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RENORMALIZED SPIN WAVES IN THE HEISENBERG FERROMAGNET

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The self-consistent renormalization of spin waves in the Heisenberg ferromagnet described in terms of Dyson's ideal magnon modes is proposed. As a first step, the renormalization of the spin wave energy and the average spin wave occupation numbers is carried out by resorting to series of bubble graphs due to dynamic and kinematic interactions. The second step consists in evaluating how the renormalization of spin waves is contributed to form diagrams being composed of one energy denominator. Such an approximation is proved to hold within the entire interval of temperatures from absolute zero to the Curie point.

1. Introduction

In one of our preceding papers [1] a procedure for figuring out series of bubble graphs due to dynamic interaction between spin waves was established. The results obtained therein agreed with those of Bloch [2] and Loly [3] which were derived along different lines.

The attempt made in [4] and [5] to include into the self-consistent renormalization of spin waves the bubble diagrams due to the kinematic interaction of magnons did not suffice for yielding the correct contribution. Indeed, the kinematic bubble graphs entered the exponents of the spin wave population numbers in the form of positive functions and thus increased the magnetization of a ferromagnet at all temperatures, which was incompatible with the experimental data.

In order to improve the renormalization due to the kinematic interaction between spin waves, we should compute a series of diagrams comprising one energy denominator. The contribution of such diagrams prevails against the disadvantegous effect of kinematic bubble graphs on the renormalization of average spin wave occupation numbers and correctly affects the spin wave energy.

There is still another side to this question. In our papers [4, 5] the ring diagrams deficient in energy denominators were not taken into account, because they were not

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adapted to our scheme of self-consistent renormalization of spin waves. It turns out, however, that the series of kinematic bubble graphs is divergent, see e.g. [6]. As the mean value of the kinematic operator over the Gibbs ensemble must be finite this divergence has to cancel out with the similar one due to the series of ring diagrams. As to the latter, they result from higher order terms of the kinematic operator. Hence, the divergence can be avoided by retaining only several kinematic bubble graphs of the lowest order.

The investigation throughout this paper is based on Matsubara [7] thermodynamical perturbation calculus. Different approaches were given in papers [8–11].

2. *The partition function*

Proceeding with the investigation of our former papers, we shall describe a cubic isotropic ferromagnet, by using the Heisenberg exchange Hamiltonian and the Zeeman term expressed in Dyson [12] ideal spin wave modes i.e.

$$\tilde{H} = E_0 + H_0 + H_I, \tag{2.1}$$

$$E_0 = -LSN - \tfrac{1}{2}JNS^2\gamma_0, \tag{2.2}$$

$$H_0 = \sum_{\lambda} (L + \varepsilon_\lambda) a_\lambda^* a_\lambda, \tag{2.3}$$

$$\varepsilon_\lambda = JS(\gamma_0 - \gamma_\lambda), \tag{2.4}$$

$$\gamma_\lambda = \sum_{\delta} \exp i\boldsymbol{\lambda}\cdot\boldsymbol{\delta}, \tag{2.5}$$

$$H_I = -\tfrac{1}{4} JN^{-1} \sum_{\lambda\varrho\sigma} \Gamma_{\varrho,\sigma}^{\lambda} a_{\sigma+\lambda}^* a_{\varrho-\lambda}^* a_\varrho a_\sigma, \tag{2.6}$$

$$\Gamma_{\varrho,\sigma}^{\lambda} = \gamma_\lambda + \gamma_{\lambda+\sigma-\varrho} - \gamma_{\lambda+\sigma} - \gamma_{\lambda-\varrho}, \tag{2.7}$$

where L is the magnetic field strength multiplied by the Bohr magneton and Landé isotropic factor, S is the quantum number of the resultant atomic spin, N is the number of lattice sites in the crystal under consideration, J is the exchange integral between nearest neighbours, ε_λ is the energy of independent spin waves, $\boldsymbol{\lambda}$, $\boldsymbol{\varrho}$ and $\boldsymbol{\sigma}$ are reciprocal lattice vectors, $\boldsymbol{\delta}$ are vectors connecting all nearest neighbours and a_λ^*, a_λ represent the creation and annihilation Bose-operators of ideal spin waves.

Reffering to [4], [7] and [13], we get for the partition function

$$Z = \mathrm{Tr}\,(e^{-\beta\tilde{H}}\hat{K}_S) = e^{-\beta E_0}\,\mathrm{Tr}\,(e^{-\beta H_0})\,\frac{\mathrm{Tr}\,(e^{-\beta H_0}\hat{S}(\beta)\hat{K}_S)}{\mathrm{Tr}\,e^{-\beta H_0}}$$

$$= \exp\left[-\beta E_0 + \sum_{\lambda} \ln(1+\bar{n}_\lambda) + \sum_{p=1}^{\infty} D_p + \sum_{p=0}^{\infty} C_p\right], \quad \beta = \frac{1}{kT}, \tag{2.8}$$

where

$$\bar{n}_\lambda = [\exp \beta(L+\varepsilon_\lambda) - 1]^{-1} \tag{2.9}$$

is the average occupation number of independent spin waves, and

$$\hat{S}(\beta) = e^{\beta H_0} e^{-\beta(H_0+H_I)} = \hat{T} \exp\left[-\int_0^\beta d\tau H_I(\tau)\right], \tag{2.10}$$

$$H_I(\tau) = e^{\tau H_0} H_I e^{-\tau H_0}. \tag{2.11}$$

The expressions

$$D_p = \frac{(-1)^p}{p!} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \ldots \int_0^\beta d\tau_p \langle \hat{T}[H_I(\tau_1)H_I(\tau_2) \ldots H_I(\tau_p)]\rangle_c, \quad p = 1, 2, 3, \ldots, \tag{2.12}$$

with

$$\langle \hat{A} \rangle = \frac{\mathrm{Tr}\,(e^{-\beta H_0}\hat{A})}{\mathrm{Tr}\,e^{-\beta H_0}} \tag{2.13}$$

we shall call dynamic graphs (diagrams). They are due to the operator H_I responsible for the dynamic interaction of spin waves.

The cross-term diagrams ($p \neq 0$)

$$C_p = \frac{(-1)^p}{p!} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \ldots \int_0^\beta d\tau_p \langle \hat{T}\{H_I(\tau_1)H_I(\tau_2) \ldots H_I(\tau_p)\,[\hat{K}_S(0)-1]\}\rangle_c \tag{2.14}$$

result from dynamic and kinematic interactions between ferromagnons. Finally,

$$C_0 = \langle[\hat{K}_S(0)-1\rangle_c \tag{2.15}$$

is the series of kinematic graphs.

In the above equations $\hat{T}$ is the Wick [14] ordering symbol and c emphasized the necessity of allowing for the connected diagrams only, as the disconnected ones appear from the expansion of $\exp\,(\sum_p D_p + \sum_p C_p)$ in a series, see e. g. [18], [19]. $\hat{K}_S$ is the kinematic projection operator ensuring that the spectrum of eigen-values of $a_f^* a_f$, where

$$a_f^* = N^{-1/2} \sum_\lambda a_\lambda^* \exp\,(i\boldsymbol{\lambda}\cdot\boldsymbol{f}), \tag{2.16a}$$

$$a_f = N^{-1/2} \sum_\lambda a_\lambda \exp\,(-i\boldsymbol{\lambda}\cdot\boldsymbol{f}) \tag{2.16b}$$

and $\boldsymbol{f}$ is the lattice vector, has to be correctly $0, 1, 2, \ldots, 2S$ instead of $0, 1, 2, \ldots \infty$. Explicitely [15, 16],

$$\hat{K}_S(0) \equiv K_S = \prod_f \theta(2S - a_f^* a_f), \tag{2.17}$$

$$\theta(2S - a_f^* a_f) \triangleq 1 - \sum_{l=0}^{\infty} (-1)^l [(2S)!\,l!\,(2S+l+1)]^{-1} (a_f^*)^{2S+l+1} a_f^{2S+l+1}, \tag{2.18}$$

$$\theta(x) = \begin{cases} 1, & x \geqslant 0, \\ 0, & x < 0. \end{cases} \tag{2.19}$$

For the cases $S = 1/2$, 1, $3/2$ etc.

$$\hat{K}_{1/2} = 1-\tfrac{1}{2}\sum_f (a_f^*)^2 a_f^2+\tfrac{1}{3}\sum_f (a_f^*)^3 a_f^3+\frac{1}{2!}(-\tfrac{1}{2})^2\sum_{f_1,f_2}(a_{f_1}^*)^2(a_{f_2}^*)^2 a_{f_1}^2 a_{f_2}^2$$

$$-\tfrac{1}{4}\sum_f (a_f^*)^4 a_f^4+(-\tfrac{1}{2})(\tfrac{1}{3})\sum_{f_1,f_2}(a_{f_1}^*)^2(a_{f_2}^*)^3 a_{f_1}^2 a_{f_2}^3+\tfrac{1}{5}\sum_f (a_f^*)^5 a_f^5+\ldots, \tag{2.20}$$

$$\hat{K}_1 = 1-\tfrac{1}{6}\sum_f (a_f^*)^3 a_f^3+\tfrac{1}{8}\sum_f (a_f^*)^4 a_f^4-\tfrac{1}{20}\sum_f (a_f^*)^5 a_f^5+\ldots, \tag{2.21}$$

$$\hat{K}_{3/2} = 1-\tfrac{1}{24}\sum_f (a_f^*)^4 a_f^4+\tfrac{1}{30}\sum_f (a_f^*)^5 a_f^5+\ldots \tag{2.22}$$

and so forth.

The diagrams (2.12), (2.14) and (2.15) can be figured out by applying the Wick [14] and Thouless [13] theorems and by using the following propagators of independent spin waves (close to their mathematical expressions are given corresponding graphical forms according to [7]):

$$a_\varrho^*(\tau_1)^\bullet a_\sigma(\tau_2)^\bullet = \delta_{\varrho,\sigma} e^{(L+\varepsilon_\varrho)(\tau_1-\tau_2)}[\theta(\tau_1-\tau_2)\bar{n}_\varrho+\theta(\tau_2-\tau_1)(\bar{n}_\varrho+1)] \tag{2.23}$$

$$a_\varrho(\tau_1)^\bullet a_\sigma^*(\tau_2)^\bullet = \delta_{\varrho,\sigma} e^{-(L+\varepsilon_\varrho)(\tau_1-\tau_2)}[\theta(\tau_1-\tau_2)(\bar{n}_\varrho+1)+\theta(\tau_2-\tau_1)\bar{n}_\varrho] \tag{2.24}$$

$$a_\varrho^*(\tau_1)^\bullet a_\sigma^*(\tau_2)^\bullet = a_\varrho(\tau_1)^\bullet a_\sigma(\tau_2)^\bullet = 0, \tag{2.25}$$

$$\left.\begin{aligned} a_\varrho^*(\tau)^\bullet a_\sigma(\tau)^\bullet &= \delta_{\varrho,\sigma}\bar{n}_\varrho,\\ a_\varrho(\tau)^\bullet a_\sigma^*(\tau)^\bullet &= \delta_{\varrho,\sigma}(\bar{n}_\varrho+1)\end{aligned}\right\} \tag{2.26}$$

wherein

$$\delta_{\varrho,\sigma} = N^{-1}\sum_f \exp i(\varrho-\sigma)\cdot f = \begin{cases}1, & \varrho = \sigma,\\ 0, & \varrho \neq \sigma.\end{cases} \tag{2.27}$$

The dots connecting every operator pair in (2.23)—(2.26) and further on denote the contraction of both operators. Such a contraction is according to the Wick [14] theorem a classical number and expresses the result of subtracting the chronological product of two operators from the normal one.

3. *The graphs due to dynamic interaction of spin waves*

In this section, we confine ourselves to one class of dynamic diagrams (2.12) namely to those deficient in energy denominators. Referring the reader to paper [1] for details, we adduce here the final results. The graphical form of those diagrams is represented in Fig. 1.

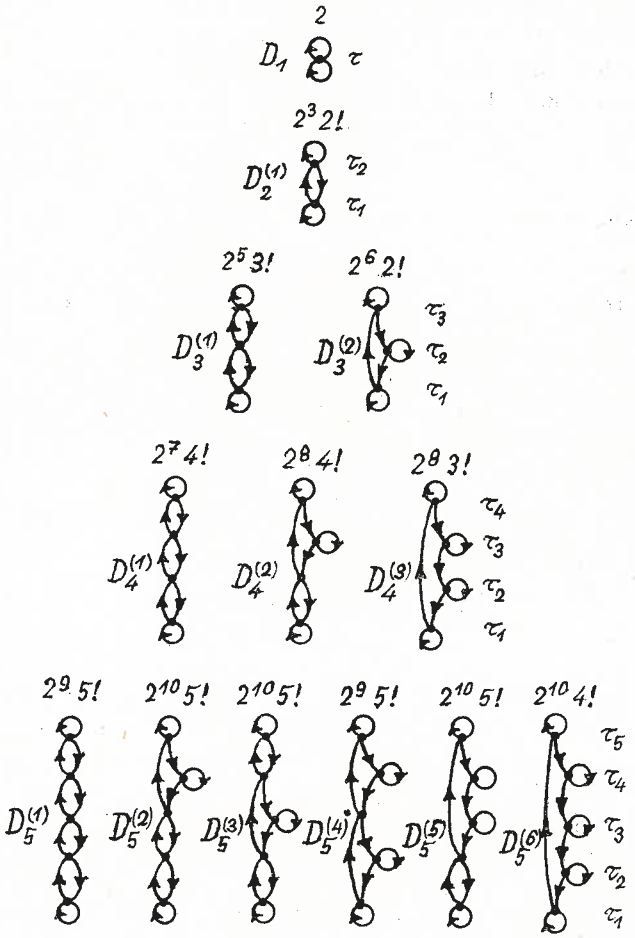

Fig. 1. Examples of the dynamic graphs

By Eq. (2.8), the free energy

$$F = \beta^{-1} \ln Z \tag{3.1}$$

including the graphs plotted in Fig. 1 is easily verified to be

$$F = E_0 - \beta^{-1} \sum_{\lambda} \ln (1+\bar{n}_\lambda) - \beta^{-1} \sum_{n=1}^{\infty} D_n$$

$$= E_0 + \sum_{\lambda} (L+\varepsilon_\lambda)\tilde{n}_\lambda - (2JS^2\gamma_0)^{-1} N^{-1} (\sum_{\varrho} \varepsilon_\varrho \tilde{n}_\varrho)^2 + \beta^{-1} \sum_{\lambda} [\tilde{n}_\lambda \ln \tilde{n}_\lambda - (1+\tilde{n}_\lambda) \ln (1+\tilde{n}_\lambda)] \tag{3.2}$$

where

$$\tilde{n}_\lambda = \{\exp \beta[L+\varepsilon_\lambda(1-Y/S)]-1\}^{-1}, \tag{3.3}$$

$$Y = N^{-1} \sum_\lambda (1-\gamma_\lambda/\gamma_0)\tilde{n}_\lambda \tag{3.4}$$

and the function Y is self-consistently determined. From Eqs (3.3) and (3.4) the Curie temperature, i.e. the temperature at which the derivative of the spontaneous magnetization with respect to temperature becomes infinite, can be numerically computed. Eq. (3.2) implies that both the internal energy and the entropy of a ferromagnet are contributed to by dynamic interaction of spin waves.

4. *Effect of kinematic bubble diagrams on the self-consistent renormalization of spin waves*

In order to derive kinematic graphs (2.15), we avail ourselves of Eqs (2.20)-(2.22) and (2.25), (2.26). For S = 1/2, we get the following diagrams (see [4]).

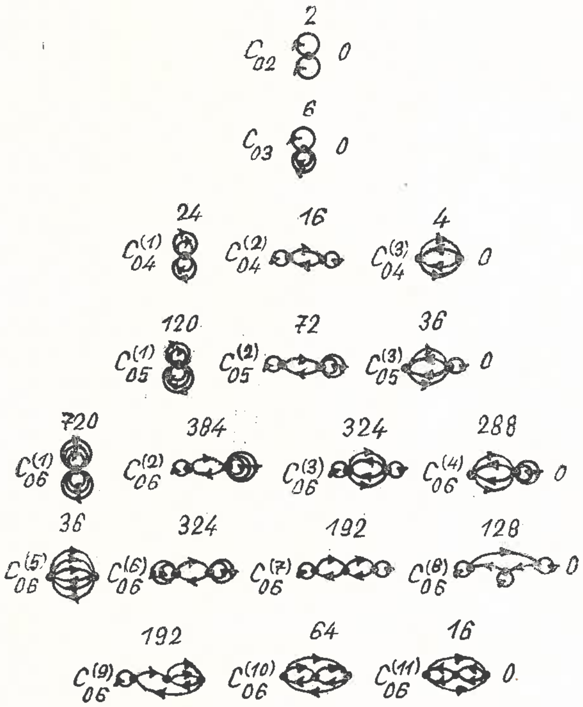

Fig. 2. Examples of diagrams due to kinematic interaction of spin waves

The number above each graph denotes the amount of topologically equivalent forms. We call C_{02}, C_{03}, $C_{04}^{(1)}$, $C_{04}^{(2)}$, $C_{05}^{(1)}$, $C_{05}^{(2)}$, $C_{06}^{(1)}$, $C_{06}^{(2)}$, $C_{06}^{(6)}$, $C_{06}^{(7)}$, $C_{06}^{(8)}$ and similar ones bubble diagrams whereas the graphs $C_{04}^{(3)}$, $C_{05}^{(3)}$, $C_{06}^{(3)}$, $C_{06}^{(4)}$, $C_{06}^{(5)}$, $C_{06}^{(9)}$, $C_{06}^{(10)}$, $C_{06}^{(11)}$ etc. are called ring diagrams.

Let us exemplify the lines of figuring out one of those diagrams. By Eqs (2.15), (2.16a), (2.16b), (2.20), (2.26) and (2.27), we have

$$C'_{04} = \left\langle \hat{T}\left[\frac{1}{2!}(-\tfrac{1}{2})^2 \sum_{f_1,f_2} (a^*_{f_1})^2 (a^*_{f_2})^2 a^2_{f_1} a^2_{f_2}\right]\right\rangle_c$$

$$= \frac{1}{2!}(-\tfrac{1}{2})^2 N^{-2} \sum_{\substack{\mu_1\mu_2\nu_1\nu_2\\ \varrho_1\varrho_2\sigma_1\sigma_2}} \delta(\mu_1+\mu_2-\nu_1-\nu_2)\delta(\varrho_1+\varrho_2-\sigma_1-\sigma_2)$$

$$\times \langle \hat{T}[a^*_{\mu_1}(0)a^*_{\mu_2}(0)a^*_{\varrho_1}(0)a^*_{\varrho_2}(0)a_{\nu_1}(0)a_{\nu_2}(0)a_{\sigma_1}(0)a_{\sigma_2}(0)]\rangle_c = C^{(2)}_{04} + C^{(3)}_{04} \tag{4.1}$$

where

$$C^{(2)}_{04} = \frac{1}{2!}(-\tfrac{1}{2})^2 N^{-2} \sum_{\substack{\mu_1\mu_2\nu_1\nu_2\\ \varrho_1\varrho_2\sigma_1\sigma_2}} \delta(\mu_1+\mu_2-\nu_1-\nu_2)\delta(\varrho_1+\varrho_2-\sigma_1-\sigma_2)$$

$$\times 16 a^*_{\mu_1}(0)^{\bullet} a^*_{\mu_2}(0)^{\bullet\bullet} a^*_{\varrho_1}(0)^{\bullet\bullet\bullet} a^*_{\varrho_2}(0)^{\bullet\bullet\bullet\bullet} a_{\nu_1}(0)^{\bullet} a_{\nu_2}(0)^{\bullet\bullet\bullet\bullet} a_{\sigma_1}(0)^{\bullet\bullet\bullet} a_{\sigma_2}(0)^{\bullet\bullet} = 2N^{-2} \sum_{\lambda\varrho\sigma} \bar{n}^2_\lambda \bar{n}_\varrho \bar{n}_\sigma, \tag{4.2}$$

$$C^{(3)}_{04} = \frac{1}{2!}(-\tfrac{1}{2})^2 N^{-2} \sum_{\substack{\mu_1\mu_2\nu_1\nu_2\\ \varrho_1\varrho_2\sigma_1\sigma_2}} \delta(\mu_1+\mu_2-\nu_1-\nu_2)\delta(\varrho_1+\varrho_2-\sigma_1-\sigma_2) 4 a^*_{\mu_1}(0)^{\bullet} a^*_{\mu_2}(0)^{\bullet\bullet}$$

$$\times a^*_{\varrho_1}(0)^{\bullet\bullet\bullet} a^*_{\varrho_2}(0)^{\bullet\bullet\bullet\bullet} a_{\nu_1}(0)^{\bullet\bullet\bullet} a_{\nu_2}(0)^{\bullet\bullet\bullet\bullet} a_{\sigma_1}(0)^{\bullet} a_{\sigma_2}(0)^{\bullet\bullet} = \tfrac{1}{2} N^{-2} \sum_{\lambda\varrho\sigma} \bar{n}_\lambda \bar{n}_\varrho \bar{n}_\sigma \bar{n}_{\varrho+\sigma-\lambda}. \tag{4.3}$$

We introduce the auxiliary functions

$$A_1 = N^{-1} \sum_\lambda \bar{n}_\lambda = N^{-1} \sum_\lambda \sum_{n=1}^{\infty} e^{-\beta n(L+\varepsilon_\lambda)}, \tag{4.4}$$

$$A_2 = -\beta^{-1} \frac{\partial A_1}{\partial L} = N^{-1} \sum_\lambda (\bar{n}^2_\lambda + \bar{n}_\lambda) = N^{-1} \sum_\lambda \sum_{n=1}^{\infty} n e^{-\beta n(L+\varepsilon_\lambda)}, \tag{4.5}$$

$$A_3 = (-\beta^{-1})^2 \frac{\partial^2 A_1}{\partial L^2} = N^{-1} \sum_\lambda (2\bar{n}^3_\lambda + 3\bar{n}^2_\lambda + \bar{n}_\lambda) = N^{-1} \sum_\lambda \sum_{n=1}^{\infty} n^2 e^{-\beta n(L+\varepsilon_\lambda)} \tag{4.6}$$

and, quite generally,

$$A_{p+1} = (-\beta^{-1})^p \frac{\partial^p A_1}{\partial L^p} = N^{-1} \sum_\lambda \sum_{n=1}^{\infty} n^p e^{-\beta n(L+\varepsilon_\lambda)}. \tag{4.7}$$

By reversal of these equations

$$N^{-1}\sum_{\lambda}\bar{n}_{\lambda} = A_1, \tag{4.8}$$

$$N^{-1}\sum_{\lambda}\bar{n}_{\lambda}^2 = A_2 - A_1, \tag{4.9}$$

$$N^{-1}\sum_{\lambda}\bar{n}_{\lambda}^3 = \tfrac{1}{2}(A_3 - 3A_2 + 2A_1) \tag{4.10}$$

etc. With the aid of Eqs (4.8) and (4.9), we obtain

$$C_{04}^{(2)} = 2N(A_1^2 A_2 - A_1^3). \tag{4.11}$$

Quite similarly one can derive the remaining kinematic bubble diagrams. As to the ring graphs, $C_4^{(3)}$ exemplifies that they form linked clusters in products of average spin wave occupation numbers. This is why they are improper to our renormalization scheme.

On the other hand, the diagrams $C_{0k}^{(1)}$ from Fig. 2 turn out to be

$$C_{0k}^{(1)} = N(-1)^{k+1}(k-1)!A_1^k, \quad k \geq 4 \tag{4.12}$$

and with $k \to \infty$ they increase to infinity. Thus, the series of kinematic bubble graphs proves to be divergent. Since the quantity C_0 from (2.15) must be finite, it can be assumed that this divergence may be removed by the series of ring diagrams. This problem is, however, very intricate and will not be considered here.

The first ring diagram $C_{04}^{(3)}$ results from the fourth order term in the kinematic operator, Eq. (2.20). As things are, we expect to obtain a plausible approximation by retaining the first few terms in a series of kinematic bubble graphs.

After straightforward computational work, we get

$$\langle [K_{1/2}(0)-1]\rangle_c = N\Big[-2(\tfrac{1}{2}A_1^2) + 2^2(\tfrac{1}{2}A_1^2A_2) - \tfrac{2}{3}A_1^4 + 2A_1^5 - 2^3\Big(\tfrac{1}{2}A_1^2A_2^2 + \frac{1}{3!}A_1^3A_3\Big) - \tfrac{52}{5}A_1^6 + \tfrac{16}{3}A_1^4A_2 + \dots\Big] \tag{4.13}$$

which because of the renormalization due to dynamic interaction becomes

$$\langle \hat{T}\{\hat{S}(\beta)\,[K_{1/2}(0)-1]\}\rangle_c = N\Big[-2(\tfrac{1}{2}\tilde{A}_1^2) + 2^2(\tfrac{1}{2}\tilde{A}_1^2\tilde{A}_2) - \tfrac{2}{3}\tilde{A}_1^4 + 2\tilde{A}_1^5 - 2^3\Big(\tfrac{1}{2}\tilde{A}_1^2\tilde{A}_2^2 + \frac{1}{3!}\tilde{A}_1^3\tilde{A}_3\Big) - \tfrac{52}{5}\tilde{A}_1^6 + \tfrac{16}{3}\tilde{A}_1^4\tilde{A}_2 + \dots\Big] \tag{4.14}$$

with

$$\tilde{A}_{p+1} = N^{-1}\sum_{\lambda}\sum_{n=1}^{\infty} n^p e^{-\beta n[L+\varepsilon_\lambda(1-2Y)]}. \tag{4.15}$$

The free energy is then

$$F = E_0 + \sum_\lambda (L+\varepsilon_\lambda)\tilde{n}_\lambda - \frac{2}{JN\gamma_0}\Big(\sum_\varrho \varepsilon_\varrho \tilde{n}_\varrho\Big)^2 + \beta^{-1}\Big\{\sum_\lambda [\tilde{n}_\lambda \ln \tilde{n}_\lambda - (1+\tilde{n}_\lambda)\ln(1+\tilde{n}_\lambda)]$$

$$+N\Big[\tilde{A}_1^2 - 2\tilde{A}_1^2\tilde{A}_2 + \tfrac{2}{3}\tilde{A}_1^4 - 2\tilde{A}_1^5 + 2^3\Big(\tfrac{1}{2}\tilde{A}_1^2\tilde{A}_2^2 + \frac{1}{3!}\tilde{A}_1^3\tilde{A}_3\Big) + \tfrac{52}{5}\tilde{A}_1^6 - \tfrac{16}{3}\tilde{A}_1^4\tilde{A}_2 + \ldots\Big]\Big\}. \tag{4.16}$$

Let us minimize this expression with respect to $\tilde{n}_\lambda$. We have

$$\frac{\partial F}{\partial \tilde{n}_\lambda} = L + \varepsilon_\lambda(1-2Y) + \beta^{-1}\Big[-\ln\frac{1+\tilde{n}_\lambda}{\tilde{n}_\lambda} + 2\tilde{A}_1 - 2\tilde{A}_1^2 + 4\tilde{A}_1^3 - 4\tilde{A}_1\tilde{A}_2$$

$$-\tfrac{46}{3}A_1^4 + 8\tilde{A}_1^2\tilde{A}_2 + \tfrac{312}{5}\tilde{A}_1^5 - \tfrac{64}{3}\tilde{A}_1^3\tilde{A}_2 + 4\tilde{A}_1^2\tilde{A}_3 + 8\tilde{A}_1\tilde{A}_2^2 + \ldots$$

$$+\tilde{n}_\lambda(-4\tilde{A}_1^2 + 8\tilde{A}_1^3 + 16\tilde{A}_1^2\tilde{A}_2 - \tfrac{32}{3}\tilde{A}_1^4 + \ldots) + \tilde{n}_\lambda^2(8\tilde{A}_1^3 + \ldots) + \ldots\Big] = 0 \tag{4.17}$$

hence the zeroth-order solution assumes the form

$$\tilde{\tilde{n}}_\lambda^{(0)} = \{\exp[2\tilde{A}_1 - 2\tilde{A}_1^2 + 4\tilde{A}_1^3 + \ldots + \beta(L+\varepsilon_\lambda(1-2Y))] - 1\}^{-1}. \tag{4.18}$$

On introducing the quantity

$$\tilde{X} = N^{-1}\sum_\lambda \tilde{n}_\lambda \tag{4.19}$$

we get

$$\tilde{\tilde{n}}_\lambda^{(0)} = \frac{1}{e^{2\tilde{X} - 2\tilde{X}^2 + 4\tilde{X}^3 + \ldots + \beta[L+\varepsilon_\lambda(1-2Y)]} - 1}. \tag{4.20}$$

At low temperature $\tilde{X}$ is proportional to $T^{3/2}$ and in the magnetization series appears a term $\sim T^3$. Moreover, close to the Curie point $\tilde{X}$ must be small and consequently the series $2\tilde{X} - 2\tilde{X}^2 + 4\tilde{X}^3 + \ldots$ reduced to its first few terms proves to be positive so that the formula for the magnetization

$$\mu(T) = -\frac{2}{N}\frac{\partial F}{\partial L} = 1 - \frac{2}{N}\sum_\lambda \tilde{\tilde{n}}_\lambda^{(0)} \tag{4.21}$$

reveals a larger magnetization than in the absence of kinematic interaction.

In order to improve self-consistent renormalization of spin waves, we must proceed in calculating further perturbation terms. Especially, we can not dispense with graphs comprising energy denominators.

5. Diagrams composed of one energy denominator

Now, we consider graphs including one energy denominator. They are represented in Fig. 3.

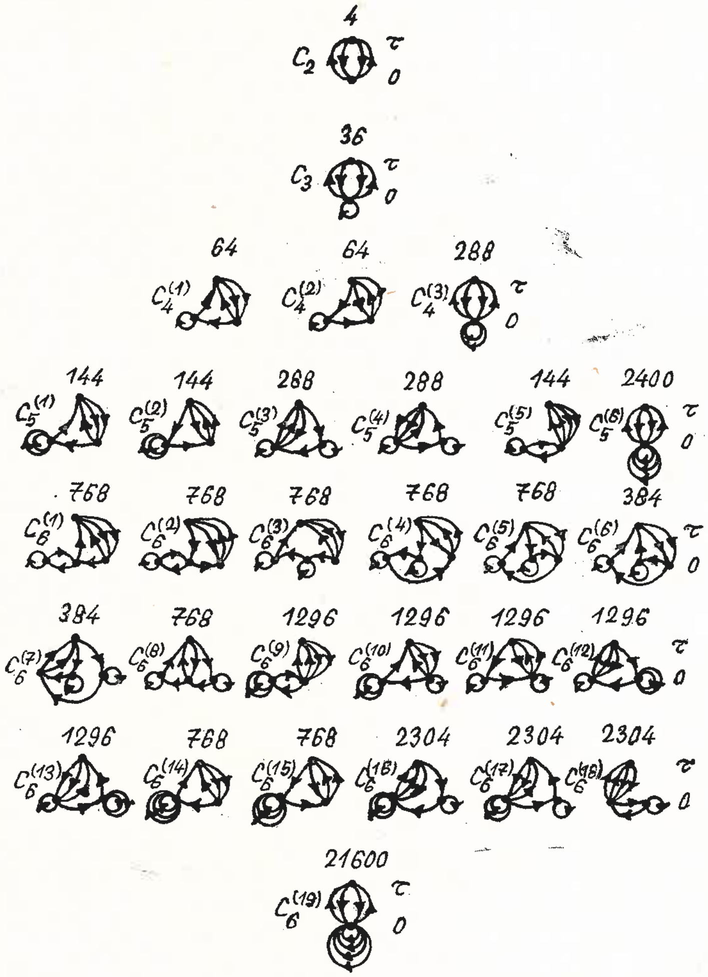

Fig. 3. The graphs due to mixed dynamic and kinematic interactions between ferromagnons

For $S = 1/2$, easy but very lengthy computation yields (see Appendix A)

$$\sum_{i,j} \tilde{C}_j^{(i)} = \frac{N}{1-2Y} [\tilde{A}_1^2 + 4\tilde{A}_1^4 - 4\tilde{A}_1^2\tilde{A}_2 - 22\tilde{A}_1^5 + 12(\tilde{A}_1^2\tilde{A}_2^2 + \tfrac{1}{3}\tilde{A}_1^3\tilde{A}_3)$$

$$+ 80\tilde{A}_1^6 - \tfrac{148}{3}\tilde{A}_1^4\tilde{A}_2 + \dots] \tag{5.1}$$

and by (2.8), (4.16) and (5.1) the free energy becomes

$$F = E_0 + \sum_{\lambda} (L+\varepsilon_\lambda)\tilde{n}_\lambda - \frac{2}{J\gamma_0 N}\left(\sum_{\varrho} \varepsilon_\varrho \tilde{n}_\varrho\right)^2$$

$$+\beta^{-1}\left\{\sum_{\lambda} [\tilde{n}_\lambda \ln \tilde{n}_\lambda - (1+\tilde{n}_\lambda)\ln(1+\tilde{n}_\lambda)]\right.$$

$$+N\left[\tilde{A}_1^2 + \tfrac{2}{3}\tilde{A}_1^4 - 2\tilde{A}_1^2\tilde{A}_2 - 2\tilde{A}_1^5 + 2^3\left(\tfrac{1}{2}\tilde{A}_1^2\tilde{A}_2^2 + \frac{1}{3!}\tilde{A}_1^3\tilde{A}_3\right) + \tfrac{52}{5}\tilde{A}_1^6 - \tfrac{16}{3}\tilde{A}_1^4\tilde{A}_2 + \ldots\right]$$

$$\left. - \frac{N}{1-2Y}[\tilde{A}_1^2 + 4\tilde{A}_1^4 - 4\tilde{A}_1^2\tilde{A}_2 - 22\tilde{A}_1^5 + 12(\tilde{A}_1^2\tilde{A}_2^2 + \tfrac{1}{3}\tilde{A}_1^3\tilde{A}_3) + 80\tilde{A}_1^6 - \tfrac{148}{3}\tilde{A}_1^4\tilde{A}_2 + \ldots]\right\} + \ldots \tag{5.2}$$

On minimizing (5.2), we obtain

$$\frac{\partial F}{\partial \tilde{n}_\lambda} = 0 = L + \varepsilon_\lambda(1-2Y) - \beta^{-1}\ln\frac{1+\tilde{n}_\lambda}{\tilde{n}_\lambda}$$

$$+\beta^{-1}N\frac{\partial}{\partial \tilde{n}_\lambda}(\tilde{A}_1^2 + \tfrac{2}{3}A_1^4 - 2\tilde{A}_1^2 A_2 + \ldots)$$

$$-\beta^{-1}\frac{N}{1-2Y}\frac{\partial}{\partial \tilde{n}_\lambda}(\tilde{A}_1^2 + 4\tilde{A}_1^4 - 4\tilde{A}_1^2\tilde{A}_2 + \ldots)$$

$$-2\beta^{-1}\frac{1-\gamma_\lambda/\gamma_0}{(1-2Y)^2}(\tilde{A}_1^2 + 4\tilde{A}_1^4 - 4\tilde{A}_1^2\tilde{A}_2 + \ldots) + \ldots \tag{5.3}$$

The sum of the fourth and fifth terms in (5.3) yields a negative contribution to the average spin wave population numbers and thus reduces the magnetization. We assume this sum to be sufficiently small so that the integral over a renormalized spin wave occupation number could be convergent. Since throughout this paper we are not in a position to account for these terms we simply neglect them. There remains

$$\beta^{-1}\ln\frac{1+\tilde{\tilde{n}}_\lambda}{\tilde{\tilde{n}}_\lambda} = L + \varepsilon_\lambda\left[1-2Y - \frac{4}{x(1-2Y)^2}(\tilde{A}_1^2 + 4\tilde{A}_1^4 - 4\tilde{A}_1^2\tilde{A}_2 + \ldots)\right] \tag{5.4}$$

with

$$x = \beta J\gamma_0. \tag{5.5}$$

Thus,

$$\tilde{\tilde{n}}_\lambda = \left\{\exp\beta\left[L + \varepsilon_\lambda\left(1-2Y - \frac{4}{x(1-2Y)^2}(\tilde{A}_1^2 + 4\tilde{A}_1^4 - 4\tilde{A}_1^2\tilde{A}_2 + \ldots)\right)\right] - 1\right\}^{-1}. \tag{5.6}$$

For $S = 1/2$,

$$\tilde{X} = N^{-1} \sum_{\lambda} \{\exp [2\tilde{X} + \tfrac{26}{3} \tilde{X}^3 + \ldots + \beta(L + \varepsilon_\lambda(1-2Y))] - 1\}^{-1}, \tag{5.7}$$

hence

$$\tilde{\tilde{n}}_\lambda = \frac{1}{e^{\beta\{L+\varepsilon_\lambda[1-2Y-4x^{-1}(1-2Y)^{-2}(\tilde{X}^2+4\tilde{X}^4-22\tilde{X}^5+\ldots)]\}}-1}. \tag{5.8}$$

The magnetization is given by the formula

$$\mu(T) = -\frac{1}{NS}\frac{d}{dL}F = -\frac{1}{NS}\left(\frac{\partial F}{\partial L} + \sum_{\lambda} \frac{\partial F}{\partial \tilde{\tilde{n}}_\lambda}\frac{\partial \tilde{\tilde{n}}_\lambda}{\partial L}\right) = -\frac{1}{NS}\frac{\partial F}{\partial L}$$

$$= 1 - (NS)^{-1} \sum_{\lambda} \tilde{\tilde{n}}_\lambda \tag{5.9}$$

i. e. for the case of $S = 1/2$.

$$\mu(T) = 1 - 2N^{-1} \sum_{\lambda} \tilde{\tilde{n}}_\lambda. \tag{5.10}$$

Quite similarly, for $S = 1$

$$\tilde{X} = N^{-1} \sum_{\lambda} \{\exp [3\tilde{X}^2 + \ldots + \beta(L + \varepsilon_\lambda(1-Y))] - 1\}^{-1}, \tag{5.11}$$

$$\tilde{\tilde{n}}_\lambda = \frac{1}{e^{\beta\{L+\varepsilon_\lambda[1-Y-x^{-1}(1-Y)^{-2}(3\tilde{X}^3-12\tilde{X}^4+51\tilde{X}^5+\ldots)]\}}-1} \tag{5.12}$$

and

$$\mu(T) = 1 - N^{-1} \sum_{\lambda} \tilde{\tilde{n}}_\lambda. \tag{5.13}$$

6. *Conclusions*

The comparison with the results obtained in papers [1–3] shows that owing to graphs resulting from cross-terms in the perturbation operator $\hat{S}(\beta)$ and comprising one energy denominator the renormalization factor

$$f_S = 1 - Y/S \tag{6.1}$$

is corrected by a quantity

$$\Delta f_S = -\frac{1}{x(1-Y/S)^2}\, g_S(\tilde{X}) \tag{6.2}$$

where the function $g_S(\tilde{X})$ is according to (5.8) and (5.12)

$$g_S(\tilde{X}) = \begin{cases} 4\tilde{X}^2 + 16\tilde{X}^4 - 88\tilde{X}^5 + \ldots, & S = 1/2, \\ 3\tilde{X}^3 - 12\tilde{X}^4 + 51\tilde{X}^5 + \ldots, & S = 1. \end{cases} \tag{6.3}$$

The renormalization correction Δf_S is negative so that it effects the decrease of the magnetization within the temperature interval from absolute zero to the Curie point. In order to show it, we had to treat the functions Y, Eq. (3.4), and $\tilde{X}$, Eqs (5.7) and (5.11), by computer "Odra 1204" whereas the method of calculating the magnetization is given in Appendix B. In Tables I and II we adduce the corresponding data and Loly's [3] values are referred to in parenthesis. Following Loly, we denoted by T_m the temperature at which the magnetization derivative becomes infinite (we called it the Curie temperature whereas for Loly the Curie point is connected with the zero value of the magnetization) and we define for brevity

$$x_m = \beta_m J\gamma_0 = \frac{J\gamma_0}{kT_m}\,. \tag{6.4}$$

TABLE I

Simple cubic lattice, $\gamma_0 = 6$

S	1/2	1
x_m	6.150 (6.144)	2.128 (2.128)
$Y(T_m)$	0.163 (0.164)	0.368 (0.368)
$\tilde{X}(T_m)$	0.187	0.160
$\mu(T_m)$	0.115 (0.220)	0.251 (0.265)

TABLE II

Body-centered cubic lattice, $\gamma_0 = 8$

S	1/2	1
x_m	6.064 (6.061)	2.115 (2.114)
$Y(T_m)$	0.160 (0.158)	0.365 (0.369)
$\tilde{X}(T_m)$	0.172	0.149
$\mu(T_m)$	0.256 (0.340)	0.345 (0.350)

From comparing Loly and our data it can be inferred that the graphs composed of one energy denominator considerably reduce the magnetization. For a simple cubic lattice this reduction amounts nearly to a half. Another decrease in magnetization may be expected from kinematic graphs and the mixed ones (the fourth and fifth terms in Eq. (5.3)). This problem is, however, very involved and it could not satisfactorily be investigated throughout this paper.

APPENDIX A

Let us exemplify the method of deriving the graphs from Fig. 3 for the case of $S = 1/2$. By Eqs (2.4), (2.6), (2.7), (2.9), (2.20), (2.23) and (2.24) and on applying the Wick [14] and Thouless [13] theorems, we obtain

$$C_2 = -\int_0^\beta d\tau \langle \hat{T}[H_I(\tau)\hat{K}^{(2)}_{1/2}(0)]\rangle_c$$

$$= -\tfrac{1}{8} JN^{-2} \sum_{\lambda\varrho\sigma} \sum_{\mu\nu\omega\zeta} \Gamma^\lambda_{\varrho,\sigma}\delta(\mu+\nu-\omega-\zeta) \int_0^\beta d\tau \langle \hat{T}[a^*_{\sigma+\lambda}(\tau)a^*_{\varrho-\lambda}(\tau)a_\varrho(\tau)a_\sigma(\tau)$$

$$\times a^*_\mu(0)a^*_\nu(0)a_\omega(0)a_\zeta(0)]\rangle_c = -\tfrac{1}{8} JN^{-2} \sum_{\lambda\varrho\sigma} \sum_{\mu\nu\omega\zeta} \Gamma^\lambda_{\varrho,\sigma}\delta(\mu+\nu-\omega-\zeta)$$

$$\times \int_0^\beta d\tau 4a^*_{\sigma+\lambda}(\tau)^{\bullet}a^*_{\varrho-\lambda}(\tau)^{\bullet\bullet}a_\varrho(\tau)^{\bullet\bullet\bullet}a_\sigma(\tau)^{\bullet\bullet\bullet\bullet}a^*_\mu(0)^{\bullet\bullet\bullet}a^*_\nu(0)^{\bullet\bullet\bullet\bullet}a_\omega(0)^{\bullet}a_\zeta(0)^{\bullet\bullet}$$

$$= -\tfrac{1}{2} JN^{-2} \sum_{\lambda\varrho\sigma} \Gamma^\lambda_{\varrho,\sigma}\bar{n}_{\sigma+\lambda}\bar{n}_{\varrho-\lambda}(\bar{n}_\varrho+1)(\bar{n}_\sigma+1)\int_0^\beta d\tau \exp \tau(\varepsilon_{\sigma+\lambda}+\varepsilon_{\varrho-\lambda}-\varepsilon_\varrho-\varepsilon_\sigma)$$

$$= -\tfrac{1}{2} JN^{-2} \sum_{\lambda\varrho\sigma} \Gamma^\lambda_{\varrho,\sigma}[(\bar{n}_{\sigma+\lambda}+1)(\bar{n}_{\varrho-\lambda}+1)\bar{n}_\varrho\bar{n}_\sigma-\bar{n}_{\sigma+\lambda}\bar{n}_{\varrho-\lambda}(\bar{n}_\varrho+1)(\bar{n}_\sigma+1)]$$

$$\times(\varepsilon_{\sigma+\lambda}+\varepsilon_{\varrho-\lambda}-\varepsilon_\varrho-\varepsilon_\sigma)^{-1}$$

$$= -\tfrac{1}{2} JN^{-2} \sum_{\lambda\varrho\sigma} (\Gamma^\lambda_{\varrho,\sigma}+\Gamma^\lambda_{\sigma+\lambda,\varrho-\lambda})(\bar{n}_{\sigma+\lambda}+1)(\bar{n}_{\varrho-\lambda}+1)\bar{n}_\varrho\bar{n}_\sigma(\varepsilon_{\sigma+\lambda}+\varepsilon_{\varrho-\lambda}-\varepsilon_\varrho-\varepsilon_\sigma)^{-1}. \quad \text{(A.1)}$$

The renormalization due to the dynamic interaction of spin waves yields

$$\tilde{C}_2 = -\tfrac{1}{2} JN^{-2} \sum_{\lambda\varrho\sigma} (\Gamma^\lambda_{\varrho,\sigma}+\Gamma^\lambda_{\sigma+\lambda,\varrho-\lambda})(\tilde{n}_{\sigma+\lambda}+1)(\tilde{n}_{\varrho-\lambda}+1)\tilde{n}_\varrho\tilde{n}_\sigma(1-2Y)^{-1}$$

$$\times(\varepsilon_{\sigma+\lambda}+\varepsilon_{\varrho-\lambda}-\varepsilon_\varrho-\varepsilon_\sigma)^{-1}. \quad \text{(A.2)}$$

We take into consideration the relation

$$\Gamma^\lambda_{\sigma+\lambda,\varrho-\lambda} = \Gamma^\lambda_{\varrho,\sigma}-(JS)^{-1}(\varepsilon_{\sigma+\lambda}+\varepsilon_{\varrho-\lambda}-\varepsilon_\varrho-\varepsilon_\sigma) \quad \text{(A.3)}$$

and using the symmetry property with respect to the transformation

$$\sigma \to \sigma-\lambda, \; \varrho \to \varrho+\lambda, \; \lambda \to -\lambda, \quad \text{(A.4)}$$

we obtain

$$\tilde{C}_2 = -\tfrac{1}{2}\frac{JN^{-2}}{1-2Y}\sum_{\lambda\varrho\sigma}(\Gamma^\lambda_{\varrho,\sigma}+\Gamma^\lambda_{\sigma+\lambda,\varrho-\lambda})(\tilde{n}_\varrho\tilde{n}_\sigma+2\tilde{n}_{\sigma+\lambda}\tilde{n}_\varrho\tilde{n}_\sigma)$$

$$\times(\varepsilon_{\sigma+\lambda}+\varepsilon_{\varrho-\lambda}-\varepsilon_\varrho-\varepsilon_\sigma)^{-1} = \frac{1}{1-2Y}N^{-2}\sum_{\lambda\varrho\sigma}(\tilde{n}_\varrho\tilde{n}_\sigma+2\tilde{n}_\lambda\tilde{n}_\varrho\tilde{n}_\sigma)$$

$$-\frac{JN^{-2}}{1-2Y}\sum_{\lambda\varrho\sigma}\Gamma^\lambda_{\varrho,\sigma}(\tilde{n}_\varrho\tilde{n}_\sigma+2\tilde{n}_{\sigma+\lambda}\tilde{n}_\varrho\tilde{n}_\sigma)(\varepsilon_{\sigma+\lambda}+\varepsilon_{\varrho-\lambda}-\varepsilon_\varrho-\varepsilon_\sigma)^{-1}. \quad \text{(A.5)}$$

We neglect the second part of (A.5) and by Eq. (4.15) we finally have

$$\tilde{C}_2 = \frac{N}{1-2Y}(\tilde{A}_1^2+2\tilde{A}_1^3). \tag{A.6}$$

Along similar lines one gets $\tilde{C}_j^{(i)}$.

APPENDIX B

In order to compute the magnetization, we use the well-known formula

$$\frac{1}{e^u-1} = \frac{1}{u} - \tfrac{1}{2} + \sum_{n=0}^{\infty} \frac{B_{2n+2}}{(2n+2)!} u^{2n+1} \tag{B.1}$$

where B_{2n+2} are Bernoulli numbers. Thus,

$$\frac{1}{NS}\sum_{\lambda} \frac{1}{e^{Z(1-\gamma\lambda/\gamma_0)}-1} \to \frac{1}{S(2\pi)^3}\iiint_0^{2\pi} \frac{d\lambda_x d\lambda_y d\lambda_z}{e^{Z(1-\gamma\lambda/\gamma_0)}-1} = \frac{1}{S8\pi^3}\iiint_0^{2\pi} d\lambda_x d\lambda_y d\lambda_z$$

$$\times\left[\frac{1}{Z(1-\gamma_\lambda/\gamma_0)} - \tfrac{1}{2} + \tfrac{1}{12} Z(1-\gamma_\lambda/\gamma_0) - \tfrac{1}{720} Z^3(1-\gamma_\lambda/\gamma_0)^3 + \tfrac{1}{30240} Z^5(1-\gamma_\lambda/\gamma_0)^5 + \ldots\right. \tag{B.2}$$

wherein

$$x = \beta J\gamma_0, \tag{B.3}$$

$$f_S' = f_S + \Delta f_S, \tag{B.4}$$

$$Z = xSf_S'. \tag{B.5}$$

For a simple cubic lattice,

$$\frac{1}{(2\pi)^3}\iiint_0^{2\pi} d\lambda_x d\lambda_y d\lambda_z (1-\gamma_\lambda/\gamma_0)^{2p-1} = \frac{1}{(2\pi)^3}\iiint_0^{2\pi} dxdydz[1-\tfrac{1}{3}(\cos x+\cos y+\cos z)]^{2p-1},$$

$$p = 0, 1, 2, 3, \ldots. \tag{B.6}$$

In particular [17],

$$\frac{1}{(2\pi)^3}\iiint_0^{2\pi} \frac{dxdydz}{1-\frac{1}{3}(\cos x+\cos y+\cos z)}$$

$$= \frac{12}{\pi^2}(18+12\sqrt{2}-10\sqrt{3}-7\sqrt{6})K^2[(2-\sqrt{3})(\sqrt{3}-\sqrt{2})] = 1.516386 \tag{B.7}$$

with $K(k)$ being the complete eliptical function

$$K(k) = \int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2 \sin^2 x}}. \tag{B.8}$$

As to a body-centered cubic lattice,

$$\frac{1}{(2\pi)^3} \iiint_0^{2\pi} d\lambda_x d\lambda_y d\lambda_z (1-\gamma_\lambda/\gamma_0)^{2p-1} = \frac{1}{(2\pi)^3} \iiint_0^{2\pi} dxdydz(1-\cos x \cos y \cos z)^{2p-1},$$

$$p = 0, 1, 2, \ldots \tag{B.9}$$

For $p = 0$ ([17])

$$\frac{1}{(2\pi)^3} \iiint_0^{2\pi} \frac{dxdydz}{1-\cos x \cos y \cos z} = \frac{4}{\pi^2} K^2 \left(\frac{\sqrt{2}}{2}\right) = 1.393249. \tag{B.10}$$

The integrals with $p = 1, 2, 3, \ldots$ can be performed by expanding subintegral expressions and integrating term by term.

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