rangle}_{\text{phot}} = n_{3,0}-\tfrac{1}{2}; \quad \overline{\left(\frac{d\langle n_3(t)\rangle}{dt}\right)}_{\text{phot}} = -\frac{1}{\langle\tau_{\text{phot}}\rangle_{\text{chaot}}}. \tag{31}$$

Making use of the last expressions the integral on the right-hand side of (23) can be computed for $t = \langle\tau_{\text{phot}}\rangle_{\text{choat}}$ as follows

$$\int\limits_0^{\langle\tau_{\text{phot}}\rangle_{\text{chaot}}} \langle n_3(t)\rangle \frac{d\langle n_3(t)\rangle}{dt} |\gamma_{11}(t)|^2 dt = -\frac{(n_{3,0}-\frac{1}{2})}{2\Gamma\langle\tau_{\text{phot}}\rangle_{\text{chaot}}}\left[1-\exp\left(-2\Gamma\langle\tau_{\text{phot}}\rangle_{\text{chaot}}\right)\right]. \tag{32}$$

By means of (25), (28), (30) and (32) and using Taylor expansion of $\exp(-2\Gamma\langle\tau_{\text{phot}}\rangle_{\text{choat}})$ we can find the final form of existence condition for realization of parametric generation process from quantum noise with chaotic pumping light

$$\tau_{\text{coh}} = \frac{1}{2\Gamma} > \frac{(2n_{3,0})^{1/2}}{g}, \tag{33a}$$

where $\tau_{\text{coh}} = 1/2\,\Gamma$ labels coherence time of pumping radiation.

The parametric generation process does not start when the reverse condition holds

$$\tau_{\text{coh}} = \frac{1}{2\Gamma} \lesssim \frac{(2n_{3,0})^{1/2}}{g}. \tag{33b}$$

Condition (33a) can never be fulfilled in real cases.

There is an apparent inconsistency in our treatment. Namely the right-hand side of equation (22) is not equal to zero when the solution of (26) is a constant $\langle n_3(t)\rangle = n_{3,0}$. This apparent inconsistency is a consequence of the fact that we have neglected the changes of $\langle n_3(t)\rangle$ less than one in our considerations.

5. *Fourth-order statistics of generated subfrequency radiations*

There are two other ways of deriving equations (12) and (22) there. Either it is possible to take directly the average of equation (7) or to derive equations for n_1 and n_2 and to take average of them similarly to the equation (10) for $\langle n_3(t)\rangle$. Both the ways certainly require a suitable assumption on statistics of generated subfrequency radiations. When suitable factorization rules are found, both the ways have to provide exactly the same results as in Sections 3 and 4. Strictly speaking this method is not fully correct from mathematical point of view, however, it leads to the good results.

The following results were found by means of the above mentioned method.

(*i*) For the case of parametric generation with coherent pumping light having Poisson statistics the generated subfrequency radiations obey the following factorization relations for photon numbers

$$\langle n_1^2(t)\rangle = \langle n_2^2(t)\rangle = \langle n_1(t)n_2(t)\rangle = \langle n_{1,2}(t)\rangle^2+\langle n_{1,2}(t)\rangle, \tag{34a}$$

$$\langle n_3(0)n_{1,2}(t)\rangle = n_{3,0}\langle n_{1,2}(t)\rangle+\langle n_{1,2}(t)\rangle, \tag{34b}$$

$$\langle n_3(t)n_{1,2}(t)\rangle = \langle n_3(t)\rangle\langle n_{1,2}(t)\rangle. \tag{34c}$$

The factorization relations (34) provide that both generated subfrequency radiations are coherent ones as well. At the same time the term describing the quantum fluctuation noise is missing in (34c), that means the pumping radiation is not correlated to both subfrequency radiations and it behaves with respect to them as a classical one.

(*ii*) For the case of chaotic pumping light with Bose–Einstein statistics, when admitting the academical chance of parametric noise amplification, both generated subfrequency radiations would obey the following factorization relations for photon numbers

$$\langle n_1^2(t)\rangle = \langle n_2^2(t)\rangle = \langle n_1(t)n_2(t)\rangle = 2\langle n_{1,2}(t)\rangle^2+\langle n_{1,2}(t)\rangle, \tag{35a}$$

$$\langle n_3(0)n_{1,2}(t)\rangle = 2[2-|\gamma_{11}(t)|^2]n_{3,0}\langle n_{1,2}(t)\rangle-2[1-|\gamma_{11}(t)|^2]n_{3,0}^2+\langle n_{1,2}(t)\rangle, \tag{35b}$$

$$\langle n_3(t)n_{1,2}(t)\rangle = 2|\gamma_{11}(t)|^2\langle n_3(t)\rangle\langle n_{1,2}(t)\rangle-2[1-|\gamma_{11}(t)|^2]\langle n_3(t)\rangle^2. \tag{35c}$$

That is clear that the generated subfrequency radiations would possess Bose–Einstein statistics, i.e. they are also chaotic ones. The correlation relations between the pump radiation and the generated subfrequency radiations, respectively, given by (35b) and (35c), are rather complicated and their detailed discussion would be rather difficult as far as the explicit solutions for $\langle n_3(t)\rangle$ and $\langle n_{1,2}(t)\rangle$ were not known.[1] We shall not deal with such a discussion because it is not important from practical point of view.

6. *Conclusion*

From the above treatment it follows that the process of parametric generation from quantum noise can start with coherent pumping radiation having the Poisson statistics. In this case both generated subfrequencies are also coherent ones with the Poisson photon number distribution and the total energy conversion is possible provided that $n_{3,0} \gg 1$.

Parametric generation process cannot start from "zero point" when pumping radiation is chaotic with Bose–Einstein statistics as condition (33a) cannot be fulfilled for real chaotic radiations. The impossibility of parametric generation with chaotic pumping light is, probably, due to the bunching effect in chaotic light that evidently acts against the decay of pump photons, whilst in the case of the second harmonic generation the bunching effect influences favourably the nonlinear process [17].

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[1] The apparent atibunching effect following from (35c) if t tends to infinity is not a real one because for this case $\langle n_{1,2}(t)\rangle$ have to be equal zero.

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