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L.S.Z. STRUCTURE OF THE (3,2) SECTOR OF THE LEE MODEL

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In this letter the Lehman-Symanzik-Zimmermann structure of the (3,2) sector of the Lee model is given. It is found that the solutions to two singular integral equations solve the entire sector.

In a recent series of papers [1], [2,] [3] the L.S.Z. formalism, the Tamm-Dancoff method, and the dispersional approach were used as a computational technique for the treatment of the (2,2) sector of the Lee model. Among these methods, the L.S.Z. formalism allows the most symmetrical treatment. With this in mind, the intention of the present letter was to show the L.S.Z. structure of a more complex sector, *viz.*, the (3,2) one. We should mention that the computational procedure is well known [4] and, therefore, do not become absorbed with a detailed treatment.

As it is well known in the L.S.Z. formalism all physical quantities may be expressed by means of the so-called τ-functions which represent vacuum expectation values of various ordered products of the field operators. In the sector considered here there are nine τ-functions, namely

$$\tau_1(t) = \langle 0|V^2(t)N(t)N^\dagger(0)(V^2(0))^\dagger|0\rangle \cdot \Theta(t), \tag{1a}$$

$$\tau_2(t;\omega) = \langle 0|V(t)N^2(t)a_k(t)N^\dagger(0)(V^2(0)^\dagger|0\rangle \cdot X^{-1}(\omega)\Theta(t), \tag{1b}$$

$$\tau_3(t;\omega,\omega') = \langle 0|N^3(t)a_k(t)a_{k'}(t)N^\dagger(0)(V^2 0)^\dagger|0\rangle \cdot X^{-1}(\omega)X^{-1}(\omega')\Theta(t), \tag{1c}$$

$$\tau_4(t;\omega) = \langle 0|V^2(t)N(t)V^\dagger(0)N^2(0)^\dagger a_k^\dagger(0)|0\rangle \cdot X^{-1}(\omega)\Theta(t), \tag{1d}$$

$$\tau_5(t;\omega,\omega') = \langle 0|V^2(t)N(t)(N^3(0))^\dagger a_k^\dagger(0)a_{k'}^\dagger|0)|0\rangle \cdot X^{-1}(\omega)X^{-1}(\omega')\Theta(t), \tag{1e}$$

$$\tau_6(t;\omega,\omega') = \langle 0|V(t)N^2(t)a_k(t)V^\dagger(0)(N^2(0))^\dagger a_{k'}^\dagger(0)|0\rangle \cdot X^{-1}(\omega)X^{-1}(\omega')\Theta(t), \tag{1f}$$

$$\tau_7(t;\omega;\omega',\omega'') = \langle 0|V(t)N^2(t)a_k(t)(N^3(0))^\dagger a_{k'}^\dagger(0)a_{k''}^\dagger(0)|0\rangle \times$$
$$\times X^{-1}(\omega)X^{-1}(\omega')X^{-1}(\omega'')\Theta(t), \tag{1g}$$

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$$\tau_8(t;\omega',\omega'';\omega) = \langle 0|N^3(t)a_{k'}(t)a_{k''}(t)V^\dagger(0)(N^2(0)^\dagger a_k^\dagger(0)|0)\rangle \times$$
$$\times X^{-1}(\omega)X^{-1}(\omega')X^{-1}(\omega'')\Theta(t), \tag{1h}$$
$$\tau_9(t;\omega,\omega';\omega'',\omega''') = \langle 0|N^3(t)a_k(t)a_{k'}(t)(N^3(0)^\dagger a_{k''}^\dagger(0)a_{k'''}^\dagger(0)|0\rangle \times$$
$$\times X^{-1}(\omega)X^{-1}(\omega')X^{-1}(\omega'')X^{-1}(\omega''')\Theta(t), \tag{1i}$$

where $V(t)$, $N(t)$ and $a_k(t)$ are the Heisenberg field operators corresponding to V, N and Θ particles. $\Theta(t)$ is the Heaviside function; $X(\omega) = u(\omega)/(2\omega)1/2$; $u(\omega)$ is a cutoff function.

With the Heisenberg field operators being defined as

$$O_H(t) = e^{iHt}O_s e^{-iHt}, \tag{2}$$

it is not difficult to see that the following equations of motion hold:

$$\left(i\frac{d}{dt} - m_0\right)V(t) = \frac{g}{Z}N(t)A(t), \quad m_0 = m_v + \delta m_v, \tag{3a}$$

$$\left(i\frac{d}{dt} - m_N\right)N(t) = gA^+(t)V(t), \tag{3b}$$

$$\left(i\frac{d}{dt} - \omega\right)a_k(t) = gX(\omega)N^+(t)V(t), \; A(t) \equiv \sum_k X(\omega)a_k(t). \tag{3c}$$

These equations have the formal solutions:

$$V(t_2) = e^{im_v(t_1-t_2)}V(t_1) - i\int_{t_1}^{t_2} e^{im_v(t-t_2)}\left[\delta m_v V(t) + \frac{g}{Z}N(t)A(t)\right]dt, \tag{4a}$$

$$N(t_2) = e^{im_N(t_1-t_2)}N(t_1) - ig\int_{t_1}^{t_2} e^{im_N(t-t_2)}V(t)A^\dagger(t)dt, \tag{4b}$$

$$a_k(t_2) = e^{i\omega(t_1-t_2)}a_k(t_1) - igX(\omega)\int_{t_1}^{t_2} e^{i\omega(t-t_2)}N^\dagger(t)V(t)dt. \tag{4c}$$

If we define the operators "in" and "out" in the sense of the L.S.Z. formalism, then from equations (4*a*, *b*, *c*) we obtain:

$$V\binom{\text{out}}{\text{in}} = V(0) - i\int_0^{\pm\infty} e^{im_v t}\left[\delta m_v V(t) + \frac{g}{Z}N(t)A(t)\right]dt, \tag{5a}$$

$$N\binom{\text{out}}{\text{in}} = N(0) - ig\int_0^{\pm\infty} e^{im_N t}V(t)A^\dagger(t)dt, \tag{5b}$$

$$a_k\binom{\text{out}}{\text{in}} = a_k(0) - igX(\omega)\int_0^{\pm\infty} e^{i\omega t}N^\dagger(t)V(t)dt. \tag{5c}$$

Now, by virtue of Eqs (4.a, b, c) the following set of integral equations may be obtained for the Fourier transforms of the $\imath$-functions:

$$(W - zm_0 - m_N)\hat{\imath}_1(W) = \frac{2!}{Z^2} + \frac{2g}{Z}\sum_k X^2(\omega)\begin{Bmatrix}\hat{\imath}_2(W;\omega)\\ \hat{\imath}_4(W;\omega)\end{Bmatrix}, \tag{6}$$

$$(W - m_0 - 2m_N - \omega)\begin{Bmatrix}\hat{\imath}_2(W;\omega)\\ \hat{\imath}_4(W;\omega)\end{Bmatrix} = 2g\hat{\imath}_1(W) + \frac{g}{Z}\sum_{k'} X^2(\omega')\begin{Bmatrix}\hat{\imath}_3(W;\omega,\omega')\\ \hat{\imath}_5(W;\omega,\omega')\end{Bmatrix}, \tag{7}$$

$$(W - 3m_N - \omega - \omega')\begin{Bmatrix}\hat{\imath}_3(W;\omega,\omega')\\ \hat{\imath}_5(W;\omega,\omega')\end{Bmatrix} = 3g\begin{Bmatrix}\hat{\imath}_2(W;\omega)+\hat{\imath}_2(W;\omega')\\ \hat{\imath}_4(W;\omega)+\hat{\imath}_4(W;\omega')\end{Bmatrix}, \tag{8}$$

$$(W - m_0 - 2m_N - \omega)\begin{Bmatrix}\hat{\imath}_6(W;\omega;\omega')\\ \hat{\imath}_5(W;\omega';\omega)\end{Bmatrix} = \frac{2}{Z}X^{-2}(\omega)\delta_{kk'} + 2g\begin{Bmatrix}\hat{\imath}_4(W;\omega')\\ \hat{\imath}_2(W;\omega')\end{Bmatrix} +$$

$$+ \frac{g}{Z}\sum_{k''} X^2(\omega'')\begin{Bmatrix}\hat{\imath}_8(W;\omega,\omega'';\omega')\\ \hat{\imath}_7(W;\omega';\omega,\omega'')\end{Bmatrix}, \tag{9}$$

$$(W - 3m_N - \omega' - \omega'')\begin{Bmatrix}\hat{\imath}_7(W;\omega;\omega',\omega'')\\ \hat{\imath}_8(W;\omega',\omega'';\omega)\end{Bmatrix} = 3g\begin{Bmatrix}\hat{\imath}_6(W;\omega;\omega')+\hat{\imath}_6(W;\omega;\omega'')\\ \hat{\imath}_6(W;\omega';\omega)+\hat{\imath}_6(W;\omega'';\omega)\end{Bmatrix}, \tag{10}$$

$$(W - 3m_N - \omega - \omega')\begin{Bmatrix}\hat{\imath}_9(W;\omega,\omega';\omega'',\omega''')\\ \hat{\imath}_9(W;\omega'',\omega''';\omega,\omega')\end{Bmatrix} = 6X^{-2}(\omega)X^{-2}(\omega')[\delta_{kk''}\delta_{k'k'''} + \delta_{kk'''}\delta_{k'k''}] +$$

$$+3g\begin{Bmatrix}\hat{\imath}_7(W;\omega';\omega'',\omega''')+\hat{\imath}_7(W;\omega;\omega'',\omega''')\\ \hat{\imath}_8(W;\omega'',\omega''';\omega')+\hat{\imath}_8(W;\omega'',\omega''';\omega)\end{Bmatrix}. \tag{11}$$

From Eqs (6), (7) and (8) we see that in order to determine $\hat{\imath}_1$, $\hat{\imath}_2$, $\hat{\imath}_3$, $\hat{\imath}_4$ and $\hat{\imath}_5$ it is necessary to solve a singular integral equation. In order to determine $\hat{\imath}_6$, $\hat{\imath}_7$, $\hat{\imath}_8$ and $\hat{\imath}_9$ from Eqs (9), (10) and (11) another singular integral equation must be solved. The solution may be obtained by reducing these two equations to the corresponding inhomogenous Hilbert problems [5]. We intend to present this in detail in a forthcoming paper.

Let us now proced to the study of scattering processes. Because of the static character of N and V particles, the only processes allowed here are:

$$3N+2\Theta \to 3N+2\Theta,$$

$$V+2N+\Theta \to V+2N+\Theta,$$

$$3N+2\Theta \to V+2N+\Theta,$$

i.e. two elastic scattering processes and a production process. The corresponding S-matrix elements may be written as follows:

$$S_1 = \frac{1}{12}\langle 0|N^3\,(\text{out})\,a_{k_1}\,(\text{out})\,a_{k_2}\,(\text{out})\,a^{\dagger}_{k_1'}\,(\text{in})\,a^{\dagger}_{k_2'}\,(\text{in})\,(N^3\,(\text{in})^{\dagger}|0\rangle, \tag{12}$$

$$S_2 = \frac{1}{2}\langle 0|V\,(\text{out})\,N^2\,(\text{out})\,a_{k_1}(\text{out})\,a^{\dagger}_{k_1'}\,(\text{in})\,(N^2\,(\text{in})^{\dagger}V^{\dagger}\,(\text{in})|0\rangle, \tag{13}$$

$$S_3 = \frac{1}{2\sqrt{6}}\langle 0|V\,(\text{out})\,N^2\,(\text{out})\,a_{k_1}(\text{out})\,(N^3\,(\text{in})^{\dagger}a^{\dagger}_{k_1'}\,(\text{in})\,a^{\dagger}_{k_2'}(\text{in})|0\rangle. \tag{14}$$

By means of the reduction technique, Eqs (3) and (5) lead, after some algebraic manipulations to the following expressions for the S-matrix elements:

$$S_1 = \frac{1}{2}(\delta_{k_1'k_1'}\delta_{k_2k_2'} + \delta_{k_1k_2'}\delta_{k_2k_1'}) - \frac{3}{2}\pi i\delta(\omega_1+\omega_2-\omega_1'-\omega_2')g^2\times \tag{15}$$

$$\times X(\omega_1)X(\omega_2)X(\omega_1')X(\omega_2')[\hat{\imath}_6(W;\omega_1;\omega_1')+\hat{\imath}_6(W;\omega_2;\omega_2')+ \tag{15}$$

$$+\hat{\imath}_6(W;\omega_1;\omega_2')+\hat{\imath}_6(W;\omega_2;\omega_1')]_{W=3m_N+\omega_1+\omega_2},$$

$$S_2 = \delta_{k_1k_1'} - \pi i g^2 X^2(\omega_1)\delta(\omega_1-\omega_1')\Bigg[4\hat{\imath}_1(W)+\frac{2}{Z}\sum_k X^2(\omega)\hat{\imath}_3(W;\omega,\omega_1)+$$

$$+\frac{2}{Z}\sum_k X^2(\omega)\hat{\imath}_5(W;\omega,\omega_1)+\frac{1}{Z^2}\sum_k\sum_{k'}X^2(\omega)X^2(\omega')\hat{\imath}_9(W;\omega,\omega_1;\omega',\omega_1')\Bigg]_{W=m_0+2m_N+\omega_1}, \tag{16}$$

$$S_3 = -\pi i g\sqrt{\frac{3}{2}}\cdot\delta(m_N+\omega_1'+\omega_2'-\omega_1-m_0)\Bigg\{\frac{1}{Z}[2(X(\omega_1')\delta_{k_1k_2'}+X(\omega_2')\delta_{k_1k_1'})+$$

$$+gX(\omega_1)X(\omega_1')X(\omega_2')\sum_k X^2(\omega)(\hat{\imath}_8(W;\omega_1,\omega;\omega_2')+\hat{\imath}_8(W;\omega_1,\omega;\omega_1'))]+$$

$$+2gX(\omega_1')X(\omega_2')X(\omega_1)(\hat{\imath}_4(W;\omega_2')+\hat{\imath}_4(W;\omega_1'))\Bigg\}_{W=2m_N+m_0+\omega_1}. \tag{17}$$

Finally, we present a short discussion on the bound states. If the coupling constant g is sufficiently strong, then, in the sector considered in this note, there may exist a bound state. The general expression for such a state is

$$|B\rangle = \frac{1}{\sqrt{12}}\sum_k\sum_{k'}\alpha(k,k')(N^3(0)^\dagger a_k^\dagger(0)a_{k'}^\dagger(0)|0\rangle+\frac{1}{\sqrt{2}}\sum_k\beta(k)V^\dagger(0)(N^2(0))^\dagger a_k^\dagger(0)|0\rangle+$$

$$+\frac{1}{\sqrt{2}}\gamma N^\dagger(0)(V^2(0))^\dagger|0\rangle, \tag{18}$$

where

$$\alpha(k,k') = \frac{1}{2\sqrt{3}}\langle 0|N^3(0)a_k(0)a_{k'}(0)|B\rangle, \tag{19}$$

$$\beta(k) = \frac{Z}{\sqrt{2}}\langle 0|V(0)N^2(0)a_k(0)|B\rangle, \tag{20}$$

$$\gamma = \frac{Z^2}{\sqrt{2}}\langle 0|V^2(0)N(0)|B\rangle, \tag{21}$$

are to be determined. Some simple manipulations yield for them the equations:

$$\alpha(k,k') = \frac{g\sqrt{3}}{Z\sqrt{2}}\cdot\frac{X(\omega)\beta(k')+X(\omega')\beta(k)}{E_B-3m_N-\omega-\omega'}, \tag{22}$$

$$\beta(k) = \frac{g}{E_B - m_0 - 2m_N - \omega} \cdot \left(\sqrt{6} \sum_{k'} X(\omega')\alpha(k, k') + \frac{2X(\omega)}{Z} \gamma \right), \tag{23}$$

$$\gamma = \frac{2g}{E_B - 2m_0 - m_N} \cdot \sum_k X(\omega)\beta(k), \tag{24}$$

where E_B is the bound state energy. Its location may be found by solving the equation

$$[\hat{\imath}_1(E_B)]^{-1} = 0. \tag{25}$$

According to Eqs (22) and (24),

$$|B\rangle = \frac{g}{2\sqrt{2}Z} \sum_k \sum_{k'} \frac{X(\omega)\beta(k') + X(\omega')\beta(k)}{E_B - 3m_N - \omega - \omega'} (N^3(0))^\dagger a_k^\dagger(0) a_{k'}^\dagger(0)|0\rangle +$$

$$+ \frac{1}{\sqrt{2}} \sum_k \beta(k) V^\dagger(0)(N^2(0)^\dagger a_k^\dagger(0)|0\rangle + g\sqrt{2} \frac{\sum_k X(\omega)\beta(k)}{E_B - 2m_0 - m_N} N^\dagger(0)(V^2(0))^\dagger|0\rangle, \tag{26}$$

and the associated creation operator is

$$B^\dagger = \frac{g}{2\sqrt{2}Z} \sum_k \sum_{k'} \frac{X(\omega)\beta(k') + X(\omega')\beta(k)}{E_B - 3m_N - \omega - \omega'} (N^3(0))^\dagger a_k^\dagger(0) a_{k'}^\dagger(0) +$$

$$+ \frac{1}{\sqrt{2}} \sum_k \beta(k) V^\dagger(0)(N^2(0))^\dagger a_k^\dagger(0) + g\sqrt{2} \frac{\sum_k X(\omega)\beta(k)}{E_B - 2m_0 - m_N} N^\dagger(0)(V^2(0))^\dagger. \tag{27}$$

The yet undetermined function $\beta(k)$ satisfies the following integral equation:

$$\beta(k) \cdot \left[Z(E_B - m_0 - 2m_N - \omega) - 3g^2 \sum_{k'} \frac{X^2(\omega')}{E_B - 3m_N - \omega - \omega'} \right]$$

$$= 2g\gamma X(\omega) + 3g^2 X(\omega) \sum_{k'} \frac{X(\omega')\beta(k')}{E_B - 3m_N - \omega - \omega'}. \tag{28}$$

It should be noted that this equation has the same structure as the integral equation which determines $\hat{\imath}_2$ and $\hat{\imath}_4$ in terms of $\hat{\imath}_1$. This is an expected result, since the coefficients α, β and γ of the bound state are expressible in terms of the first five $\hat{\imath}$-functions.

The conclusion is that an exhaustive solution of the (3,2) sector consist in solving two singular integral equations or, equivalently, the corresponding Hilbert problems.

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