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CHARGED IMPURITY CONCENTRATION WAVES IN He II AS A TYPE OF SECOND SOUND WAVES

By Z. K. Petru

Institute of Theoretical Physics, University of Wrocław*

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The set of conservation laws for the Bose superfluid system with Fermi impurities is derived in exact operator form. The theory of the charge waves phenomenon (waves of a concentration of charged impurities generated by temperature waves) in superfluid helium is presented. A method for its experimental verification is suggested.

1. *Introduction*

The behaviour of charged and neutral impurities in superfluid helium was studied extensively in the recent years, since Careri's experiment in 1959 [1].

The purpose of this paper is to suggest a new method of an experimental verification of the concentration waves phenomenon which was recently predicted theoretically for the ^{3}He-HeII solutions [2].

Measurements of the change of charge seem to be easier to perform than those, proposed in [2], of measuring the change of ^{3}He concentration in the ^{4}He superfluid.

We shall give in this paper a theoretical description of waves of the charged impurity concentration.

The two component quantum hydrodynamic approach enables us to consider many types of interparticle interactions, especially if we use the exact method of deriving conservation laws as given in [3]. This gives us the possibility to study a wider class of components than those in [2]. On the other hand (a) normal component includes all impurities in HeII [1] and (b) normal component (phonon gas) is a carrier of the second sound. Thus, some connections between thermal waves and motions of any impurities should exist (as a result of a two fluid HeII model) and we shall derive them.

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* Address: Instytut Fizyki Teoretycznej, Cybulskiego 36, 50-205 Wrocław, Poland.

2. Conservation laws for two-component system

Consider a system of superfluid bosons (B) and normal fermions (F) with arbitrary boson-boson, boson-fermion and fermion-fermion interactions. This solution can be described by the following Hamiltonian (see *e. g.* [4, 5])

$$H = H_B + H_F + H_{BF}^{\text{int}} + \delta H_t \equiv H^0 + \delta H_t, \tag{1}$$

where

$$H_B = \frac{1}{4m_B}\int [\Delta\varphi^+(tr)\cdot\varphi(tr) + \varphi^+(tr)\Delta\varphi(tr)]dr - \lambda_B \int \hat{\varrho}_B(tr)dr +$$
$$+ \tfrac{1}{2}\int V_B(|r-r'|)\varphi^+(tr)\hat{\varrho}_B(tr')\varphi(tr)drdr', \tag{2}$$

$$H_F = \frac{1}{4m_F}\sum_s \int [\Delta\psi^+(trs)\cdot\psi(trs) + \psi^+(trs)\Delta\psi(trs)]dr -$$
$$-\lambda_F \int \hat{\varrho}_F(tr)dr + \tfrac{1}{2}\int V_F(|r-r'|)\sum_s \psi^+(trs)\hat{\varrho}_F(tr')\psi(trs)drdr'. \tag{3}$$

H_{BF}^{int} is the interaction between components

$$H_{BF}^{\text{int}} = \tfrac{1}{2}\int V_{BF}(|r-r'|)\,[\hat{\varrho}_B(tr)\hat{\varrho}_F(tr') + \hat{\varrho}_F(tr)\hat{\varrho}_B(tr')]drdr'. \tag{4}$$

The last term in (1) introduces Bogolubov's sources of particles $\eta(tr)$ and $\eta^*(tr)$, characteristic for the superfluid Bose system [6] and an external scalar potential $\delta U(tr)$,

$$\delta H_t = \int dr[\eta(tr)\varphi^+(tr) + \eta^*(tr)\varphi(tr) + \delta U(tr)\hat{\varrho}(tr)], \tag{5}$$

and they are given time and space dependent functions.

In Eqs (2)–(4) we introduced the Bose and Fermi densities of particles

$$\hat{\varrho}_B(tr) = \varphi^+(tr)\varphi(tr); \qquad \hat{\varrho}_F(tr) = \sum_s \psi^+(trs)\psi(trs) \tag{6}$$

and a total density operator

$$\hat{\varrho} = \hat{\varrho}_B + \hat{\varrho}_F \tag{6'}$$

With the aid of Hamiltonian (1) we obtain the equation of motion for the boson field operator, $\varphi(tr)$, and for the fermion one, $\psi(trs)$,

$$i\partial_t\varphi(tr) = -\frac{\Delta\varphi}{2m_B} + (\delta U - \lambda_B)\varphi + \eta +$$
$$+ \int [V_B(r-r')\hat{\varrho}_B(tr') + V_{BF}(r-r')\hat{\varrho}_F(tr')]dr'\varphi(tr), \tag{7}$$

$$i\partial_t\psi(trs) = -\frac{\Delta\psi}{2m_B} + (\delta U - \lambda_F)\psi +$$
$$+ \int [V_F(r-r')\hat{\varrho}_F(tr') + V_{BF}(r-r')\hat{\varrho}_B(tr')]dr'\psi(trs). \tag{8}$$

Using Eqs (7) and (8) we derive four conservation laws in the exact operator form. For the mass-density operator

$$\hat{\varrho}^m = m_B\hat{\varrho}_B + m_F\hat{\varrho}_F \tag{9}$$

we have the continuity equation

$$\partial_t\hat{\varrho}^m = -\nabla\hat{j} + im_B(\varphi\eta^* - \varphi^+\eta). \tag{10}$$

For the mass-current density operator

$$\hat{j} = \hat{j}_B + \hat{j}_F = \frac{i}{2}(\nabla\varphi^+\cdot\varphi - \varphi^+\cdot\nabla\varphi) + \frac{i}{2}\sum_s(\nabla\psi^+\cdot\psi - \psi^+\cdot\nabla\psi) \tag{11}$$

we have the conservation law

$$\partial_t\hat{j}_\alpha = -\sum_\beta T_{\alpha\beta,\beta} + \hat{\varrho}\partial_\alpha\delta U +$$

$$+\tfrac{1}{2}(\eta\overset{+}{\varphi}_{,\alpha} + \eta^*\varphi_{,\alpha} - \eta_{,\alpha}\varphi^+ - \overset{*}{\eta}_{,\alpha}\varphi). \tag{12}$$

The symmetric stress tensor $T_{\alpha\beta}$ has the form (see Appendix of Ref. [7])

$$T_{\alpha\beta} \equiv T^{(\mathrm{I})}_{\alpha\beta} + T^{(\mathrm{II})}_{\alpha\beta},$$

$$T^{(\mathrm{I})}_{\alpha\beta} = \frac{1}{m_B}(\overset{+}{\varphi}_{,\alpha}\varphi_{,\beta} + \overset{+}{\varphi}_{,\beta}\varphi_{,\alpha}) + \frac{1}{m_F}\sum_s(\overset{+}{\psi}_{,\alpha}\psi_{,\beta} + \overset{+}{\psi}_{,\beta}\psi_{,\alpha}) -$$

$$-\partial_\alpha\partial_\beta\left(\frac{\varrho^B}{4m_B} + \frac{\varrho^F}{4m_F}\right),$$

$$T^{(\mathrm{II})}_{\alpha\beta} = -\tfrac{1}{4}\int d\boldsymbol{R}\,\frac{R_\alpha R_\beta}{R}\int_{-1}^{1}dx\left[2\frac{\partial V_{BF}(R)}{\partial R}\hat{\varrho}_B(r_+)\hat{\varrho}_F(r_-) + \right.$$

$$\left. + \frac{\partial V_B(R)}{\partial R}\varphi^+(r_+)\hat{\varrho}_B(r_-)\varphi(r_+) + \frac{\partial V_F(R)}{\partial R}\sum_s\psi^+(r_+, s)\hat{\varrho}_F(r_-)\psi(r_+, s)\right], \tag{13}$$

where

$$r_\pm \equiv \left(t, \boldsymbol{r} + \frac{x\pm 1}{2}\boldsymbol{R}\right).$$

We define the energy density operator as follows

$$\int d\boldsymbol{r}(\widehat{\varrho^m\varepsilon})(t\boldsymbol{r}) \equiv H^0$$

and

$$\partial_t(\widehat{\varrho^m \varepsilon}) = -\sum_\alpha \Big\{ I_{\alpha,\alpha} + \Big(\frac{1}{m_B} j_\alpha^B + \frac{1}{m_F} j_\alpha^F\Big)\partial_\alpha \delta U -$$

$$-(i/4m_B)\,[(\eta\varphi^+_{,\alpha\alpha} + \eta_{,\alpha\alpha}\varphi^+) - \text{h.c.}]\Big\} +$$

$$+\frac{i}{2}\int dr'\{\eta^*(tr)\,[V_B(r-r')\varrho_B(tr') + V_{BF}(r-r')\varrho_F(tr')]\varphi(tr) - \text{h.c.} +$$

$$+\eta^*(tr')\,[V_B(r-r')\varrho_B(tr) + V_{BF}(r-r')\varrho_F(tr)]\varphi(tr') - \text{h.c.}\} \tag{14}$$

where energy current I_α

$$I_\alpha \equiv I_\alpha^{(\mathrm{I})} + I_\alpha^{(\mathrm{II})},$$

$$I_\alpha^{(\mathrm{I})} = -\frac{i}{8}\partial_\alpha \sum_\beta \Big[\frac{1}{m_B^2}(\varphi^+_{,\beta\beta}\varphi - \varphi^+\varphi_{,\beta\beta}) + \frac{1}{m_F^2}\sum_s (\psi^+_{,\beta\beta}\psi - \psi^+\psi_{,\beta\beta})\Big] -$$

$$-\frac{i}{4}\sum_\beta \Big(\frac{1}{m_B^2}(\varphi^+_{,\alpha}\varphi_{,\beta\beta} - \varphi^+_{,\beta\beta}\varphi_{,\alpha}) + \sum_s \frac{1}{m_F^2}(\psi^+_{,\alpha}\psi_{,\beta\beta} - \psi^+_{,\beta\beta}\psi_{,\alpha})\Big) +$$

$$+\tfrac{1}{2}\int dr' \Big[V_B(r-r')\varphi^+(tr')\frac{1}{m_B} j_\alpha^B(tr)\varphi(tr') +$$

$$+V_F(r-r')\sum_{s'} \psi^+(tr's')\frac{1}{m_F} j_\alpha^F(tr)\psi(tr's') +$$

$$+V_{BF}(r-r')\Big(\varrho^B(tr')\frac{1}{m_F} j_\alpha^F(tr) + \varrho^F(tr')\frac{i}{m_B} j_\alpha^B(tr)\Big)\Big],$$

$$I_\alpha^{(\mathrm{II})} = -\sum_\beta \frac{1}{4}\int dR\,\frac{R_\alpha R_\beta}{R}\int_{-1}^{1} dx \Big\{\frac{\partial V^B}{\partial R}\varphi^+(r_+)\frac{1}{m_B} j_\beta^B(r_-)\varphi(r_+) +$$

$$+\frac{\partial V^F}{\partial R}\sum_s \psi^+(r_+, s)\frac{1}{m_F} j_\beta^F(r_-)\psi(r_+, s) +$$

$$+\frac{\partial V^{BF}}{\partial R}\Big(\varrho^B(r_+)\frac{1}{m_F} j_\beta^F(r_-) + \varrho^F(r_+)\frac{1}{m_B} j_\beta^B(r_-)\Big)\Big\}. \tag{15}$$

Now we introduce the concentration of Fermi particles by the relation

$$c(tr)\hat{\varrho}^m(tr) = m_F\hat{\varrho}_F(tr). \tag{16}$$

$\partial_t \hat{\varrho}_F$ can be obtained from (8) in analogy to (10) and then we can find

$$\partial_t(c\hat{\varrho}^m) = -\nabla \hat{j}_F. \tag{17}$$

If we consider the superfluid bosons, we must add to Eqs (10), (12), (14) and (17) the fifth equation for the superfluid velocity v_s derived in a standard way [6] by splitting the equation of motion for the complex quantity $\langle\varphi\rangle$ into two equations: one for the real amplitude $a(tr)$ and one for the real phase $\chi(tr)$

$$\langle\varphi(tr)\rangle = a\cdot e^{i\chi} \tag{18}$$

$\langle\ldots\rangle$ denotes the nonequilibrium expectation value with the Hamiltonian (1). For the superfluid $\langle\varphi\rangle \neq 0$. From (7) we have

$$\partial_t\chi(tr) = \frac{\Delta a}{2am_B} - \frac{(\nabla\chi)^2}{2m_B} - \delta U + \lambda_B - \frac{1}{2a}(\zeta+\zeta^*) -$$

$$-a^{-2}\int d\boldsymbol{R}\,\mathrm{Re}\,X_t^{BF}(\boldsymbol{r},\boldsymbol{R}), \tag{19}$$

where

$$X_t^{BF}(\boldsymbol{r},\boldsymbol{r}'-\boldsymbol{r}) \equiv \langle\varphi^+(tr)\rangle\,[V_B(\boldsymbol{r}-\boldsymbol{r}')\,\langle\hat{\varrho}_B(tr')\varphi(tr)\rangle +$$

$$+V_{BF}(\boldsymbol{r}-\boldsymbol{r}')\,\langle\hat{\varrho}_F(tr')\varphi(tr)\rangle],$$

$$\zeta(tr) \equiv \eta(tr)\exp\{-i\chi(tr)\}. \tag{20}$$

The integral term can be calculated from Eq. (19) in the equilibrium and, if we assume the so-obtained formula to hold also within a small deviation from the equilibrium [6], then we find the equation for $\partial_t v_s$

$$\partial_t \boldsymbol{v}_s(tr) = \partial_t \frac{1}{m_B}\nabla\chi(tr). \tag{21}$$

The essential differences between Eqs (12), (14), (20) and their counterparts in [4] appear in (13), (15) and (20). Namely: (*i*) the above equations describe an arbitrary two component system with interactions of boson-boson, V_B, fermion-fermion, V_F, and boson--fermion, V_{BF}, types, contrary to [4] where $V_B = V_F = V_{BF}$; (*ii*) formulae for operators $T_{\alpha\beta}$ and I_α, (13) and (15), are exact here and were calculated without the restriction of [4] that the interparticle forces must be short-range ones.

3. *Charge concentration waves in* HeII

With the aid of Eqs (10), (12), (14), (17) and (21) we can study the behaviour of a wide class of solutions. In [2, 4] dilute solution ^{3}He in HeII was examined. In this section we shall apply the above theory to describe the motion of charged impurities in HeII.

Consider a system of helium ions, $^4He^+$, in the superfluid ^{4}He. In the real case of small impurity concentration, $c \ll 1$, we can neglect the interaction term V_F in Eqs (3) and following equations. In the opposite case the integral $\int\int dr\,dr'\,V_F\psi^+\hat{\varrho}_F\psi$ gives us the unstability of the infinite system of charged particles (if we did not introduce any other additional assumptions which preserve the convergence of this integral, as *e. g.*, boundary conditions). We also put

$$m_F \approx m_B = m$$

in all formulae, which simplifies our relations, *e. g.*: $\hat{\varrho}^m = m\hat{\varrho}$ *etc.*

Now we take the nonequilibrium expectation values from Eqs (10)–(17), derive the formula for pressure, $P\delta_{\alpha\beta} = \langle T_{\alpha\beta} \rangle$, and for the energy-current along the same method as that in [2]. In such a way we find five hydrodynamic equations of motion (for $\varrho \equiv \langle \hat{\varrho} \rangle$, $\boldsymbol{j} \equiv \langle \hat{\boldsymbol{j}} \rangle$, c, $\varrho\varepsilon \equiv \langle \hat{\varrho}\varepsilon \rangle$, $\boldsymbol{v}_s$). From this point our considerations are quite parallel to those of [4] and after passing from (14) to the formula of the entropy conservation law [6], we obtain a set of five equations (having the same external form as the equations in [2, 4]) connecting together five thermodynamic parameters: ϱ, c, $\boldsymbol{v}_s$, $\boldsymbol{v}_n$, θ — where $\boldsymbol{v}_n(t\boldsymbol{r})$ is the velocity of the normal component and $\theta(t\boldsymbol{r})$ is the local temperature.

We are interested in the deviation of the system from equilibrium when we adiabatically switch of the external scalar potential perturbation $\delta U(t\boldsymbol{r})$. To obtain the response of the system linear in U, we assume that all thermodynamic parameters vary very slowly in space and time (hydrodynamic approximation) and we linearize the hydrodynamic equations by putting: $c(t\boldsymbol{r}) = c_0 + c(t\boldsymbol{r})$, $\theta(t\boldsymbol{r}) = \theta_0 + \delta\theta(t\boldsymbol{r})$, *etc.* (acoustic approximation). Now we can simply obtain solutions (in Fourier transforms) for $\delta c(\omega\boldsymbol{k})$, $\delta\theta(\omega\boldsymbol{k})$, *etc.* [2]. Comparing δc and $\delta\theta$ we obtain

$$\delta c(\omega, \boldsymbol{k}) = \delta\theta(\omega, \boldsymbol{k}) \frac{c_0 \dfrac{\partial s}{\partial \theta}}{s_0 - c_0 \dfrac{\partial s}{\partial c}}, \tag{22}$$

where s is the entropy per particle.

Eq. (22) suggests that if we generate temperature waves in our system we obtain simultaneously the impurity concentration waves. Because impurities consist of $^4He^+$ ions, waves of the charge density shall appear. Especially, in nodes of the standing second sound wave we can observe oscillations of the charge density. We hope that this effect can be measured, for example, by the capacitor method.

We must emphasize that the sort of impurity ions (*i. e.* the type of V_{BF} interactions) does not play any important role in this theory. This is a consequence of the fact that all therms with arbitrary interactions are included within definitions of the pressure and energy density which includes $\langle T_{\alpha\alpha} \rangle$, Eq. (13), and $\langle I_\alpha \rangle$, Eq. (15), and any change of the type of interaction can only modify the pressure (and energy density), but cannot change the explicit form of conservation laws (10), (12), (14), (19), and thus, cannot change the final result (22). This suggests that the relation (22) is mainly the consequence of the two--fluid-model theory of superfluid helium. Note, that for an ordinary two component solution, such as HeI with impurities ($\boldsymbol{v}_s = 0$), when we put

$$\boldsymbol{j}_F = m\boldsymbol{v}_n \varrho_F, \qquad \varrho_F = c\varrho$$

into (17) and substract (10) from (17) we obtain

$$\partial_t c(t\boldsymbol{r}) = -m\varrho(t\boldsymbol{r})\boldsymbol{v}_n(t\boldsymbol{r})\nabla c(t\boldsymbol{r}). \tag{23}$$

Therefore, if we want to linearize (23) we find that

$$\partial_t \delta c = -m\varrho_0 \delta \boldsymbol{v}_n \nabla \delta c \sim O(\delta^2) \tag{24}$$

rather than (22).

This theory directly confirms the conclusion drawn from the famous Careri's experiment [1], that "foreign bodies" in HeII move with the normal component. We hope that in a similar way the experimental verification of the charge concentration waves in superfluid helium can be also performed.

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Note added in proof: If we take into account that the mobility of the ions is very sensitive to temperature, we can detect stationary temperature waves using ions as probes. The charge concentration might then be calculated from the intensity of the current being detected and the known temperature dependence of the mobility [8].

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