ON THE REAL SPIN WAVE THEORY OF FERROMAGNETISM. CORRECTIONS TO THE MAGNETIZATION

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A way of deriving corrections to the free energy and magnetization of a cubic Heisenberg ferromagnet due to kinematical and dynamical interactions of real spin waves is shown. The results obtained are valid within the entire range of temperatures from absolute zero up to the Curie point.

The thermodynamical perturbation calculus and diagrammatical representation of perturbation terms are applied throughout this investigation.

1. *Introduction*

In an earlier paper by this author (1970b) there has been calculated the contribution to the sum-over-states and thereupon to the free energy of a cubic Heisenberg ferromagnet from one type of dynamical interaction graphs, namely from those without energy denominators. We were thus able to renormalize the spin wave energy. A matter of course, this renormalization deviated from the one obtained by the Green function, as both methods are quite different.

We have called the approach that we apply hereinafter the real spin wave theory of ferromagnetism. It is well known that the usual spin wave theory works efficiently at low temperatures, *i.e.* for spin waves very long compared with the lattice constant. For this case, magnons are well defined quasi-particles and their life-times prove to be almost infinite. Things change when the temperature increases. Near the Curie point magnons are no longer quasi-particles and their life-times become finite. Computation of them requires resorting to the Green function method and calculating the mass operator, what is by no means easy.

Nevertheless, we assume that the spin wave theory holds for temperatures up to the Curie point. Of course, we have to stipulate that "bare", *i.e.* free, magnons must feel their dynamical and kinematical interactions, that is, they have to become "dressed". Along these lines we shall get some new corrections to the sum-over-states and thereupon to the free energy and magnetization.

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The procedure followed here, although exact, is very tedious, so that at present we can only solve this problem in its theoretical aspect. Numerical results including feasible computer calculations will appear in a subsequent paper.

2. *Graphs due to dynamical interaction*

The theory proposed here is a continuation of the investigation carried out in our previous paper (1970b), denoted henceforth by I; the reader is referred thereto for necessary details.

Dyson's Hamiltonian of a cubic Heisenberg ferromagnet (1956), expressed in modes of ideal spin waves, has the form

$$\mathscr{H} = E_0 + \mathscr{H}_0 + \mathscr{H}_I, \tag{2.1}$$

$$E_0 = -LSN - \tfrac{1}{2} JNS^2 \gamma_0, \tag{2.2}$$

$$\mathscr{H}_0 = \sum_{\lambda} (L + \varepsilon_\lambda) a_\lambda^* a_\lambda, \tag{2.3}$$

$$\varepsilon_\lambda = JS(\gamma_0 - \gamma_\lambda), \tag{2.4}$$

$$\gamma_\lambda = \sum_{\delta} \exp i\boldsymbol{\lambda} \cdot \boldsymbol{\delta}, \tag{2.5}$$

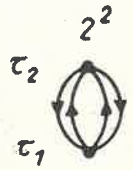
$$\mathscr{H}_I = -\tfrac{1}{4} JN^{-1} \sum_{\lambda\varrho\sigma} \Gamma^\lambda_{\varrho,\sigma} a^*_{\sigma+\lambda} a^*_{\varrho-\lambda} a_\varrho a_\sigma, \tag{2.6}$$

$$\Gamma^\lambda_{\varrho,\sigma} = \gamma_\lambda + \gamma_{\lambda+\sigma-\varrho} - \gamma_{\lambda+\sigma} - \gamma_{\lambda-\varrho}, \tag{2.7}$$

$$[a^*_\varrho, a_\sigma] = \delta_{\varrho,\sigma}, \tag{2.8}$$

where L is the magnetic field strength multiplied by Bohr's magneton and Landè's factor, S is the atomic spin quantum number, N is the number of lattice points, J is the exchange integral between nearest neighbours, $\boldsymbol{\lambda}$ is the reciprocal lattice vector, and ε_λ is spin wave energy. Summation over $\boldsymbol{\delta}$ covers all nearest neighbours and a^*_λ, a_λ stand for creation and annihilation operators of ideal magnons.

Recurring to Eqs (2.5–(2.7), (I4.8)–(I4.14), (I4.18), (IB.4), (IB.5) and (IB.7), and restricting computation to graphs containing terms $\sim D^{-n}_{\lambda\varrho\sigma}$, $n = 1, 2, 3, \ldots$, where $D_{\lambda\varrho\sigma}$ is the energy denominator, we get the following diagrams:

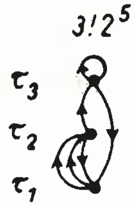


$$= \tfrac{1}{8} \beta J^2 N^{-2} \sum_{\lambda\varrho\sigma} \Gamma^\lambda_{\varrho,\sigma} \Gamma^\lambda_{\sigma+\lambda,\varrho-\lambda} D^{-1}_{\lambda\varrho\sigma} \times$$

$$\times [(\bar{n}_{\sigma+\lambda}+1)(\bar{n}_{\varrho-\lambda}+1)\bar{n}_\varrho \bar{n}_\sigma - \bar{n}_{\sigma+\lambda}\bar{n}_{\varrho-\lambda}(\bar{n}_\varrho+1)(\bar{n}_\sigma+1)], \tag{2.9}$$

$$\beta = (kT)^{-1}, \tag{2.10}$$

$$D_{\lambda\varrho\sigma} = \varepsilon_{\sigma+\lambda} + \varepsilon_{\varrho-\lambda} - \varepsilon_\varrho - \varepsilon_\sigma, \tag{2.11}$$

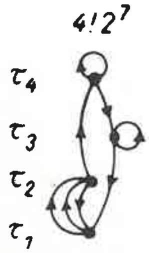


$$= \tfrac{1}{2} J^3 \gamma_0 m_1 N^{-2} \sum_{\lambda\varrho\sigma} \Gamma^{\lambda}_{\varrho\sigma} \cdot \Gamma^{\lambda}_{\sigma+\lambda,\varrho-\lambda}(1-x_\sigma)\times$$

$$\times\{\beta^2 D^{-1}_{\lambda\varrho\sigma}\bar{n}_\sigma(\bar{n}_\sigma+1)[\bar{n}_\varrho(\bar{n}_{\sigma+\lambda}+1)(\bar{n}_{\varrho-\lambda}+1)-(\bar{n}_\varrho+1)\bar{n}_{\sigma+\lambda}\bar{n}_{\varrho-\lambda}]+$$
$$+\beta D^{-2}_{\lambda\varrho\sigma}[(\bar{n}_\varrho+1)(\bar{n}_\sigma+1)\bar{n}_{\sigma+\lambda}\bar{n}_{\varrho-\lambda}-\bar{n}_\varrho\bar{n}_\sigma(\bar{n}_{\sigma+\lambda}+1)(\bar{n}_{\varrho-\lambda}+1)]\}, \qquad (2.12)$$

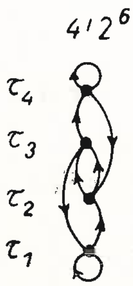
$$= \tfrac{1}{2} J^4 \gamma_0^2 m_1 m_2 N^{-2} \sum_{\lambda\varrho\sigma} \Gamma^{\lambda}_{\varrho,\sigma} \cdot \Gamma^{\lambda}_{\sigma+\lambda,\varrho-\lambda}(1-x_\sigma)\times$$

$$\times\{\beta^3 D^{-1}_{\lambda\varrho\sigma}\bar{n}_\sigma(\bar{n}_\sigma+1)[\bar{n}_\varrho(\bar{n}_{\sigma+\lambda}+1)(\bar{n}_{\varrho-\lambda}+1)-(\bar{n}_\varrho+1)\bar{n}_{\sigma+\lambda}\bar{n}_{\varrho-\lambda}]+$$
$$+\beta^2 D^{-2}_{\lambda\varrho\sigma}[(\bar{n}_\varrho+1)(\bar{n}_\sigma+1)\bar{n}_{\sigma+\lambda}\bar{n}_{\varrho-\lambda}-\bar{n}_\varrho\bar{n}_\sigma(\bar{n}_{\sigma+\lambda}+1)(\bar{n}_{\varrho-\lambda}+1)]\}, \qquad (2.13)$$



$$= \tfrac{1}{4} J^4 \gamma_0^2 m_1^2 N^{-2} \sum_{\lambda\varrho\sigma} \Gamma^{\lambda}_{\varrho,\sigma} \cdot \Gamma^{\lambda}_{\sigma+\lambda,\varrho-\lambda}(1-x_\sigma)^2\times$$

$$\times\{\beta^3 D^{-1}_{\lambda\varrho\sigma}\bar{n}_\sigma(\bar{n}_\sigma+1)(2\bar{n}_\sigma+1)[\bar{n}_\varrho(\bar{n}_{\sigma+\lambda}+1)(\bar{n}_{\varrho-\lambda}+1)-(\bar{n}_\varrho+1)\bar{n}_{\sigma+\lambda}\bar{n}_{\varrho-\lambda}]+$$
$$+2\beta^2 D^{-2}_{\lambda\varrho\sigma}\bar{n}_\sigma(\bar{n}_\sigma+1)[(\bar{n}_\varrho+1)\bar{n}_{\sigma+\lambda}\bar{n}_{\varrho-\lambda}-\bar{n}_\varrho(\bar{n}_{\sigma+\lambda}+1)(\bar{n}_{\varrho-\lambda}+1)]+$$
$$+2\beta D^{-3}_{\lambda\varrho\sigma}[(\bar{n}_{\sigma+\lambda}+1)(\bar{n}_{\varrho-\lambda}+1)\bar{n}_\varrho\bar{n}_\sigma-\bar{n}_{\sigma+\lambda}\bar{n}_{\varrho-\lambda}(\bar{n}_\varrho+1)(\bar{n}_\sigma+1)]\}, \qquad (2.14)$$

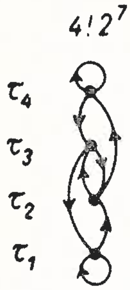


$$= \tfrac{1}{4} J^4 \gamma_0^2 m_1^2 N^{-2} \sum_{\lambda\varrho\sigma} \Gamma^{\lambda}_{\varrho,\sigma}\Gamma^{\lambda}_{\sigma+\lambda,\varrho-\lambda}(1-x_\varrho)(1-x_\sigma)\times$$

$$\times\{\beta^3 D_{\lambda\varrho\sigma}\bar{n}_\varrho(\bar{n}_\varrho+1)\bar{n}_\sigma(\bar{n}_\sigma+1)[(\bar{n}_{\sigma+\lambda}+1)(\bar{n}_{\varrho-\lambda}+1)-\bar{n}_{\sigma+\lambda}\bar{n}_{\varrho-\lambda}]+$$

$$+2\beta^2 D^{-2}_{\lambda\varrho\sigma}\bar{n}_\sigma(\bar{n}_\sigma+1)[(\bar{n}_\varrho+1)\bar{n}_{\sigma+\lambda}\bar{n}_{\varrho-\lambda}-\bar{n}_\varrho(\bar{n}_{\sigma+\lambda}+1)(\bar{n}_{\varrho-\lambda}+1)]+$$

$$+2\beta D^{-3}_{\lambda\varrho\sigma}[(\bar{n}_{\sigma+\lambda}+1)(\bar{n}_{\varrho-\lambda}+1)\bar{n}_\varrho\bar{n}_\sigma-\bar{n}_{\sigma+\lambda}\bar{n}_{\varrho-\lambda}(\bar{n}_\varrho+1)(\bar{n}_\sigma+1)]\}, \tag{2.15}$$



$$= \tfrac{1}{2} J^4\gamma_0^2 m_1^2 N^{-2} \sum_{\lambda\varrho\sigma} \Gamma^\lambda_{\varrho,\sigma}\Gamma^\lambda_{\sigma+\lambda,\varrho-\lambda}(1-x_{\varrho-\lambda})(1-x_\varrho)\times$$

$$\times\{\beta^3 D^{-1}_{\lambda\varrho\sigma}\bar{n}_{\varrho-\lambda}(\bar{n}_{\varrho-\lambda}+1)\bar{n}_\varrho(\bar{n}_\varrho+1)[(\bar{n}_{\sigma+\lambda}+1)\bar{n}_\sigma-\bar{n}_{\sigma+\lambda}(\bar{n}_\sigma+1)]+$$

$$+2\beta^2 D^{-2}_{\lambda\varrho\sigma}\bar{n}_\varrho(\bar{n}_\varrho+1)[(\bar{n}_{\sigma+\lambda}+1)(\bar{n}_{\varrho-\lambda}+1)\bar{n}_\sigma-\bar{n}_{\sigma+\lambda}\bar{n}_{\varrho-\lambda}(\bar{n}_\sigma+1)]+$$

$$+2\beta D^{-3}_{\lambda\varrho\sigma}[\bar{n}_{\sigma+\lambda}\bar{n}_{\varrho-\lambda}(\bar{n}_\varrho+1)(\bar{n}_\sigma+1)-(\bar{n}_{\sigma+\lambda}+1)(\bar{n}_{\varrho-\lambda}+1)\bar{n}_\varrho\bar{n}_\sigma]\} \tag{2.16}$$

and so forth. For the derivation of these diagrams see Appendix A. The numbers above graphs indicate their multiplicity. As easily seen, all these graphs can be summed to yield

$$G_1 = \frac{\beta J^2}{8\left(1-\dfrac{1}{S}\,Y\right)} N^{-2} \sum_{\lambda\varrho\sigma} \Gamma^\lambda_{\varrho,\sigma}\Gamma^\lambda_{\sigma+\lambda,\varrho-\lambda} D^{-1}_{\lambda\varrho\sigma}\times$$

$$\times[(\tilde{n}_{\sigma+\lambda}+1)(\tilde{n}_{\varrho-\lambda}+1)\tilde{n}_\varrho\tilde{n}_\sigma-\tilde{n}_{\sigma+\lambda}\tilde{n}_{\varrho-\lambda}(\tilde{n}_\varrho+1)(\tilde{n}_\sigma+1)], \tag{2.17}$$

with

$$\tilde{n}_\lambda = \left\{\exp\beta\left[L+\varepsilon_\lambda\left(1-\frac{1}{S}\,Y\right)\right]-1\right\}^{-1}. \tag{I4.25}$$

$$Y = N^{-1}\sum_\lambda (1-x_\lambda)\tilde{n}_\lambda = \frac{1}{JSN\gamma_0}\sum_\lambda \varepsilon_\lambda\tilde{n}_\lambda. \tag{I.4.26}$$

The genuineness of (2.17) becomes evident if we take into account that

$$\tilde{n}_\lambda = \sum_{n=1}^{\infty}\exp\left\{-\beta n\left[L+\varepsilon_\lambda\left(1-\frac{1}{S}\,Y\right)\right]\right\} = \bar{n}_\lambda + \frac{1}{1!}\,x(1-x_\lambda)\bar{n}_\lambda(\bar{n}_\lambda+1)\,Y+$$

$$+\frac{1}{2!}\,x^2(1-x_\lambda)^2\bar{n}_\lambda(\bar{n}_\lambda+1)(2\bar{n}_\lambda+1)\,Y^2+O(x^3), \qquad x=\beta J\gamma_0, \tag{2.18}$$

whence

$$\tfrac{1}{8}\beta J^2N^{-2}\sum_{\lambda\varrho\sigma}\Gamma^{\lambda}_{\varrho,\sigma}\Gamma^{\lambda}_{\sigma+\lambda,\varrho-\lambda}D^{-1}_{\lambda\varrho\sigma}[(\tilde{n}_{\sigma+\lambda}+1)(\tilde{n}_{\varrho-\lambda}+1)\tilde{n}_{\varrho}\tilde{n}_{\sigma}-\tilde{n}_{\sigma+\lambda}\tilde{n}_{\varrho-\lambda}(\tilde{n}_{\varrho}+1)(\tilde{n}_{\sigma}+1)]$$

$$=\tfrac{1}{8}\beta J^2N^{-2}\sum_{\lambda\varrho\sigma}\Gamma^{\lambda}_{\varrho,\sigma}\Gamma^{\lambda}_{\sigma+\lambda,\varrho-\lambda}D^{-1}_{\lambda\varrho\sigma}\times$$

$$\times\{[(\bar{n}_{\sigma+\lambda}+1)(\bar{n}_{\varrho-\lambda}+1)\bar{n}_{\varrho}\bar{n}_{\sigma}-\bar{n}_{\sigma+\lambda}\bar{n}_{\varrho-\lambda}(\bar{n}_{\varrho}+1)(\bar{n}_{\sigma}+1)]+$$

$$+4xm_1(1-x_{\sigma})\bar{n}_{\sigma}(\bar{n}_{\sigma}+1)[\bar{n}_{\varrho}(\bar{n}_{\sigma+\lambda}+1)(\bar{n}_{\varrho-\lambda}+1)-(\bar{n}_{\varrho}+1)\bar{n}_{\sigma+\lambda}\bar{n}_{\varrho-\lambda}]+$$

$$+4x^2m_1m_2(1-x_{\sigma})\bar{n}_{\sigma}(\bar{n}_{\sigma}+1)[\bar{n}_{\varrho}(\bar{n}_{\sigma+\lambda}+1)(\bar{n}_{\varrho-\lambda}+1)-(\bar{n}_{\varrho}+1)\bar{n}_{\sigma+\lambda}\bar{n}_{\varrho-\lambda}]+$$

$$+2x^2m_1^2(1-x_{\sigma})^2\bar{n}_{\sigma}(\bar{n}_{\sigma}+1)(2\bar{n}_{\sigma}+1)[\bar{n}_{\varrho}(\bar{n}_{\sigma+\lambda}+1)(\bar{n}_{\varrho-\lambda}+1)-(\bar{n}_{\varrho}+1)\bar{n}_{\sigma+\lambda}\bar{n}_{\varrho-\lambda}]+$$

$$+2x^2m_1^2(1-x_{\varrho})(1-x_{\sigma})\bar{n}_{\varrho}(\bar{n}_{\varrho}+1)\bar{n}_{\sigma}(\bar{n}_{\sigma}+1)[(\bar{n}_{\sigma+\lambda}+1)(\bar{n}_{\varrho-\lambda}+1)-\bar{n}_{\sigma+\lambda}\bar{n}_{\varrho-\lambda}]+$$

$$+4x^2m_1^2(1-x_{\varrho-\lambda})(1-x_{\varrho})\bar{n}_{\varrho-\lambda}(\bar{n}_{\varrho-\lambda}+1)\bar{n}_{\varrho}(\bar{n}_{\varrho}+1)[(\bar{n}_{\sigma+\lambda}+1)\bar{n}_{\sigma}-\bar{n}_{\sigma+\lambda}(\bar{n}_{\sigma}+1)]+$$

$$+O(x^3)\}. \tag{2.19}$$

We then have the sum of first parts of graphs (2.9)–(2.16). Second parts $\sim D^{-2}_{\lambda\varrho\sigma}$ result from the expression

$$-\tfrac{1}{2}\beta J^3\gamma_0 YN^{-2}\sum_{\lambda\varrho\sigma}\Gamma^{\lambda}_{\varrho,\sigma}\Gamma^{\lambda}_{\sigma+\lambda,\varrho-\lambda}(1-x_{\sigma})D^{-2}_{\lambda\varrho\sigma}\times$$

$$\times[(\tilde{n}_{\sigma+\lambda}+1)(\tilde{n}_{\varrho-\lambda}+1)\tilde{n}_{\varrho}\tilde{n}_{\sigma}-\tilde{n}_{\sigma+\lambda}\tilde{n}_{\varrho-\lambda}(\tilde{n}_{\varrho}+1)(\tilde{n}_{\sigma}+1)] \tag{2.20}$$

which with the help of the factor $(1-x_{\sigma})$ can be transformed to the term $\sim D^{-1}_{\lambda\varrho\sigma}$. Quite similarly,

$$-\tfrac{1}{2}\beta J^4\gamma_0^2Y^2N^{-2}\sum_{\lambda\varrho\sigma}\Gamma^{\lambda}_{\varrho,\sigma}\Gamma^{\lambda}_{\sigma+\lambda,\varrho-\lambda}[(1-x_{\sigma})^2+(1-x_{\varrho})(1-x_{\sigma})-$$

$$-2(1-x_{\varrho-\lambda})(1-x_{\varrho})]D^{-3}_{\lambda\varrho\sigma}[(\tilde{n}_{\sigma+\lambda}+1)(\tilde{n}_{\varrho-\lambda}+1)\tilde{n}_{\varrho}\tilde{n}_{\sigma}-\tilde{n}_{\sigma+\lambda}\tilde{n}_{\varrho-\lambda}(\tilde{n}_{\varrho}+1)(\tilde{n}_{\sigma}+1)] \tag{2.21}$$

reduces to the function $\sim D^{-1}_{\lambda\varrho\sigma}$. Thus, we obtain (2.17).

Let us call (2.9) the basic graph and diagrams (2.12)–(2.16) plus similar ones in higher orders — its branch. It is clear that such a branch arises due to the proliferation of the basic graph. In mathematical terms this means that renormalization of the spin wave energy has to occur, *i.e.*

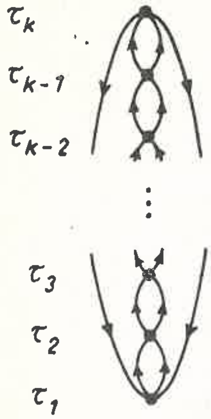
$$\varepsilon_{\lambda}\to\varepsilon_{\lambda}\left(1-\frac{1}{S}Y\right). \tag{2.22}$$

We have thus procured the general rule for deriving diagrams contributing to the sum-over-states. Namely, we only want to find basic graphs and by renormalization of the energy to render appropriate corrections to them at all temperatures between absolute zero and the Curie point.

Assuming that

$$(NS)^{-1}\sum_{\lambda}\tilde{n}_{\lambda}\ll 1 \tag{2.23}$$

and restricting computation to terms proportional to the double product of renormalized spin wave population numbers, we can obtain all basic graphs due to the dynamical interaction. They are:



$$G_k = \frac{1}{2^{k+1}} \frac{\beta(JN^{-1})^{k+1}}{\left(1-\frac{1}{S}Y\right)^k} \sum_{\varrho\sigma} \sum_{\lambda_1,\lambda_2,\dots\lambda_k} \Gamma^{\lambda_1}_{\varrho,\sigma} \times$$
$$\times \Gamma^{-\lambda_1+\lambda_2}_{\varrho-\lambda_1,\sigma+\lambda_1} \Gamma^{-\lambda_2+\lambda_3}_{\varrho-\lambda_2,\sigma+\lambda_2} \dots \Gamma^{-\lambda_{k-1}+\lambda_k}_{\varrho-\lambda_{k-1},\sigma+\lambda_{k-1}} \Gamma^{-\lambda_k}_{\varrho-\lambda_k,\sigma+\lambda_k} \times$$
$$\times (\varepsilon_{\sigma+\lambda_1}+\varepsilon_{\varrho-\lambda_1}-\varepsilon_\varrho-\varepsilon_\sigma)^{-1}(\varepsilon_{\sigma+\lambda_2}+\varepsilon_{\varrho-\lambda_2}-\varepsilon_\varrho-\varepsilon_\sigma)^{-1} \times$$
$$\times \dots (\varepsilon_{\sigma+\lambda_{k-1}}+\varepsilon_{\varrho-\lambda_{k-1}}-\varepsilon_\varrho-\varepsilon_\sigma)^{-1}(\varepsilon_{\sigma+\lambda_k}+\varepsilon_{\varrho-\lambda_k}-\varepsilon_\varrho-\varepsilon_\sigma)^{-1} \times$$
$$\times \tilde{n}_\varrho \tilde{n}_\sigma, \; k = 1, 2, 3, \dots, \infty. \tag{2.24}$$

Graphs (2.24) have been derived in the paper (1962a) by this author. The renormalized diagram (2.17) proves to be a special case of (2.24) for $k = 1$.

As the graphs G_k do not easily submit to further calculations, we leave their analysis to a subsequent paper.

3. *Graphs due to kinematical interaction*

In the previous paper (1970b) we established a set of kinematical operators which fascilitate finding arbitrary average thermodynamical quantities of the spin wave system without always having to remember that spin deviations are limited to $2S+1$ values. Such kinematical operators were defined by

$$\text{Tr}\,(e^{-\beta\mathscr{H}}\hat{C})_{\text{cut-off}} = \text{Tr}\,(e^{-\beta\mathscr{H}}\hat{C}\hat{K}_S), \tag{3.1}$$

with $\hat{C}$ being an arbitrary operator quantity. The trace on the left-hand side of (3.1) admits the physical states only, whereas on the right-hand side of (3.1) all possible states may contribute. Thus, $\hat{K}_S$ has to be the operator projecting the Hilbert space of state vectors on the subspace of physical states. Just as every projection operator, it is bound to satisfy the relation

$$\hat{K}_S^2 = \hat{K}_S. \tag{3.2}$$

Let us examine the matrix elements of $\hat{K}_S$. Recurring to Eqs (IA.10), (IA.13) and (IA.16), we get

$$\text{Tr}\,[e^{-\beta\mathscr{H}}(1-\hat{K}_S)]_{\text{cut-off}} = 0 = \text{Tr}\,[e^{-\beta\mathscr{H}}(1-\hat{K}_S)\hat{K}_S]. \tag{3.3}$$

We suppose the Hamiltonian to be Hermitian, what can be achieved by using the Holstein-Primakoff (1940) relations between the spin components and the oscillatory operators, whence

$$\mathscr{H}|m) = E_m|m) \tag{3.4}$$

and

$$\mathrm{Tr}\,[e^{-\beta\mathscr{H}}\hat{K}_S(1-\hat{K}_S)] = \sum_m (m|e^{-\beta\mathscr{H}}\hat{K}_S(1-\hat{K}_S)|m)$$

$$= \sum_{m,n} e^{-\beta E_m}(\hat{K}_S)_{m,n}[\delta_{m,n}-(\hat{K}_S)_{n,m}] = 0. \tag{3.5}$$

Since according to (IA.11), (IA.14), (IA.15) *etc.* $\hat{K}_S$ proves to have diagonal matrix elements only,

$$(\hat{K}_S)_{m,m}[1-(\hat{K}_S)_{m,m}] = 0 \tag{3.6}$$

for every m and S. We finally obtain

$$(\hat{K}_S)_{m,m} = \begin{cases} 1 \text{ if } m \in \text{physical states,} \\ 0 \text{ otherwise,} \end{cases} \tag{3.7}$$

q.e.d.

Let us now proceed to a derivation of basic kinematical graphs. As in the case of diagrams due to dynamical interaction, we only allow for graphs proportional to $\tilde{n}_\varrho\tilde{n}_\sigma$. This approximation just necessitates the use of the first term of the operator $K_{1/2}-1$, *i.e.*

$$\hat{K}_{1/2}-1 = -\tfrac{1}{2}\sum_f (a_f^*)^2 a_f^2 + \ldots = -\tfrac{1}{2}N^{-1}\sum_{\mu\nu\varrho\sigma} \delta_{\mu+\nu,\varrho+\sigma} a_\mu^* a_\nu^* a_\varrho a_\sigma + \ldots, \tag{IA.11}$$

as the remaining ones denoted by dots contribute as $\sim \tilde{n}_\lambda\tilde{n}_\varrho\tilde{n}_\sigma$ and in the form of higher order products. The same refers to $\hat{K}_S$, $S = 1, 3/2, 2.$ *etc.* We then have to deal with the spin $S = 1/2$ only.

Rewriting the expression for kinematical graphs from our earlier paper (1970b)

$$C_n(1/2) = \frac{(-1)^n}{n!}\int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \ldots \int_0^\beta d\tau_n \langle \hat{T}[\mathscr{H}_I(\tau_1)\mathscr{H}_I(\tau_2)\ldots\mathscr{H}_I(\tau_n)(\hat{K}_{1/2}-1)]\rangle_c \tag{3.8}$$

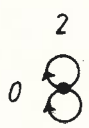
and inserting (IA.11) in (3.8), we obtain

$$C_n(1/2) = \frac{(-1)^{n+1}}{2n!}N^{-1}\sum_{\mu\nu\varrho\sigma}\delta_{\mu+\nu,\varrho+\sigma}\int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \ldots \int_0^\beta d\tau_n \langle \hat{T}[a_\mu^*(0)a_\nu^*(0)\times$$

$$\times a_\varrho(0)a_\sigma(0)\mathscr{H}_I(\tau_1)\mathscr{H}_I(\tau_2)\ldots\mathscr{H}_I(\tau)\rangle_c. \tag{3.9}$$

We interchanged the order of operators in (3.9), what under the sign of Wick's (1950) ordering symbol is permissible.

The lowest order basic graph is now



$$= -\tfrac{1}{2}N^{-1}\sum_{\mu\nu\varrho\sigma}\delta_{\mu+\nu,\varrho+\sigma}\langle \hat{T}[(a_\mu^*(0)a_\nu^*(0)a_\varrho(0)a_\sigma(0)]\rangle_c = -N^{-1}\sum_{\varrho\sigma}\tilde{n}_\varrho\tilde{n}_\sigma \tag{3.10}$$

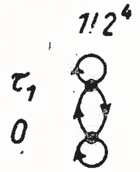
and by renormalizing the energy,

$$\Gamma_1 = -N^{-1} \sum_{\varrho\sigma} \tilde{n}_\varrho \tilde{n}_\sigma. \tag{3.11}$$

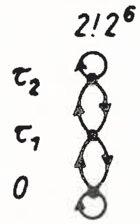
Indeed, introducing the quantities

$$o_i = N^{-1} \sum_{\lambda} \sum_{n=1}^{\infty} n^{i-1}(1-x_\lambda)^{i-1} e^{-\beta n(L+\varepsilon_\lambda)}, \quad i = 1, 2, 3, \ldots, \infty, \tag{3.12}$$

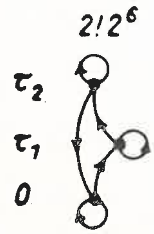
we easily prove the graph (3.10) to proliferate in the following forms:



$$= -2Nxm_1o_1o_2, \tag{3.13}$$

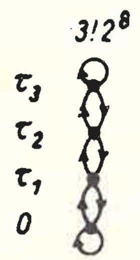


$$= -2Nx^2m_1m_2o_1o_2, \tag{3.14}$$

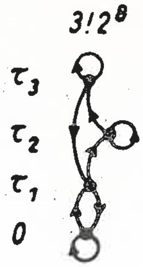


$$= -Nx^2m_1^2o_1o_3, \tag{3.15}$$

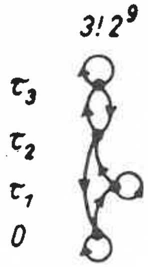
$$= -Nx^2m_1^2o_2^2, \tag{3.16}$$



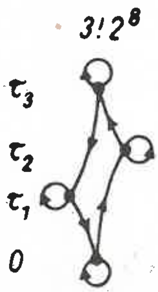
$$= -2Nx^3m_1m_2^2o_1o_2, \tag{3.17}$$



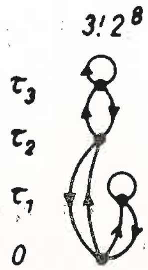
$$= -Nx^3m_1^2m_3o_1o_2, \tag{3.18}$$



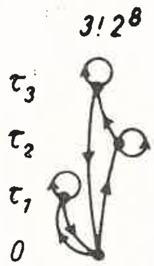
$$= -2Nx^3m_1^2m_2o_1o_3, \tag{3.19}$$



$$= -\tfrac{1}{3}Nx^3m_1^3o_1o_4, \tag{3.20}$$



$$= -2Nx^3m_1^2m_2o_2^2, \tag{3.21}$$



$$= -Nx^3m_1^3o_2o_3 \tag{3.22}$$

etc., see Appendix B. Taking into consideration that

$$Y = m_1 + x m_1 m_2 + x^2 \left(m_1 m_2^2 + \frac{1}{2!} m_1^2 m_3 \right) +$$

$$+ x^3 \left(m_1 m_2^3 + \frac{3}{2} m_1^2 m_2 m_3 + \frac{1}{3!} m_1^3 m_4 \right) + O(x^4), \tag{IB.14}$$

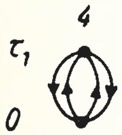
whereupon, by Eq. (2.18),

$$N^{-1} \sum_{\lambda} \tilde{n}_\lambda = o_1 + x m_1 o_2 + x^2 \left(m_1 m_2 o_2 + \frac{1}{2!} m_1^2 o_3 \right) +$$

$$+ x^3 \left(m_1 m_2^2 o_2 + \frac{1}{2!} m_1^2 m_3 o_2 + m_1^2 m_2 o_3 + \frac{1}{3!} m_1^3 \sigma_4 \right) + O(x^4), \tag{3.23}$$

we get (3.11).

The next basic graph is



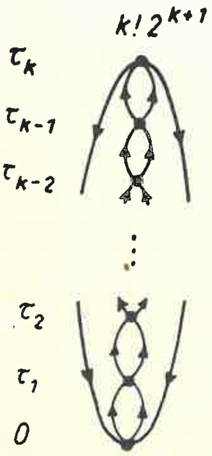
$$= -\tfrac{1}{2} J N^{-2} \sum_{\lambda\varrho\sigma} \Gamma^{\lambda}_{\varrho,\sigma} D^{-1}_{\lambda\varrho\sigma} [(\bar{n}_{\sigma+\lambda}+1)(\bar{n}_{\varrho-\lambda}+1)\bar{n}_\varrho \bar{n}_\sigma - \bar{n}_{\sigma+\lambda}\bar{n}_{\varrho-\lambda}(\bar{n}_\varrho+1)(\bar{n}_\sigma+1)], \tag{3.24}$$

wherefrom, by renormalization,

$$\Gamma_2 = -\frac{1}{2(1-2Y)} J N^{-2} \sum_{\lambda\varrho\sigma} \Gamma^{\lambda}_{\varrho,\sigma} D^{-1}_{\lambda\varrho\sigma} \times$$

$$\times [(\tilde{n}_{\sigma+\lambda}+1)(\tilde{n}_{\varrho-\lambda}+1)\tilde{n}_\varrho \tilde{n}_\sigma - \tilde{n}_{\sigma+\lambda} - \tilde{n}_{\varrho-\lambda}(\tilde{n}_\varrho+1)(\tilde{n}_\sigma+1)]. \tag{3.25}$$

Quite generally,



$$\Gamma_{k+1} = -\frac{J^k N^{-(k+1)}}{2^k (1-2Y)^k} \sum_{\varrho\sigma} \sum_{\lambda_1,\lambda_2,\ldots,\lambda_k} \sum{}' [\Gamma^{\lambda_1}_{\varrho,\sigma} \times$$

$$\times \Gamma^{-\lambda_1+\lambda_2}_{\varrho-\lambda_1,\sigma+\lambda_1} \Gamma^{-\lambda_2+\lambda_3}_{\varrho-\lambda_2,\sigma+\lambda_2} \cdots \Gamma^{-\lambda_{k-1}+\lambda_k}_{\varrho-\lambda_{k-1},\sigma+\lambda_{k-1}} \Gamma^{-\lambda_k}_{\varrho-\lambda_k,\sigma+\lambda_k}] \times$$

$$\times (\varepsilon_{\sigma+\lambda_1} + \varepsilon_{\varrho-\lambda_1} - \varepsilon_\varrho - \varepsilon_\sigma)^{-1} (\varepsilon_{\sigma+\lambda_2} + \varepsilon_{\varrho-\lambda_2} - \varepsilon_\varrho - \varepsilon_\sigma)^{-1} \times$$

$$\times \ldots (\varepsilon_{\sigma+\lambda_{k-1}} + \varepsilon_{\varrho-\lambda_{k-1}} - \varepsilon_\varrho - \varepsilon_\sigma)^{-1} (\varepsilon_{\sigma+\lambda_k} + \varepsilon_{\varrho-\lambda_k} - \varepsilon_\varrho - \varepsilon_\sigma)^{-1} \times$$

$$\times \tilde{n}_\varrho \tilde{n}_\sigma, \quad k = 1,2,3, \ldots, \infty, \tag{3.26}$$

with

$$\sum' [\Gamma^{\lambda_1}_{\varrho,\sigma}\Gamma^{-\lambda_1+\lambda_2}_{\varrho-\lambda_1,\sigma+\lambda_1}\Gamma^{-\lambda_2+\lambda_3}_{\varrho-\lambda_2,\sigma+\lambda_2}\ldots\Gamma^{-\lambda_{k-1}+\lambda_k}_{\varrho-\lambda_{k-1},\sigma+\lambda_{k-1}}\Gamma^{-\lambda_k}_{\varrho-\lambda_k,\sigma+\lambda_k}] \tag{3.27}$$

being the sum of products. In each of them the first, the second, the third, *etc.* term must be replaced by unity, *e. g.*, for $k = 2$

$$\sum' [\Gamma^{\lambda_1}_{\varrho,\sigma}\Gamma^{-\lambda_1+\lambda_2}_{\varrho-\lambda_1,\sigma+\lambda_1}\Gamma^{-\lambda_2}_{\varrho-\lambda_2,\sigma+\lambda_2}] = \Gamma^{-\lambda_1+\lambda_2}_{\varrho-\lambda_1,\sigma+\lambda_1}\Gamma^{-\lambda_2}_{\varrho-\lambda_2,\sigma+\lambda_2} + \Gamma^{\lambda_1}_{\varrho,\sigma}\Gamma^{-\lambda_2}_{\varrho-\lambda_2,\sigma+\lambda_2} + \Gamma^{\lambda_1}_{\varrho,\sigma}\Gamma^{-\lambda_1+\lambda_2}_{\varrho-\lambda_1,\sigma+\lambda_1}.$$

The proof of (3.26) is very tedious and we refrain from adducing it here.

In contradistinction to (2.24), at low temperatures the graphs (3.26) do not affect the sum-over-states up to the term $\sim T^4$. The problem of their behaviour near the Curie point is extremely intricate and will not be considered here.

4. *Free energy and magnetization*

Collecting the results obtained hitherto, we get by virtue of (I4.22), (I4.24), (2.24), (3.11) and (3.26)

$$\begin{aligned}
F = {} & E_0 + \sum_\lambda (L+\varepsilon_\lambda)\tilde{n}_\lambda - \frac{1}{2JS^2\gamma_0} N^{-1} \sum_{\varrho\sigma} \varepsilon_\varrho\varepsilon_\sigma\tilde{n}_\varrho\tilde{n}_\sigma + \beta^{-1}\sum [\tilde{n}_\lambda \ln \tilde{n}_\lambda - \\
& -(1+\tilde{n}_\lambda)\ln(1+\tilde{n}_\lambda)] - \sum_{k=1}^{\infty} \frac{(JN^{-1})^{k+1}}{2^{k+1}\left(1-\frac{1}{S}Y\right)^k} \sum_{\varrho\sigma}\sum_{\lambda_1,\lambda_2,\ldots,\lambda_k} \Gamma^{\lambda_1}_{\varrho,\sigma}\Gamma^{+\lambda_1+\lambda_2}_{\varrho-\lambda_1,\sigma+\lambda_1}\times \\
& \times\Gamma^{-\lambda_2+\lambda_3}_{\varrho-\lambda_2,\sigma+\lambda_2}\ldots\Gamma^{-\lambda_{k-1}+\lambda_k}_{\varrho-\lambda_{k-1},\sigma+\lambda_{k-1}}\Gamma^{-\lambda_k}_{\varrho-\lambda_k,\sigma+\lambda_k}(\varepsilon_{\sigma+\lambda_1}+\varepsilon_{\varrho-\lambda_1}-\varepsilon_\varrho-\varepsilon_\sigma)^{-1}\times \\
& \times(\varepsilon_{\sigma+\lambda_2}+\varepsilon_{\varrho-\lambda_2}-\varepsilon_\varrho-\varepsilon_\sigma)^{-1}\ldots(\varepsilon_{\sigma+\lambda_k}+\varepsilon_{\varrho-\lambda_k}-\varepsilon_\varrho-\varepsilon_\sigma)^{-1}\tilde{n}_\varrho\tilde{n}_\sigma + \\
& +\beta^{-1}\delta_{S,1/2}N^{-1}\sum_{\varrho\sigma}\tilde{n}_\varrho\tilde{n}_\sigma\Bigg\{1+\sum_{k=1}^{\infty}\frac{1}{2^k\left(1-\frac{1}{S}Y\right)^k}J^kN^{-k}\times \\
& \times\sum_{\lambda_1,\lambda_2,\ldots,\lambda_k}\sum' [\Gamma^{\lambda_1}_{\varrho,\sigma}\Gamma^{-\lambda_1+\lambda_2}_{\varrho-\lambda_1,\sigma+\lambda_1}\ldots\Gamma^{-\lambda_{k-1}+\lambda_k}_{\varrho-\lambda_{k-1},\sigma+\lambda_{k-1}}\Gamma^{-\lambda_k}_{\varrho-\lambda_k,\sigma+\lambda_k}]\times \\
& \times(\varepsilon_{\sigma+\lambda_1}+\varepsilon_{\varrho-\lambda_1}-\varepsilon_\varrho-\varepsilon_\sigma)^{-1}(\varepsilon_{\sigma+\lambda_2}+\varepsilon_{\varrho-\lambda_2}-\varepsilon_\varrho-\varepsilon_\sigma)^{-1}\ldots(\varepsilon_{\sigma+\lambda_k}+\varepsilon_{\varrho-\lambda_k}-\varepsilon_\varrho-\varepsilon_\sigma)^{-1}\Bigg\}.
\end{aligned} \tag{4.1}$$

This is the free energy of a system of real spin waves in a cubic Heisenberg ferromagnet.

Thus, we have formulated the spin wave theory of ferromagnetism for temperatures between absolute zero and the Curie point. Evidently, such theory will hold within the entire interval if the condition (2.23) is satisfied.

Actually, apart from the third and fourth terms in (4.1), the form of real spin wave theory resembles the conventional spin wave approach, only the energy of non-interacting magnons must be replaced by the renormalized one according to (2.22), what is equivalent to "dressing" the particles.

Combination of (I4.25) and (I4.26) yields

$$\frac{\partial Y}{\partial L}=\frac{-\beta N^{-1}\sum_{\lambda}(1-x_{\lambda})\tilde{n}_{\lambda}(\tilde{n}_{\lambda}+1)}{1-xN^{-1}\sum_{\lambda}(1-x_{\lambda})^{2}\tilde{n}_{\lambda}(\tilde{n}_{\lambda}+1)}. \tag{4.2}$$

Neglecting the products $\sim\tilde{n}_{\lambda}(\tilde{n}_{\lambda}+1)\tilde{n}_{\varrho}\tilde{n}_{\sigma}$ and similar higher-order terms, we get from (I4.27) and (4.1) the relative magnetization

$$\mu(T)=1-\frac{1}{NS}\sum_{\lambda}\tilde{n}_{\lambda}-\sum_{k=1}^{\infty}\frac{\beta J^{k+1}}{2^{k}S\left(1-\frac{1}{S}Y\right)^{k}}N^{-(k+2)}\sum_{\varrho\sigma}\sum_{\lambda_1,\lambda_2,\ldots,\lambda_k}\Gamma^{\lambda_1}_{\varrho,\sigma}\Gamma^{-\lambda_1+\lambda_2}_{\varrho-\lambda_1,\sigma+\lambda_1}\times$$

$$\times\Gamma^{-\lambda_2+\lambda_3}_{\varrho-\lambda_2,\sigma+\lambda_2}\ldots\Gamma^{-\lambda_{k-1}+\lambda_k}_{\varrho-\lambda_{k-1},\sigma+\lambda_{k-1}}\Gamma^{-\lambda_k}_{\varrho-\lambda_k,\sigma+\lambda_k}(\varepsilon_{\sigma+\lambda_1}+\varepsilon_{\varrho-\lambda_1}-\varepsilon_{\varrho}-\varepsilon_{\sigma})^{-1}\times$$

$$\times(\varepsilon_{\sigma+\lambda_2}+\varepsilon_{\varrho-\lambda_2}-\varepsilon_{\varrho}-\varepsilon_{\sigma})^{-1}\ldots(\varepsilon_{\sigma+\lambda_k}+\varepsilon_{\varrho-\lambda_k}-\varepsilon_{\varrho}-\varepsilon_{\sigma})^{-1}\tilde{n}_{\varrho}\tilde{n}_{\sigma}(\tilde{n}_{\sigma}+1)+$$

$$+2\delta_{S,\frac{1}{2}}S^{-1}N^{-1}\sum_{\varrho\sigma}\tilde{n}_{\varrho}\tilde{n}_{\sigma}(\tilde{n}_{\sigma}+1)\left\{1+\sum_{k=1}^{\infty}\frac{J^{k}}{2^{k}\left(1-\frac{1}{S}Y\right)^{k}}\times\right.$$

$$\times N^{-k}\sum_{\lambda_1,\lambda_2,\ldots,\lambda_k}\sum{}'\left[\Gamma^{\lambda_1}_{\varrho,\sigma}\Gamma^{-\lambda_1+\lambda_2}_{\varrho-\lambda_1,\sigma+\lambda_1}\ldots\Gamma^{-\lambda_{k-1}+\lambda_k}_{\varrho-\lambda_{k-1},\sigma+\lambda_{k-1}}\Gamma^{-\lambda_k}_{\varrho-\lambda_k,\sigma+\lambda_k}\right]\times$$

$$\left.\times(\varepsilon_{\sigma+\lambda_1}+\varepsilon_{\varrho-\lambda_1}-\varepsilon_{\varrho}-\varepsilon_{\sigma})^{-1}(\varepsilon_{\sigma+\lambda_2}+\varepsilon_{\varrho-\lambda_2}-\varepsilon_{\varrho}-\varepsilon_{\sigma})^{-1}\ldots(\varepsilon_{\sigma+\lambda_k}+\varepsilon_{\varrho-\lambda_k}-\varepsilon_{\varrho}-\varepsilon_{\sigma})^{-1}\right\}. \tag{4.3}$$

The first two terms in (4.3) have been derived by Bloch (1962). They are partially due to dynamical interaction between magnons. The third term complements the effect of the dynamical interaction and the third one allows for kinematical restrictions for spins.

APPENDIX A

Let us derive (2.16). We get

$$G_4^{(4)}=\frac{4!2^7}{4!2^8}J^4N^{-4}\sum_{\substack{\lambda_1\varrho_1\sigma_1\\ \lambda_2\varrho_2\sigma_2\\ \lambda_3\varrho_3\sigma_3\\ \lambda_4\varrho_4\sigma_4}}\Gamma^{\lambda_1}_{\varrho_1,\sigma_1}\Gamma^{\lambda_2}_{\varrho_2,\sigma_2}\Gamma^{\lambda_3}_{\varrho_3,\sigma_3}\Gamma^{\lambda_4}_{\varrho_4,\sigma_4}\times$$

$$\times\int_0^{\beta}d\tau_1\int_0^{\beta}d\tau_2\int_0^{\beta}d\tau_3\int_0^{\beta}d\tau_4\,[a^{*}_{\sigma_1+\lambda_1}(\tau_1)^{\bullet}a_{\sigma_1}(\tau_1)^{\bullet}]\,[a^{*}_{\varrho_1-\lambda_1}(\tau_1)^{\bullet}a_{\varrho_2}(\tau_2)^{\bullet}]\times$$

$$\times[a_{\varrho_1}(\tau_1)^{\bullet}a^{*}_{\varrho_3-\lambda_3}(\tau_3)^{\bullet}]\,[a^{*}_{\sigma_2+\lambda_2}(\tau_2)^{\bullet}a_{\sigma_3}(\tau_3)^{\bullet}]\,[a_{\sigma_2}(\tau_2)^{\bullet}a^{*}_{\sigma_3+\lambda_3}(\tau_3)^{\bullet}]\times$$

$$\times[a^{*}_{\varrho_2-\lambda_2}(\tau_2)^{\bullet}a_{\varrho_4}(\tau_4)^{\bullet}]\,[a_{\varrho_3}(\tau_3)^{\bullet}a^{*}_{\varrho_4-\lambda_4}(\tau_4)^{\bullet}]\,[a^{*}_{\sigma_4+\lambda_4}(\tau_4)^{\bullet}a_{\sigma_4}(\tau_4)^{\bullet}], \tag{A.1}$$

wherein the integrations must be carried out over propagation functions (I4.8)—(I4.14).

With their help we have

$$G_4^{(4)} = \frac{1}{2} J^4 N^{-4} \sum_{\substack{\lambda_1\varrho_1\sigma_1\\ \dots \\ \lambda_4\varrho_4\sigma_4}} \Gamma^{\lambda_1}_{\varrho_1,\sigma_1} \dots \Gamma^{\lambda_4}_{\varrho_4,\sigma_4} \delta_{\lambda_1,0} \delta_{\varrho_1,\varrho_2} \delta_{\sigma_3,\sigma_2+\lambda_2} \times$$

$$\times \delta_{\lambda_3,-\lambda_2} \delta_{\varrho_4,\varrho_2-\lambda_2} \delta_{\varrho_3,\varrho_4} \delta_{\lambda_4,0} \bar{n}_{\sigma_1} \bar{n}_{\sigma_4} \int_0^\beta d\tau_1 \dots \int_0^\beta d\tau_4 \exp{(\tau_2-\tau_3)(\varepsilon_{\sigma_2+\lambda_2}+\varepsilon_{\varrho_2-\lambda_2}-\varepsilon_{\varrho_2}-\varepsilon_{\sigma_2})} \times$$

$$\times [\theta_{1,2}\bar{n}_{\varrho_2}+\theta_{2,1}(\bar{n}_{\varrho_2}+1)][\theta_{1,3}(\bar{n}_{\varrho_2}+1)+\theta_{3,1}\bar{n}_{\varrho_2}] \times$$

$$\times [\theta_{2,3}\bar{n}_{\sigma_2+\lambda_2}(\bar{n}_{\sigma_2}+1)+\theta_{3,2}(\bar{n}_{\sigma_2+\lambda_2}+1)\bar{n}_{\sigma_2}] \times$$

$$\times [\theta_{2,4}\bar{n}_{\varrho_2-\lambda_2}+\theta_{4,2}(\bar{n}_{\varrho_2-\lambda_2}+1)][\theta_{3,4}(\bar{n}_{\varrho_2-\lambda_2}+1)+\theta_{4,3}\bar{n}_{\varrho_2-\lambda_2}]$$

$$= \frac{1}{2} J^4 N^{-4} \sum_{\lambda\mu\nu\varrho\sigma} \Gamma^0_{\varrho,\mu} \Gamma^0_{\varrho-\lambda,\nu} \Gamma^\lambda_{\varrho,\sigma} \Gamma^\lambda_{\sigma+\lambda,\varrho-\lambda} \bar{n}_\mu \bar{n}_\nu \times$$

$$\times \int_0^\beta d\tau_1 \dots \int_0^\beta d\tau_4 \exp{(\tau_2-\tau_3)(\varepsilon_{\sigma+\lambda}+\varepsilon_{\varrho-\lambda}-\varepsilon_\varrho-\varepsilon_\sigma)} \times$$

$$\times [\theta_{1,2}\bar{n}_\varrho+\theta_{2,1}(\bar{n}_\varrho+1)][\theta_{1,3}(\bar{n}_\varrho+1)+\theta_{3,1}\bar{n}_\varrho][\theta_{2,3}\bar{n}_{\sigma+\lambda}(\bar{n}_\sigma+1)+\theta_{3,2}(\bar{n}_{\sigma+\lambda}+1)\bar{n}_\sigma] \times$$

$$\times [\theta_{2,4}\bar{n}_{\varrho-\lambda}+\theta_{4,2}(\bar{n}_{\varrho-\lambda}+1)](\theta_{3,4}(\bar{n}_{\varrho-\lambda}+1)+\theta_{4,3}\bar{n}_{\varrho-\lambda}]. \tag{A.2}$$

Using the auxiliary relations

$$N^{-2} \sum_{\mu\nu} \Gamma^0_{\varrho,\mu} \Gamma^0_{\varrho-\lambda,\nu} \bar{n}_\mu \bar{n}_\nu = \gamma_0^2(1-x_{\varrho-\lambda})\,(1-x_\varrho) N^{-2} \sum_{\mu\nu} (1-x_\mu)\bar{n}_\mu \times$$

$$\times (1-x_\nu)\bar{n}_\nu = \gamma_0^2 m_1^2\,(1-x_{\varrho-\lambda})\,(1-x_\varrho), \tag{A.3}$$

$$\int_0^\beta d\tau_1 [\theta_{1,2}\bar{n}_\varrho+\theta_{2,1}(\bar{n}_\varrho+1)][\theta_{1,3}(\bar{n}_\varrho+1)+\theta_{3,1}\bar{n}_\varrho]$$

$$= \beta\bar{n}_\varrho(\bar{n}_\varrho+1)+(\tau_2-\tau_3)[\theta_{2,3}(\bar{n}_\varrho+1)+\theta_{3,2}\bar{n}_\varrho], \tag{A.4}$$

$$\int_0^\beta d\tau_4 [\theta_{2,4}\bar{n}_{\varrho-\lambda}+\theta_{4,2}(\bar{n}_{\varrho-\lambda}+1)][\theta_{3,4}(\bar{n}_{\varrho-\lambda}+1)+\theta_{4,3}\bar{n}_{\varrho-\lambda}]$$

$$= \beta\bar{n}_{\varrho-\lambda}(\bar{n}_{\varrho-\lambda}+1)-(\tau_2-\tau_3)\,[\theta_{2,3}\bar{n}_{\varrho-\lambda}+\theta_{3,2}\,(\bar{n}_{\varrho-\lambda}+1)], \tag{A.5}$$

we obtain (2.16).

Equations (2.9), (2.12), (2.13), (2.14) and (2.15) can be found in the same way.

APPENDIX B

In order to exemplify the derivation of the diagrams (3.13)—(3.22), let us first compute (3.20). We have

$$C_4^{(4)} = -\frac{3\,!2^8}{2\cdot 3\,!2^6} J^3 N^{-4} \sum_{\substack{\mu\nu\omega\zeta\\ \lambda_1\varrho_1\sigma_1\\ \lambda_2\varrho_2\sigma_2\\ \lambda_3\varrho_3\sigma_3}} \delta_{\mu+\nu,\omega+\zeta} \Gamma^{\lambda_1}_{\varrho_1\sigma_1} \Gamma^{\lambda_2}_{\varrho_2,\sigma_2} \Gamma^{\lambda_3}_{\varrho_3,\sigma_3} \times$$

$$\times\int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \int_0^\beta d\tau_3\, [a^*_\mu(0)^\bullet a_\omega(0)^\bullet]\, [a^*_\nu(0)^\bullet a_{\varrho_2}(\tau_2)^\bullet]\, [a_\zeta(0)^\bullet a^*_{\varrho_1-\lambda_1}(\tau_1)^\bullet]\times$$

$$\times[a^*_{\sigma_1+\lambda_1}(\tau_1)^\bullet a_{\sigma_1}(\tau_1)^\bullet]\, [a_{\varrho_1}(\tau_1)^\bullet a^*_{\varrho_3-\lambda_3}(\tau_3)^\bullet]\, ([a^*_{\sigma_2+\lambda_2}(\tau_2)^\bullet a_{\sigma_2}(\tau_2)]\times$$

$$\times[a^*_{\varrho_2-\lambda_2}(\tau_2)^\bullet a_{\varrho_3}(\tau_3)^\bullet]\, [a^*_{\sigma_3+\lambda_3}(\tau_3)^\bullet a_{\sigma_3}(\tau_3)^\bullet] = -2J^3N^{-4}\times$$

$$\times\sum_{\substack{\mu\nu\omega\zeta\\ \lambda_1\varrho_1\sigma_1\\ \lambda_2\varrho_2\sigma_2\\ \lambda_3\varrho_3\sigma_3}} \delta_{\mu+\nu,\omega+\zeta}\Gamma^{\lambda_1}_{\varrho_1,\sigma_1}\Gamma^{\lambda_2}_{\varrho_2,\sigma_2}\Gamma^{\lambda_3}_{\varrho_3,\sigma_3}\delta_{\mu,\omega}\delta_{\nu,\varrho_1}\delta_{\zeta,\varrho_1}\delta_{\varrho_2,\varrho_1}\delta_{\lambda_1,0}\times$$

$$\times\delta_{\varrho_3,\varrho_1}\delta_{\lambda_2,0}\delta_{\lambda_3,0}\bar n_\mu \bar n_{\varrho_1}(\bar n_{\varrho_1}+1)\, \bar n_{\sigma_1}\bar n_{\sigma_2}\bar n_{\sigma_3}\times$$

$$\times\int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \int_0^\beta d\tau_3[\theta_{1,3}(n_{\varrho_1}+1)+\theta_{3,1}\, \bar n_{\varrho_1}]\, [\theta_{2,3}\, \bar n_{\varrho_1}+\theta_{3,2}(\bar n_{\varrho_1}+1)]$$

$$= -2J^3N^{-4}\sum_{\lambda\mu\nu\varrho\sigma}\Gamma^0_{\varrho,\sigma}\Gamma^0_{\varrho,\mu}\Gamma^0_{\varrho,\nu}\bar n_\lambda \bar n_\varrho(\bar n_\varrho+1)\, \bar n_\sigma \bar n_\mu \bar n_\nu\times$$

$$\times\int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \int_0^\beta d\tau_3[\theta_{1,3}(\bar n_\varrho+1)\theta_{3,1}\bar n_\varrho]\, [\theta_{2,3}\, \bar n_\varrho+\theta_{3,2}(\bar n_\varrho+1)]. \qquad \text{(B.1)}$$

Because of

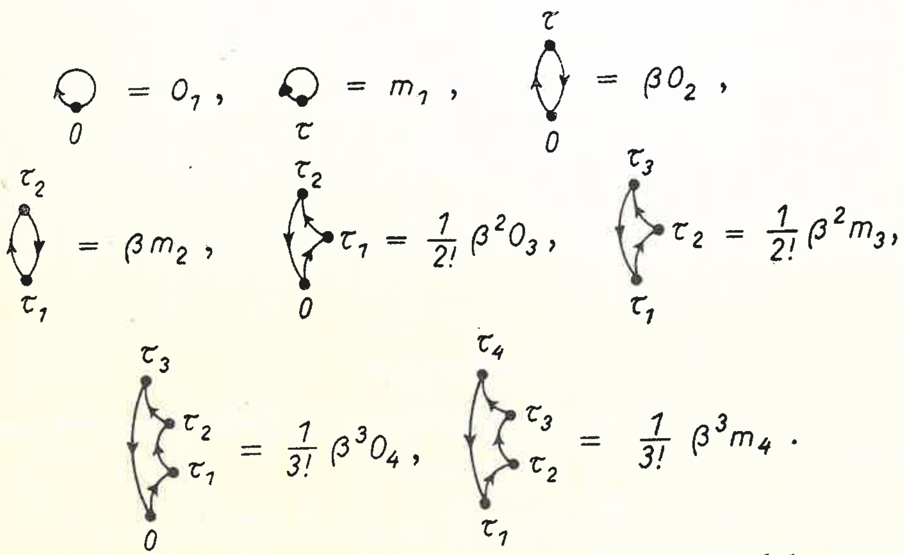
$$\int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \int_0^\beta d\tau_3[\theta_{1,3}(\bar n_\varrho+1)+\theta_{3,1}\bar n_\varrho][\theta_{2,3}\bar n_\varrho+\theta_{3,2}(\bar n_\varrho+1)] = \tfrac{1}{6}\beta^3[6\bar n_\varrho(\bar n_\varrho+1)1], \qquad \text{(B.2)}$$

we get

$$C_4^{(4)} = -\tfrac{1}{3}\beta^3J^3\gamma_0^3N^{-4}\sum_{\lambda\mu\nu\varrho\sigma}\bar n_\lambda(1-x_\varrho)^3[6\bar n_\varrho^2(\bar n_\varrho+1)^2+\bar n_\varrho(\bar n_\varrho+1)]\times$$

$$\times(1-x_\sigma)\bar n_\sigma(1-x_\mu)\bar n_\mu(1-x_\nu)\bar n_\nu = -\tfrac{1}{3}x^3Nm_1^3o_1o_4. \qquad \text{(B.3)}$$

Let us now coordinate graphical and mathematical symbols in the following way:



(B.4)

etc. Resorting to this coordination and multiplying each graph by $[-(J\gamma_0)^{k-1}]/2(k-1)!4^{k-1}$, where k is the number of vertices, we can compute all diagrams (3.13)—(3.22).

REFERENCES

Bloch, M., *Phys. Rev. Letters*, **9**, 286 (1962).
Dyson, F. J., *Phys. Rev.*, **102**, 1217, 1230 (1956).
Holstein, T. and Primakoff, H., *Phys. Rev.*, **58**, 1098 (1940).
Matsubara, T., *Progr. Theor. Phys.*, **14**, 351 (1955).
Szaniecki, J., *Acta Phys. Polon.*, **22**, 389 (1962a); **A38**, 33 (1970b).
Thouless, D. J., *Phys. Rev.*, **107**, 1162 (1957).
Wick, G. C., *Phys. Rev.*, **80**, 268 (1950).