ON PHENOMENOLOGICAL DEFINITIONS OF MEAN HEAT CONDUCTIVITY FOR NONHOMOGENEOUS MEDIA

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The definitions of mean heat conductivity are discussed for media characterized by constant heat conductivity, temperature-dependent heat conductivity, and coordinate dependent heat conductivity.

It is pointed out that media with temperature-dependent heat conductivity are particularly complicated, since the change in the temperature difference applied to the boundary surfaces of the specimen results in a change in the effective heat conductivity, so that the resultant heat conductivity of the specimen is each time different.

1. *Introduction*

The Fourier postulate formulated in 1822 concerned in its original form media homogeneous from the point of view of heat conduction, and thus media with constant heat conductivity.

The postulated proportionality of the heat flux density for heat flowing between two isothermal surfaces, to the mean temperature gradient between these two surfaces has been written in terms of the difference equation

$$q = -K_0 \frac{\Delta T}{\Delta x} . \tag{1}$$

The factor K_0 was thus treated as a constant characterizing the material from the point of view of heat conduction and called heat conductivity.

When experiments showed that the proportionality factor in Eq. (1) is temperature-dependent the original form of the Fourier postulate (1) was generalized for media with temperature-dependent heat conductivity to the form:

$$q = -\overline{K(\Delta T)} \frac{\Delta T}{\Delta x} , \tag{2}$$

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where $\overline{K(\Delta T)}$ is the mean thermal conductivity understood as the integral mean in the temperature interval ΔT.

On the basis of the difference equations (1) and (2) it is possible to determine experimentally the heat conductivity of media characterized by constant conductivity (Eq. (1)) and with temperature-dependent heat conductivity (Eq. (2)).

2. *Formulation of the problem*

By nonhomogeneous medium in the sense of heat conduction we mean such a medium which is characterized by non-vanishing heat conductivity gradient, *i. e.*, grad $K \neq 0$.

Among the nonhomogeneous media one can distinguish media which are monotonically nonhomogeneous, *i. e.*, where the gradient of heat conductivity is constant or monotonic.

The case of constant heat conductivity gradient is the simplest case of monotonic nonhomogeneity, which in this case may be called linear nonhomogeneity (the heat conductivity is then a linear function of the space coordinate).

A non-vanishing heat conductivity gradient grad K may be present in a medium even then when the latter is in thermal equilibrium, *i. e.*, if the temperature gradient — grad T — in the medium equals zero. We then say that the medium is substantially nonhomogeneous. A characteristic property of a substantially nonhomogeneous medium is that grad K is independent of grad T applied externally when the medium under consideration is not in thermal equilibrium.

So far the technology of the preparation of samples did not succeed in obtaining samples with *a priori* given and controlled substance nonhomogeneity. Therefore there are at present no data on the heat conduction in substantially nonhomogeneous media. It turns out, however, that a medium which is homogeneous in thermal equilibrium may become nonhomogeneous under the influence of external temperature gradient, *i. e.*, when there is no thermal equilibrium, owing to the temperature-dependence of heat conductivity. In such a medium a nonvanishing grad K will appear which, however, vanishes, when the externally applied temperature gradient vanishes.

Such media are called nonhomogeneous in the sense of temperature-dependent heat conductivity and we always deal with them if the heat conductivity of the investigated material that is temperature-dependent. Media which are nonhomogeneous in the sense of temperature-dependent heat conductivity are thus quite often investigated, since many samples whose conductivity was measured show a distinct dependence on temperature (*e. g.* on the low-temperature and the high temperature side of the maximum heat conductivity at low temperatures). A characteristic feature of a medium which is nonhomogeneous due to temperature-dependent heat conductivity is a rigorous dependence of grad K on the externally applied grad T which thus results in grad $K \neq 0$. This feature can be used to distinguish these media from those which are substantially nonhomogeneous, in which grad K is a property of the particular material and is independent of the existence or non-existence of externally applied grad T.

For the sake of clarity we shall consider isotropic media (in the sense of crystal structure) and the only directional quantities will be grad K and grad T. Our experiments [1] and [2] dealt with the study of heat conductivity of samples which are nonhomogeneous owing to temperature dependent heat conductivity, and this in the region where the temperature dependence of heat conductivity is very strong. Theoretical explanation on the experimental results was given in Refs [3] and [4].

The purpose of the present paper is a comparative analysis of the definitions of the mean heat conductivity for media with constant, temperature-dependent, and coordinate--dependent heat conductivity.

3. Media with constant heat conductivity

The characteristic feature of a medium with constant heat conductivity is that the proportionality coefficient between the heat flux density and the temperature gradient is the same at each point of a heat-conducting medium. It follows from the continuity condition for heat flow that the local value of the temperature gradient is the same at each point of the medium, and thus the mean value of the gradient is equal to the local value at any point. The mean value of the gradient is always defined as the ratio of the temperature difference ΔT between two isothermal surfaces bounding a heat conducting specimen to the length of the specimen Δx.

Owing to this property the difference equation (1) is equivalent to the differential equation

$$q = -K_0 \frac{dT}{dx} \,. \tag{3}$$

The Fourier postulate written in the form of a differential equation is assumed to be valid for arbitrary heat conducting media and thus also for media with temperature-dependent or substance varying heat conductivity. However in case of media with constant conductivity the replacement of the Fourier postulate written in the form of a difference equation (1) by the differential equation (3) does not bring any new physical sense.

For media with constant heat conductivity the local direct proportionality of heat flux to the temperature gradient (3) coincides with the proportionality of heat flux to the mean temperature gradient (1), the proportionality factors being equal for the same sample.

4. Media with temperature-dependent heat conductivity

When the medium is characterized by temperature-dependent heat conductivity then after the application of an external temperature gradient it becomes not only thermodynamically nonhomogeneous, but also nonhomogeneous due to temperature-dependent heat conductivity. The heat conduction process through such medium is described by the difference equation (2). One can also write a differential equation for those media whose heat conductivity changes with temperature:

$$q = -K(T) \cdot \frac{dT}{dx}(T). \tag{4}$$

The comparison of the difference equation (2) with the differential equation (4) leads to the necessity of defining the mean heat conductivity and the mean temperature gradient in Eq. (2). The mean temperature gradient in Eq. (2) is defined as $\frac{\Delta T}{\Delta x}$ which mean that only the temperature values at the boundaries of the sample and the distance Δx between these two isothermal surfaces are considered. Such a definition of the mean temperature gradient is equivalent to a replacement of the real temperature distribution by a linear one. On the other hand the mean heat conductivity in Eq. (2) is defined in terms of an integral mean value and is written in the form [5]:

$$\overline{K(\Delta T)} = \frac{1}{\Delta T} \int_{T_0}^{T_l} K(T) dT. \tag{5}$$

Thus the Fourier postulate written in the form of a difference equation for a medium with temperature dependent heat conductivity has the form

$$q = -\left(\frac{1}{\Delta T} \int_{T_0}^{T_l} K(T) dT\right) \frac{\Delta T}{\Delta x}. \tag{6}$$

For media for which the heat conductivity is temperature-dependent, the differential equation (4) is thus not identical with the difference equation (6), and the methods of determining the mean temperature gradient and the mean heat conductivity are different.

5. Media with heat conductivity varying in the material

If the medium is characterized by a dependence of heat conductivity on the coordinate it is called substantially nonhomogeneous. This nonhomogeneity is the property of the medium independently of the temperature gradient applied, and for the same of simplicity we can assume that there is no additional dependence of heat conductivity on temperature.

The conduction of heat energy through this type of medium can be described by the difference equation

$$q = -\overline{K(\Delta x)} \frac{\Delta T}{\Delta x}. \tag{7}$$

The differential equation, however, for media with coordinate dependent heat conductivity becomes

$$q = -K(x) \frac{dT}{dx}(x). \tag{8}$$

The comparison of the difference equation (7) with the differential equation (8) leads also to the necessity of defining the mean heat conductivity and the mean temperature gradient in Eq. (7).

The mean temperature gradient in Eq. (7) is defined as $\Delta T/\Delta x$ which is equivalent to replacing the existing temperature distribution by linear distribution.

The mean heat conductivity in Eq. (7) is defined analogously to (5) also as an integral mean and is written[1]

$$\overline{K(\Delta x)} = \frac{1}{\Delta x} \int_0^{\Delta x} K(x) dx. \tag{9}$$

Accordingly, for a medium whose heat conductivity changes with the coordinate, the Fourier postulate written in the form of a difference equation is

$$q = -\left(\frac{1}{\Delta x} \int_0^{\Delta x} K(x) dx\right) \frac{\Delta T}{\Delta x} . \tag{10}$$

For media with coordinate-dependent heat conductivity the difference equation (8) is not identical with the difference equation (10) similarly as in the case of the relationships (4) and (6). The definitions of the mean temperature gradient and the mean heat conductivity are different in analogy to the media with temperature-dependent heat conductivity.

6. *Discussion*

For media with constant heat conductivity the local proportionality of the heat flux to the temperature gradient is identical in the entire volume of the medium and is equal to the resultant proportionality for the whole heat-conducting medium. An increase in the difference between the temperatures applied between the bounding surfaces of a specimen with constant heat conductivity, does not change the heat conducting properties of the specimen and gives rise only to a proportional increase in the flow of heat conducted by the specimen. Changes of the direction of externally applied temperature gradient are equivalent and thus the same temperature differences will give rise to the same heat flux densities.

In case of substantially nonhomogeneous media the local relation between the density of heat flux and the temperature gradient is not simultaneously identical in the entire volume of the medium. There remains, however, the local direct proportionality between the local heat flux density and the local value of the temperature gradient independently of the externally applied effective temperature gradient. The increase in the temperature difference applied between the boundary surfaces of the investigated specimen with constant resulting heat conductivity does not change the effective heat-conducting properties of this particular specimen and causes the increase of heat flux flowing through this sample. There remains, however, the question whether this increase of heat flux is proportional to the applied temperature difference, and thus proportional to the temperature gradient un-

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[1] Note, however, that starting from (4) we can obtain (5) but starting from (8) we cannot obtain (9). This will be the subject of separate considerations.

derstood as the ratio of the difference between the temperatures at the boundaries of the specimen to the length of the specimen.

Formally there is another possibility, that the increase in the heat flux in consideration is proportional to the effective gradient for the whole specimen defined as the integral mean,

$$\overline{(\mathrm{grad}\ T)}_{\Delta x} = \frac{1}{\Delta x}\int_{0}^{\Delta x} \frac{dT}{dx}(x)dx. \tag{11}$$

The possibility of defining the effective gradient in the form (11) is suggested by analogy to the corresponding definition (9) of the mean heat conductivity. There are so far no data in literature available concerning the measurements of heat conductivity for substantially nonhomogeneous media, and thus there is no possibility of experimental check of the suggestion (11).

Since substantially nonhomogeneous media are characterized by non-zero gradient of heat conductivity we have every reason to believe that the reversal of the externally applied temperature gradient does not lead to an equivalent but only reversed heat flow.

In both cases there will be different heat flux densities, *i.e.*, we shall deal with a heat rectifying effect. The existence of heat rectifying has been proved beyond doubt by the recently published results [6] concerning heat rectification by systems of brass discs whose bounding planes are on one side polished and on the other rough. The physical principle of heat rectification consists here on the existence of asymmetry of heat resistivity of the contacting surfaces with respect to the direction of heat flow. Another heat rectifier proposed in [7] is based on a different principle. It is composed of a system of thin stainless steel layers and a metal block (*e.g.* of aluminium) subjected to constant external pressure. The heat flow through this system for the same temperature difference between its boundaries will depend on the direction of the temperature difference applied, owing to different thermal expansion of the metal block which will differently press together the thin stainless steel layers.

For media with temperature-dependent heat conductivity one should take into account the fact that the increase in the externally applied temperature difference results always in a change in the mean heat conductivity defined by Eq. (5). The application of a greater and greater temperature difference to the boundaries of a specimen with temperature-dependent heat conductivity means that every time we deal with a specimen with other effective heat conductivity.

For the low-temperature branch in the region of the maximum temperature dependence of the heat conductivity the increase in the effective temperature gradient applied to the specimen results always in an increase in the mean heat conductivity of the specimen defined by Eq. (5).

The increase in the heat flux density will thus be due both to the increasing effective temperature gradient and to increasing mean heat conductivity, as can be seen from Eq. (6).

For the high-temperature branch in the region of the maximum of the temperature dependence of heat conductivity, the increase in the effective gradient of temperature applied to the sample always results in a decrease in the mean heat conductivity of the specimen (5). In this particular case the increase in heat flux density is due to increasing effective gradient of temperature in spite of decreasing mean heat conductivity.

This is the way in which the difference between the low- and the high-temperature branch in the region of the maximum of the temperature dependence of heat conductivity is manifested in the determination of mean heat conductivity (5).

In the case of media which are nonhomogeneous in the sense of temperature-dependent heat conductivity there is no direct-proportionality of heat flux density to temperature gradient, since a change in the temperature difference applied results in a change in the value of mean heat conductivity expressed by Eq. (5).

It should be pointed out that the mean heat conductivity defined in Eq. (5) does not depend on the length of the specimen. In other words two specimens of different length made of the same material, and having the same temperatures at the bounding surfaces will have the same mean heat conductivity according to Eq. (5). This means that the term $K(T)$ appearing in the integral of Eq. (5) does not depend on the temperature gradient.

There remains also the question whether the increase in the heat flux in Eq. (6) is due to the increase in the effective gradient defined as $\Delta T/\Delta x$ or to the increase in the mean gradient defined as the following integral mean:

$$\overline{(\mathrm{grad}\ T)_{\Delta T}} = \frac{1}{\Delta T}\int_{T_0}^{T_l} \frac{dT}{dx}(T)\,dT. \tag{12}$$

There are so far no experimental data mentioned in literature avaliable which could be used for checking the suggestion expressed by Eq. (12).

7. Conclusions

Media with constant heat conductivity are characterized by direct proportionality between heat flux density and the temperature gradient. This proportionality is identical at every point of the medium and thus the local proportionality coincides with the resultant proportionality between the total heat flux flowing through the specimen and the effective temperature gradient understood as the ratio of the temperature difference between the boundary surfaces to the length of the specimen.

The coefficients of the above-mentioned local and resultant proportionalities are identical.

Substantially nonhomogeneous media are characterized by local direct proportionality between heat flux density and local temperature gradient, but only at arbitrary fixed point in space (referred to the thermometer mounted on the specimen). The resultant proportionality between the total heat flux and the effective temperature gradient is also fulfilled.

The proportionality factors in the local and resultant proportionalities, are however not identical in this case. The effective temperature gradient for substantially nonhomogeneous media may be either defined as the ratio of the temperature difference to the length or one can propose the definition (11). This requires additional experimental study.

Media which are nonhomogeneous in the sense of temperature-dependent heat conductivity do not exhibit for changing temperature any local direct proportionality between the heat flux density and the local temperature gradient with respect to the thermometer mounted on the specimen (*i.e.* for a fixed point in space).

For changing temperature the resultant proportionality between the total heat flux through the specimen and the effective gradient is also not fulfilled, since the effective heat conductivity changes with changing temperature according to Eq. (5).

For media which are nonhomogeneous owing to temperature-dependent heat conductivity the local and effective conductivities are both temperature-dependent and different.

The effective temperature gradient fot the media in consideration is defined as the ratio of the temperature difference between the boundaries of the specimen to the length, or can be defined by Eq. (12). The final choice of definition can be made only after additional experimental investigations.

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