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A GENERAL "DYNAMICAL" MODEL FOR A CLASS OF STATISTICAL DISTRIBUTIONS (II)*

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Within the preceeding paper on this subject [1], a successful attempt was made in order to give a differential form to the law assumed at the basis of a wide class of statistical distributions (the "generalized gamma distribution" [2]).

In [1], an evident analogy of formalism with Schroedinger's theory for Quantum Mechanics was noted and underlined, but more detailed interpretation and criticism were delayed to a further investigation.

This note contains only the first, simple stage of such a research, so that, necessarily, no effort has been made to attain complete mathematical rigour.

Studies have been undertaken in such a direction.

1

It may be useful to the reader starting the present note by recalling for a moment the deduction of formula (3) contained in [1]; this will serve to the continuity and, above all, to the clarification of an essential aspect of the mentioned formula. Someone, in fact, has observed[1] that equation (3) of [1] may appear less and less general than it actually is; this depends only on the choice of symbols.

The point is just the following: in [1] one specifies that the variable t, on which the probability P depends, is understood to be of a completely general physical meaning; however, for the sake of brevity, one refers to it as time. Unfortunately, the same symbol t is used in the continuation of the paper to represent the very variable time. (Nevertheless, a remark is made in order to emphasize such a distinction: we refer to the analogies involving the Schroedinger equation.)

For this reason we want now to improve those considerations, by using two different variables t and T, and we shall call them time and temperature respectively. It will result clearly that these are not ... engaging denominations.

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[1] C. Ferreri, private communication.

Thus, let

$$(dP)_0 = \psi(t_0)dT$$

be the probability of finding the system S with the temperature T at the instant t_0. (Obviously, in a general case ψ will depend on t and T). At the instant $t_0+\Delta t$ this probability becomes:

$$(dP)_{0,\Delta t} = \psi(t_0+\Delta t)dT,$$

hence the relative variation of probability will be:

$$\frac{\psi(t_0+\Delta t)-\psi(t_0)}{\psi(t_0)}.$$

Now, if we call $H(t_0)$ the sum of all the stimulations to which the system succumbs at time t_0, we are able to formulate the following hypothesis:

the relative variation of probability is proportional, whithin an infinitesimal of higher order than Δt, to the sum of all the stimulations and to the interval of time Δt.

In quantitative terms:

$$\psi(t_0+\Delta t)-\psi(t_0) \propto H(t_0) \cdot \psi(t_0)\Delta t;$$

hence, dividing by Δt and letting $\Delta\ t \to 0$, one has at any time t:

$$\frac{d\psi}{dt} = kH\psi,$$

that is, equation (3) in [1]. (Of course, H will depend too, in general, on t and T).

Repeating these arguments, but dealing with one variable only, one obtains conclusions which are formally identical to those drawn in [1]; a slight change of language is required, since no difference must be made between time and temperature.

Concluding, it should appear without any possibility of misunderstanding that the considerations in [1] possess a wider amount of generality than one could think on the basis of the above-mentioned interpretation of symbols.

Besides, referring to the partial differential equation (6) of [1], we stress that it follows from the same kind of arguments we have shown here; it is clear that the variable here called T must be replaced there by x, while variable t assumes, within that relation, the proper meaning of time.

2

We wish now to pursue criticism and interpretation of the hypothesis expressed by the law of the m.g.g.d.. Following this line, it is worth noting that analogies between the result relative to the m.g.g.d. and Quantum Mechanics refer not only to the particular differential equation:

$$-i\hbar\frac{\partial\psi}{\partial t} = \frac{p^2}{2m}\psi,$$

but involve profoundly the operator structure introduced by Schroedinger at the basis of Quantum Mechanics. In other terms, the idea of Schroedinger, which consists essentially in the correspondence:

$$-i\hbar \frac{\partial}{\partial t} \leftrightarrow H,$$

H being the Hamiltonian of the system, appears again — unchanged — at the background of the general law of the m.g.g.d.: it results immediately from a simple inspection of its formulation.

In effect, one can accomplish the construction of a Quantum Mechanics equation (no matter whether relativistic or not) starting from the abovesaid formulation.

Thus, let $|\psi(x, t_0)|^2 \cdot dx$ be the probability of detecting (by experiments on infinitely many samples) a particle at the position x at time t_0; further, denote by $|\psi(x, t_0+\Delta t)|^2 \cdot dx$ the same probability at time $t_0+\Delta t$. Let us suppose, too, that this probability follows from a function ψ, which we assume to be a complex-valued one; more precisely:

$$|\psi|^2 = \psi^*\psi.$$

Finally, as it is reasonable, let us ascribe to the total energy of the particle the meaning of the sum of all the stimulations to which the system succumbs and denote by H such an energy, expressed by a real function of x and t.

The law of variation of probability gives then:

$$\frac{|\psi(x, t_0+\Delta t)|^2 - |\psi(x, t_0)|^2}{|\psi(x, t_0)|^2} = kH\Delta t + \ldots, \tag{I}$$

where one has neglected higher order infinitesimals.

The above relation is equivalent to the other:

$$\frac{|\psi(x, t_0+\Delta t)|^2 - |\psi(x, t_0)|^2}{\Delta t} = kH|\psi(x, t_0)|^2 + \ldots \tag{II}$$

hence, by taking the limit as $\Delta t \to 0$:

$$\frac{\partial|\psi(x, t_0)|^2}{\partial t} = kH \cdot |\psi(x, t_0)|^2;$$

i.e., at any time t:

$$\frac{\partial|\psi|^2}{\partial t} = kH|\psi|^2. \tag{*}$$

We now consider the equation:

$$\frac{\partial\psi}{\partial t} = cH\psi$$

and its complex conjugate:

$$\frac{\partial \psi^*}{\partial t} = c^* H^* \psi^*.$$

If one multiplies these equations by ψ^* and ψ respectively, one obtains, after addition:

$$\frac{\partial |\psi|^2}{\partial t} = kH \cdot |\psi|^2,$$

namely equation (*), if one takes $k = 2\,\mathrm{Re}\,(c)$ and $H(x, t)$ real.

Thus, the structure of the Schroedinger formalism falls effectively under the general aw expressed by the m.g.g.d., at least in the case of multiplicative energy operators.

3

It must be observed that the above procedure cannot apply to the case of proper operators acting in the space of the functions $|\psi|^2$, due essentially to two reasons:

1) equation (I), written in such a way, has no significance if H is a non-multiplicative operator in the space of the functions $|\psi|^2$;

2) deduction of (II) from (I) is possible if, and only if, one can ordinarily multiply both members of (I) by $|\psi|^2$.

This circumstance, which seems to introduce an a priori difficulty, actually arises only because of the particular choice we have made for the mathematical form of the sum of all the stimulations to which the system S succumbs. In other words, the mathematical translation of the general physical concept of stimulation has been based, so far, on the assumption of multiplicative operators; however, it is obvious that this restriction is not based on an intrinsic feature of the m.g.g.d. hypothesis.

In effect, as we shall prove, it is possible, by means of a wider mathematical interpretation of the concept of stimulation, to insert the general case of proper and non--proper operators into the "dynamical" model of the m.g.g.d.

A useful picture of this fact is schematically furnished by Fig. 1, to which we shall refer in the following.

In this figure, a family of curves is sketched and every curve is assumed to represent a given probability distribution as a function of a (real) variable t. Then, let us consider the curve marked by $|\psi_1|^2$ in the figure and, more precisely, let us look at those points which correspond to t_0 and to $t_0+\Delta t$ respectively.

The value of $|\psi_1|^2$ at the point $t_0+\Delta t$, which we shall denote by $|\psi_1|^2_{t_0+\Delta t}$, is just the same as the value that function $|\psi_2|^2$ assumes at the point t_0; hence, the unit variation of probability is:

$$|\psi_1|^2_{t_0+\Delta t} - |\psi_1|^2_{t_0} = |\psi_2|^2_{t_0} - |\psi_1|^2_{t_0}; \tag{1}$$

thus, corresponding to a given value of Δt, the following equality can be written:

$$|\psi_1|^2_{t_0+\Delta t} - |\psi_1|^2_{t_0} = H_0|\psi_1|^2_{t_0} - |\psi_1|^2_{t_0} = (H_0 - I)|\psi_1|^2_{t_0}, \tag{2}$$

where, clearly, H_0 is a proper or non-proper operator, acting in the space of the functions $|\psi|^2$; furthermore, at any time t, the last equality from (2) gives:

$$H_0|\psi_1|_t^2-|\psi_1|_t^2 = (H_0-I)|\psi_1|_t^2. \tag{3}$$

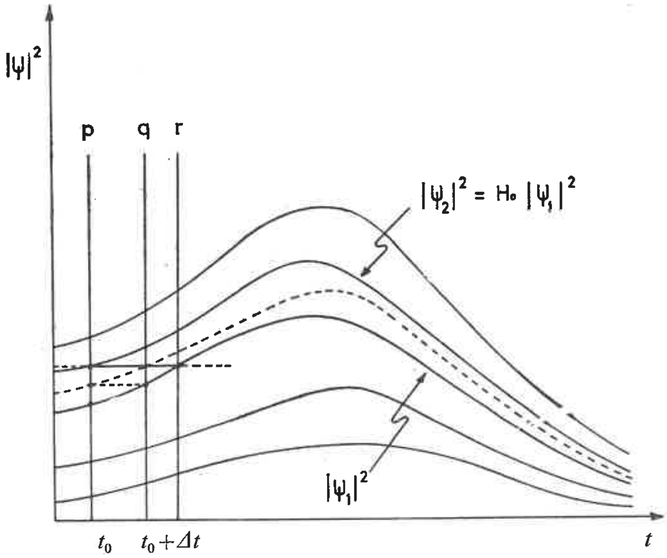

Fig. 1

At this point, (1) is immediately written in the form:

$$|\psi_1|_{t+\Delta t}^2-|\psi_1|_t^2 = (H_0-I)|\psi_1|_t^2 \tag{4}$$

and to the total stimulation the following definition must apply:

$$H_t = \frac{(H_0-I)|\psi_1|_t^2}{|\psi_1|_t^2}. \tag{5}$$

This is seen as follows. Divide (4) by $|\psi_1|_t^2$, to obtain:

$$\frac{|\psi_1|_{t+\Delta t}^2-|\psi_1|_t^2}{|\psi_1|_t^2} = \frac{(H_0-I)|\psi_1|_t^2}{|\psi_1|_t^2} \tag{6}$$

and this, apart from a numerical constant k, is just the formulation of the m.g.g.d. law, provided one remembers that its right-hand side depends implicitly on Δt: hence the left--hand side will increase proportionally with Δt, at least within the first approximat on we assumed at the beginning.[2]

Thus equality (6), by emphasizing this dependence on Δt, is written more generally:

$$\frac{|\psi_1|_{t+\Delta t}^2-|\psi_1|_t^2}{|\psi_1|_t^2} = \frac{(H_0-I)|\psi_1|_t^2}{|\psi_1|_t^2}\,\Delta t \tag{7}$$

and this gives the expression for the stimulation H proposed in (5).

[2] This circumstance is graphically expressed in our scheme: in fact, note that, relative to a different Δt (from line p to line q, instead of p and r), a different transition (from $|\psi_1|^2$ to the dashed curve, instead of this from $|\psi_1|^2$ to $|\psi_2|^2$) must be performed. Of course proportionality is a further precisation.

We have still to note that (5) reduces to $H_0 - I$ for non-proper operators, so giving an effective generalisation of the mathematical meaning before taken for H. Further, in this particular case, the definition (5) furnishes a form for the stimulation which is independent on $|\psi_1|_t^2$ and this is a very reasonable requirement for the construction of a physical theory. The same is not true for the strictly operatorial interpretation we have just explained. A further slight modification, besides being completely acceptable, allows us to overcome this last difficulty; the way is the following.

Formula (7), taking the limit when $\Delta t \to 0$, clearly becomes:

$$\frac{\partial |\psi_1|_t^2}{\partial t} = (H_0 - I)|\psi_1|_t^2$$

and this falls into the scheme furnished by

$$\frac{\partial |\psi|^2}{\partial t} = k\hat{H}|\psi|^2$$

if one assumes the operator

$$\hat{H} = \frac{1}{k}(H_0 - I)$$

to represent the stimulation, instead of right-hand side of (5).

Thus we arrived at the required generalisation of (3) in [1], by including proper operators in the structure of the hypothesis giving place to the m.g.g.d.

Of course, all the above considerations ought to be rearranged in a completely rigorous mathematical theory; at present, although studies have been undertaken in such a direction, the entire scheme we have just concluded can be accepted only as a simple hint for an exhaustive analysis of the undeniable analogy between statistical and quantum-mechanical formalism. Furthermore, an unambiguous definition of stimulation is still required, even within a merely physical approach.

We hope to attain the completion of this program with some present and future investigations.

Finally, we wish to express our gratitude to Professor C. Ferreri, who first encouraged this research.

REFERENCES

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[2] J. H. Lienhard, P. L. Meyer, *Quart. Appl. Math.*, **25**, 330 (1967).