ON THE MARGINALLY SINGULAR N/D EQUATIONS AND THE STRIP APPROXIMATION

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The marginally singular kernels theory is applied to the "old strip" approximation of Chew and Frautschi. A set of coupled integral equations is obtained similar to those derived by Contogouris and Atkinson in the context of "new strip" approximation. This permits a discussion of the analytic properties and uniqueness of the solution.

1. It is well known that if one considers a left-hand cut discontinuity with an asymptotic behaviour consistent with unitarity, *i.e.* an asymptotically constant left-cut discontinuity, then the kernel of the N/D equations is "marginally" singular [1–4].

In order to handle these non-Fredholm equations one splits the kernel into two parts: a singular symmetric kernel and a Fredholm-type kernel. This splitting of kernels (and equations) is closely connected with the decomposition of the left-hand cut discontinuity into a "long-range" part and a "short-range" part.

But an analogous cut-off is employed in the strip approximation which assumes that the distant part of the left-hand cut discontinuity does not appreciably affect the scattering amplitude for not too high energies. This "nearby singularities hypothesis" is supported by the idea that the short-range forces bear little weight in the low energy region.

It follows therefore, that it would be interesting to consider the connections between the marginally singular N/D equations and the strip approximation. We must note that a similar problem has already been treated by Contogouris and Atkinson [2] in the context of the "new strip" approximation of Chew and co-workers (*i.e.* a D function with a finite part of the right hand cut and a Regge asymptotic behaviour) [57].

What we intend to study in this note is the possible connection between the marginally singular N/D equations and the "old strip" approximation of Chew and Frautschi [8], [9]. This will permit us to explicitly consider the crossing-symmetry properties of the scattering amplitude.

We will obtain a set of coupled integral equations analogous to those derived in the marginally singular kernels theory. This lets us discuss the analytic properties and uniqueness of the solution.

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II. We remember the basic idea of the strip approximation proposed by Chew and Frautschi [8]. The double spectral functions are negligible everywhere except within the shaded regions of Fig. 1 (s_1, t_1, u_1 are the values which separate the low energy and high energy regions). In other words, for high s, for instance, only the low contribution is important.

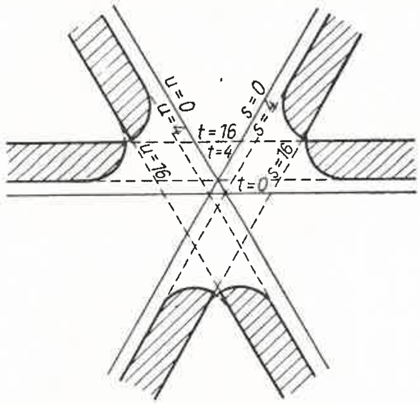

Fig. 1. The strip approximation. The shaded regions are assumed to be dominant

By using the Mandelstam representation and the strip approximation we obtain (we restrict ourselves to elastic scattering of two spinless particles of equal mass)

$$A^{\mathrm{I}}(s,t)=\frac{1}{\pi}\int_4^{s_1} ds'\,\frac{A_s^{\mathrm{I}}(s',t)}{s'-s}+\frac{1}{\pi}\int_4^{t_1} dt'\,\frac{A_t^{\mathrm{I}}(t',s)}{t'-t}+\frac{1}{\pi}\int_4^{u_1} du'\,\frac{A_{u'}^{\mathrm{I}}(u',s)}{u'-u} \tag{1}$$

where $A_s^{\mathrm{I}}(A_t^{\mathrm{I}}, A_u^{\mathrm{I}})$ is the absorbtive part of the scattering amplitude in the $s\,(t, u)$ channel.

If we make partial wave expansion and project out the l-th partial wave, we can write

$$A_l^{\mathrm{I}}(\nu)=\frac{\nu^l}{\pi}\int_0^{\nu_1} d\nu'\,\frac{\operatorname{Im} A_l^{\mathrm{I}}(\nu')}{\nu'^l(\nu'-\nu)}+F_l^{\mathrm{I}}(\nu) \tag{2}$$

where

$$F_l^{\mathrm{I}}(\nu)=V_l^{\mathrm{I}}(\nu)+\text{“}\textit{contribution of waves} > l\text{”} \tag{3}$$

and

$$V_l^{\mathrm{I}}(\nu)=\frac{1+(-1)^{l+\mathrm{I}}}{2\pi\nu}\int_4^{t_1} dt'\,A_t^{\mathrm{I}}(t',4\nu+4)\,Q_l\left(1+\frac{t'}{2\nu}\right). \tag{4}$$

Let us return now to the “marginal hypothesis”

$$\operatorname{Im} A_l^{\mathrm{I}}(\nu)\underset{\nu\to\infty}{=} g_{\mathrm{I}}+0(\nu^{-\varepsilon}),\qquad \varepsilon>0 \tag{5}$$

We shall show that the term *contribution of waves* $> l$ is connected with (5). To make this assertion evident we write the N/D equations in the strip approximation

$$D_l^{\mathrm{I}}(\nu) = 1 - \frac{1}{\pi} \int_0^{\nu_1} d\nu' \, \frac{\varrho_l^{\mathrm{I}}(\nu')\, N_l^{\mathrm{I}}(\nu')}{\nu' - \nu} \tag{6}$$

$$N_l^{\mathrm{I}}(\nu) = F_l^{\mathrm{I}}(\nu) D_l^{\mathrm{I}}(\nu) + \frac{\nu^l}{\pi} \int_0^{\nu_1} d\nu' \, \frac{\varrho_l^{\mathrm{I}}(\nu') F_l^{\mathrm{I}}(\nu') N_l^{\mathrm{I}}(\nu')}{\nu'^l(\nu' - \nu)} . \tag{7}$$

Let us introduce the notation

$$B_l^{\mathrm{I}}(\nu) = \frac{1}{\pi} \left[\int_{-\infty}^{-1} + \int_{\nu_1}^{\infty} \right] d\nu' \, \frac{\mathrm{Im}\, A_l^{\mathrm{I}}(\nu')}{\nu'^l(\nu' - \nu)} . \tag{8}$$

From the above equations one can easily derive

$$F_l^{\mathrm{I}}(\nu) = B_l^{\mathrm{I}}(\nu) \tag{9}$$

and

$$N_l^{\mathrm{I}}(\nu) = B_l^{\mathrm{I}}(\nu) + \frac{1}{\pi} \int_0^{\nu_1} \frac{B_l^{\mathrm{I}}(\nu') - B_l^{\mathrm{I}}(\nu)}{\nu' - \nu} \, \varrho_l^{\mathrm{I}}(\nu') \nu'^l N_l^{\mathrm{I}}(\nu') . \tag{10}$$

$B_l^{\mathrm{I}}(\nu)$ is analytic in the interval $0 \leqslant \nu < \nu_1$ except for a logarithmic branch point at $\nu = \nu_1$. From (5) one has

$$B_l^{\mathrm{I}}(\nu) = - \frac{g_{\mathrm{I}}}{\pi \nu_1^l \varrho(\nu_1)} \log(\nu_1 - \nu) + V_l^{\mathrm{I}}(\nu) \tag{11}$$

Substituting (11) into the kernel of Eq. (10) one obtains

$$N_l^{\mathrm{I}}(\nu) = B_l^{\mathrm{I}}(\nu) - \frac{g_{\mathrm{I}}}{\pi^2} \int_0^{\nu_1} d\nu' \, \frac{\log [(\nu_1 - \nu')/(\nu_1 - \nu)]}{\nu' - \nu} \, N_l^{\mathrm{I}}(\nu') +$$

$$+ \frac{g_{\mathrm{I}}}{\pi^2} \int_0^{\nu_1} d\nu' H_l^{\mathrm{I}}(\nu, \nu') N_l^{\mathrm{I}}(\nu') , \tag{12}$$

One can easily see that $H_l^{\mathrm{I}}(\nu, \nu')$ is a Fredholm-type kernel.

If we introduce the new variable

$$\varkappa = \frac{\nu_1}{\nu_1 - \nu} \tag{13}$$

and the definitions

$$\frac{\nu_1}{\nu_1 - \nu} N_l^{\mathrm{I}}(\nu) = \mathcal{N}_l^{\mathrm{I}}(\varkappa), \quad \frac{\nu_1}{\nu_1 - \nu} B_l^{\mathrm{I}}(\nu) = \mathcal{B}_l^{\mathrm{I}}(\varkappa)$$

the equation (12) becomes

$$\mathcal{N}_l^{\mathrm{I}}(\varkappa) = \mathcal{N}_l^{0\mathrm{I}}(\varkappa) + \frac{g_{\mathrm{I}}}{\pi^2} \int_1^\infty d\varkappa' \frac{\log \varkappa'/\varkappa}{\varkappa' - \varkappa} \mathcal{N}_l^{\mathrm{I}}(\varkappa') \tag{14}$$

$$\mathcal{N}_l^{0\mathrm{I}}(\varkappa) = \mathcal{B}_l^{\mathrm{I}}(\varkappa) + \frac{g^{\mathrm{I}}}{\pi^2} \int_1^0 d\varkappa' \mathcal{K}_l^{\mathrm{I}}(\varkappa, \varkappa') \mathcal{N}_l^{\mathrm{I}}(\varkappa'). \tag{15}$$

Equation (14) can be diagonalised by a shifted Mehler trnasform [2]. We will, however, employ another method to diagonalise Eq. (14). For this, let us observe that Eq. (14) possesses a kernel depending on a difference. Therefore the well known Wiener-Hopf method [10] can be applied.

Using the techniques described in [10] one obtaines for the resolvent of Eq. (14) the following expression

$$\mathcal{R}(\varkappa, \varkappa'; g) = \frac{1}{2} \int_c d\nu \frac{\mathrm{tgh}\, \pi\nu}{\cosh^2 \pi\nu - g} P_{-\frac{1}{2}+i\nu}(2\varkappa - 1) P_{-\frac{1}{2}i+\nu}(2\varkappa' - 1) +$$

$$+ P_{-\frac{1}{2}+p_0}(2\varkappa - 1) H(\varkappa', g). \tag{16}$$

$$p_0 = (1/\pi) \arccos \sqrt{g_{\mathrm{I}}}, \qquad \mathrm{arc} < \pi/2$$

Obviously, the solution is a multivalued function in the cut coupling plane. By suitably choosing the contour in Eq. (16) (see Fig. 2) one can obtain a solution which is regular

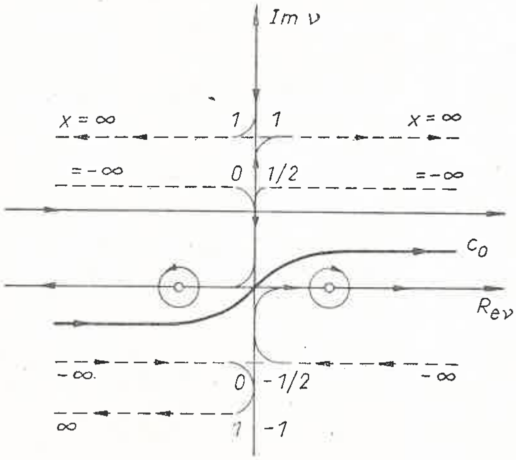

Fig. 2. The integration contours for the resolvent functions

and approaches the kernel in the weak coupling limit. The resolvent can be made to have one branch point only at $g_{\mathrm{I}} = 1$ on the first sheet of the g_{I} — Riemann surface.

Therefore the general solutions is

$$\mathcal{N}_l^{\mathrm{I}}(\varkappa) = \mathcal{N}_l^{0\mathrm{I}}(\varkappa) + g_{\mathrm{I}} \int d\varkappa' \mathcal{R}(\varkappa, \varkappa'; g_{\mathrm{I}}) \mathcal{N}_l^{0\mathrm{I}}(\varkappa') + A(g_{\mathrm{I}}) P_{-\frac{1}{2}p_0}(2\varkappa - 1) \tag{17}$$

where $A(g_{\mathrm{I}})$ is an arbitrary function of g_{I}.

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