2nd Symposium on New Trends in Nuclear and Medical Physics, Poland, September 24–26, 2025

The Weak Decay Constant of Positronium

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The positronium, as the lightest, purely leptonic bound state, provides an ideal testing ground for the quantum field-theoretical description of composite systems. While its electromagnetic annihilation is well understood as the dominant decay channel, the weak interaction sector of the positronium is strongly suppressed. In this work, we extend the composite quantum field-theoretical framework developed earlier for the two-photon decay and introduce the concept of a weak decay constant for para-positronium. This constant, defined in analogy to those of pseudoscalar mesons, quantifies the coupling of the positronium field to the weak axial current and serves as a measure of its internal structure in the electroweak domain. Its numerical value $f_{\mathrm{P}} \simeq \alpha^{3/2} m_e/(2\sqrt{\pi}) = 89.8593$ eV is, as expected, small. Nevertheless, the resulting expression and the related comparison with quarkonium systems allows us to determine the vertex function linking the positronium to its own constituents.

topics: Jagiellonian positron emission tomography (J-PET), positronium, weak decay constant

1. Introduction

The positronium, composed of an electron and a positron, represents the simplest neutral bound state described by quantum electrodynamics (QED) [1–6]. Its theoretical and experimental understanding is remarkable, reaching levels of precision far beyond those attainable for hadronic systems [1, 5–8]. In recent years, the positronium has also attracted renewed attention in applied research, particularly in medical physics. The ability to form positronium atoms in biological and porous materials enables novel imaging techniques, and studies performed by the Jagiellonian positron emission tomography (J-PET) collaboration have demonstrated its potential as a diagnostic probe in positron emission tomography and related fields [9–14]. Nevertheless, positronium remains of interest not only as a test case of QED but also as a conceptual link to composite particles in quantum chromodynamics (QCD). In particular, the para-positronium (p-Ps) is a pseudoscalar system ($J^{\mathrm{PC}} = 0^{-+}$) and, in this respect, closely resembles the neutral pion, the lightest pseudoscalar meson in QCD, as well as to the lightest charm–anticharm state, the η_c meson. However, this analogy concerns only the quantum numbers and the composite-field structure. While para-positronium can be described within the framework of nonrelativistic quantum electrodynamics (NRQED), heavy quarkonia such as η_c are treated within the corresponding nonrelativistic quantum chromodynamics (NRQCD), whereas the pion requires a fully relativistic chiral description.

This analogy leads to a discussion of the weak decay constant, widely used in hadron physics, for positronium states. While the weak decay of the positronium is very small, the research exercise we have undertaken is nevertheless useful because it helps to understand how to model the coupling of the positronium state to its own constituents, more specifically, the vertex function. As expected, this vertex function is proportional to the wave function, but a comparison with known results about quarkonium states allows us to determine this connection.

More specifically, a composite quantum field teoretical (QFT) model [15, 16] has been successfully applied to the two-photon decay of para-positronium, using a triangle-loop diagram with virtual electrons and a nonlocal vertex function linked to the bound-state wave function [17, 18]. Such an approach reproduces the well-known QED decay width quite well and provides a natural connection to similar processes in mesons. Building on this formalism, one can proceed one step further and couple the positronium field to the weak current.

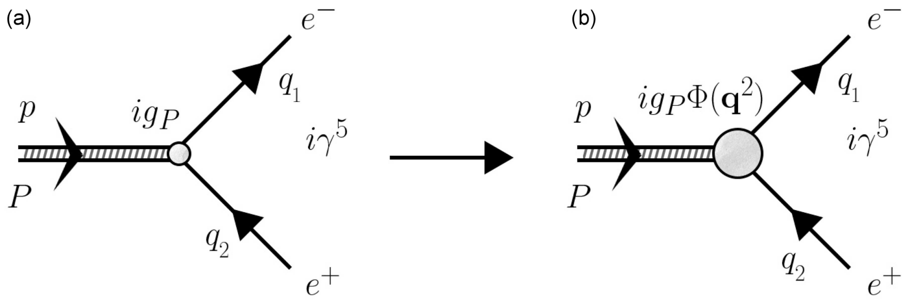

Fig. 1. Transition from the local (a) to the nonlocal (b) interaction vertex involving the positronium field and its electron–positron constituents.

2. Weak decay constants in QCD — brief recall

In QCD, the pseudoscalar decay constant $f_{\rm P}^Q$ (where $Q = \bar{q}q$ stands for a generic pseudoscalar quarkonium state) plays a central role. It is defined through the matrix element

$$\langle 0 \mid J_w^\mu \mid P(p)\rangle = {\rm i}\, f_{\rm P}^Q\, p^\mu , \tag{1}$$

which links the pseudoscalar meson state $|P(p)\rangle$ to the weak axial-vector current, see e.g. [19–21], as

$$J_w^\mu = \bar{\psi}_q \gamma^\mu\, (1-\gamma_5)\, \psi_q . \tag{2}$$

Here, p^μ denotes the four momentum of the meson, ψ_q represents the quark field, and γ^μ and γ_5 are the usual Dirac matrices. The constant $f_{\rm P}^Q$ characterizes the strength of the coupling between the meson and the weak current; it determines weak decay rates and encodes the overlap of the bound-state wave function with the vacuum.

In contrast, the para-positronium — being a purely leptonic system — is governed almost exclusively by electromagnetic interactions, and its decays are typically described within the NRQED framework [1, 5–7]. Yet, from a theoretical standpoint, the positronium can also be regarded as a composite pseudoscalar field. We thus introduce the weak positronium decay constant, denoted in the following as $f_{\rm P}$, by replacing the quark current with the corresponding leptonic one, $J_w^\mu = \bar{\psi}_e\, \gamma^\mu\, (1-\gamma_5)\, \psi_e$ [22–24].

3. Composite QFT model for the positronium

To describe the weak decay constant for the positronium system, we employ the same composite QFT framework that has been successfully used to study its electromagnetic annihilation channels [15, 16, 22, 23, 25–27]. In this approach, the positronium is treated as an effective pseudoscalar field $P(x)$ representing the bound state of an electron and a positron. In the local limit, its interaction with the constituent fermions is described by the effective Lagrangian

$$\mathcal{L}_{\rm P} = g_{\rm P}\, P(x)\, \bar{\psi}_e(x)\; {\rm i}\; \gamma_5\, \psi_e(x) \tag{3}$$

where $g_{\rm P}$ is the coupling constant and $\psi_e(x)$ denotes the electron field. The pseudoscalar nature of the positronium state is reflected in the factor ${\rm i}\gamma_5$, which ensures the correct parity and charge-conjugation quantum numbers ($J^{\rm PC} = 0^{-+}$). Due to the composite nature of positronium, the Lagrangian is promoted to a nonlocal form [17] — in analogy with quark–antiquark systems [25–27]. In general, the vertex is modulated by a spacial vertex function $\tilde{\Phi}(x_1, x_2, x_3)$, which describes the smeared, nonlocal interaction of the constituents (schematically $\mathcal{L}_{\rm P} \to \mathcal{L}_{\rm P}^{\rm nonlocal}$). In the rest frame of the positronium, one has $p = (\sqrt{s}, 0)$, and thus

$${\rm i}g_{\rm P} \to {\rm i}\; g_{\rm P}\; \Phi(\boldsymbol{q}^2) \tag{4}$$

with $\boldsymbol{q} = (\boldsymbol{q}_1 - \boldsymbol{q}_2)/2 = (0, \boldsymbol{q})$ and where $\Phi(\boldsymbol{q}^2)$ emerges as the Fourier transform of the vertex function. See Fig. 1 for a pictorial representation of the transition from the local to the nonlocal case.

This result can also be achieved in a covariant manner, see details in [23]. The vertex function $\Phi(\boldsymbol{q}^2)$ must be proportional to the positronium momentum wave function $A(\boldsymbol{q}^2)$, but the question that we aim to discuss here concerns the precise relation between the mentioned functions. We investigate this relation by using the weak decay constant.

The coupling constant $g_{\rm P}$ in (3) is not a free parameter, but is calculated using the compositeness condition [15]. For this reason, the electron–positron loop is calculated using Feynman diagrams (see Fig. 2), which leads to (in the rest frame of P)

$$\Sigma(s) = {\rm i} \int \frac{{\rm d}^4 q}{(2\pi)^4} \frac{{\rm Tr}\left[{\rm i}\gamma^5\, (\not{q}_1 + m_e)\, {\rm i}\gamma^5\, (\not{q}_2 + m_e)\right]}{(\boldsymbol{q}_1^2 - m_e^2 + {\rm i}\varepsilon)\, (\boldsymbol{q}_2^2 - m_e^2 + {\rm i}\varepsilon)}\, \Phi^2\left(\boldsymbol{q}^2\right) . \tag{5}$$

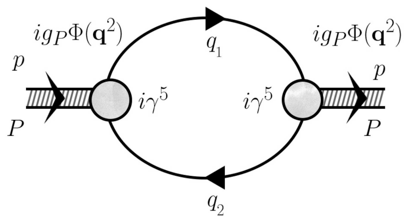

Fig. 2. Positronium self-energy diagram with the vertex function $\Phi\left(\boldsymbol{q}^2\right)$.

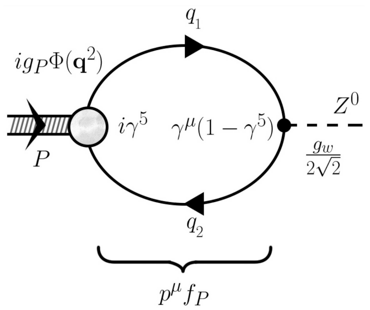

Fig. 3. Weak transition $P \to Z^0$ defining the constant f_{P}.

Taking into account that

$$\mathrm{Tr}\left[\mathrm{i}\gamma^5\left(\not{q}_1+m_e\right)\mathrm{i}\gamma^5\left(\not{q}_2+m_e\right)\right]=4\Big(\left(\boldsymbol{q}_1\cdot\boldsymbol{q}_2\right)-m_e^2\Big), \tag{6}$$

and performing the integral over q^0, the function $\Sigma(s)$ takes the compact form

$$\Sigma(s)=\int\frac{\mathrm{d}^3q}{(2\pi)^3}\,\frac{8\sqrt{\boldsymbol{q}^2+m_e^2}}{\left[4\left(\boldsymbol{q}^2+m_e^2\right)-s\right]}\,\Phi^2\left(\boldsymbol{q}^2\right). \tag{7}$$

Hence, its first derivative is

$$\Sigma'(s)=\int\frac{\mathrm{d}^3q}{(2\pi)^3}\,\frac{8\sqrt{\boldsymbol{q}^2+m_e^2}}{\left[4\left(\boldsymbol{q}^2+m_e^2\right)-s\right]^2}\,\Phi^2\left(\boldsymbol{q}^2\right), \tag{8}$$

which allows to determine g_{P} as

$$g_{\mathrm{P}}=\frac{1}{\sqrt{\Sigma'(s{=}m_{\mathrm{P}}^2)}}, \tag{9}$$

where $m_{\mathrm{P}}=2m_e-m_e\,\alpha^2/4$ is the mass of the para-positronium (at order α^2).

4. Weak decay constant of the positronium

The weak decay constant of the positronium describes the transition of P into the weak boson Z^0. To this end, we introduce the weak interaction

$$\mathcal{L}_w=\frac{g_w}{2\sqrt{2}}\,Z_\mu^0\,\Big[\bar{\psi}_e\gamma^\mu(1{-}\gamma^5)\psi_e+\bar{\psi}_{\nu_e}\gamma^\mu(1{-}\gamma^5)\psi_{\nu_e} +\ldots\Big] \tag{10}$$

where g_w is the weak coupling constant. Upon considering the Lagrangian $\mathcal{L}_{\mathrm{P}}^{\mathrm{nonlocal}}+\mathcal{L}_w$, the weak decay constant of the field P emerges as (see Fig. 3)

$$p^\mu\,f_{\mathrm{P}}=g_{\mathrm{P}}\left[\int\frac{\mathrm{d}^4q}{(2\pi)^4}\,\frac{\mathrm{Tr}\left[\mathrm{i}\gamma^5(\not{q}_1+m_e)\,\gamma^\mu\,(1-\gamma^5)(\not{q}_2+m_e)\right]}{(\boldsymbol{q}_1^2-m_e^2+\mathrm{i}\varepsilon)\,(q_2^2-m_e^2+\mathrm{i}\varepsilon)}\right]\Phi\left(\boldsymbol{q}^2\right). \tag{11}$$

It is interesting to note that the same formalism, when the weak vertex $\gamma^\mu(1{-}\gamma_5)$ is replaced by the electromagnetic one $e\gamma^\mu(\not{q}_3{-}m_e)^{-1}\,e\gamma^\nu$, reproduces the well-known amplitude for the two-photon decay of the para-positronium. The weak and electromagnetic channels thus arise from a common field-theoretical structure, differing only in the nature of the external current. This correspondence reinforces the interpretation of the positronium as a convenient theoretical laboratory for exploring the interplay between the composite dynamics and the gauge interactions.

After calculating the traces and the integral over q^0, the weak decay constant of the positronium is

$$f_{\mathrm{P}}=4g_{\mathrm{P}}m_e\int\frac{\mathrm{d}^3q}{(2\pi)^3}\,\frac{\Phi\left(\boldsymbol{q}^2\right)}{\sqrt{\boldsymbol{q}^2+m_e^2}\left[4\left(\boldsymbol{q}^2+m_e^2\right)-m_{\mathrm{P}}^2\right]}. \tag{12}$$

This is the main analytical result of this work. In turn, the effective coupling of the positronium to the weak neutral gauge boson reads $\mathcal{L}_{eff}=\frac{g_w}{2\sqrt{2}}\,f_{\mathrm{P}}\partial_\mu(PZ^0)$, where g_w is the weak coupling.

Next, let us consider a single quark–antiquark state, such as the pseudoscalar meson η_c. The weak decay constant for this state is expressed as [19–21, 28]

$$f_{\mathrm{P}}^{(Q=\eta_c)}=f_{\eta_c}\sim\int\frac{\mathrm{d}^3q}{(2\pi)^3}\,\frac{A_{\eta_c}\left(\boldsymbol{q}^2\right)}{\sqrt{\boldsymbol{q}^2+m_c^2}}, \tag{13}$$

where $A_{\eta_c}(\boldsymbol{q}^2)$ is the wave function of η_c and m_c the (constituent) charm mass.

A comparison implies the following choice for the positronium vertex function $\Phi(\boldsymbol{q}^2)$

$$\Phi\left(\boldsymbol{q}^2\right)=\frac{A(\boldsymbol{q}^2)}{A(0)}\,\frac{4\left(\boldsymbol{q}^2+m_e^2\right)-m_{\mathrm{P}}^2}{4m_e^2-m_{\mathrm{P}}^2}, \tag{14}$$

where the normalization $\Phi(0)=1$ has been chosen. In turn,

$$\Phi\left(\boldsymbol{q}^2\right)=\frac{1}{\left(1+\frac{\boldsymbol{q}^2}{\gamma^2}\right)^2}\left(\frac{\gamma^2+\boldsymbol{q}^2}{\gamma^2}\right)=\frac{\gamma^2}{\gamma^2+\boldsymbol{q}^2}, \tag{15}$$

$$\gamma^2=m_e^2-\frac{m_{\mathrm{P}}^2}{4}. \tag{16}$$

Note, this result agrees with the Ansatz of [29]. Numerically, the value of the positronium weak decay constant is

$$f_{\mathrm{P}} = 89.4432 \text{ eV}. \tag{17}$$

This result should be compared with $f_\pi \simeq 130$ MeV for the pion [30], or even with the value 300 MeV for η_c [31].

It is also instructive to study the non-relativistic limit. First, note that

$$f_{\mathrm{P}} \simeq \frac{g_{\mathrm{P}}}{\gamma^2 A(0)} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} A(\boldsymbol{q}^2) = \frac{g_{\mathrm{P}}}{\gamma^2 A(0)} \psi(0). \tag{18}$$

On the other hand, we take into account that the derivative of the loop reads

$$\Sigma'(m_{\mathrm{P}}^2) = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{8\sqrt{\boldsymbol{q}^2+m_e^2}}{\left(4\left(\boldsymbol{q}^2+m_e^2\right)-m_{\mathrm{P}}^2\right)^2} \frac{A^2(\boldsymbol{q}^2)}{A^2(0)} \frac{\left(4\left(\boldsymbol{q}^2+m_e^2\right)-m_{\mathrm{P}}^2\right)^2}{\left(4m_e^2-m_{\mathrm{P}}^2\right)^2} = \frac{1}{\left(4m_e^2-m_{\mathrm{P}}^2\right)^2 A^2(0)}$$
$$\times \int \frac{\mathrm{d}^3 q}{(2\pi)^3} 8\left(\sqrt{\boldsymbol{q}^2+m_e^2}\right) A^2(\boldsymbol{q}^2) \simeq \frac{8m_e}{(4m_e^2-m_{\mathrm{P}}^2)^2 A^2(0)} \int \frac{d^3 q}{(2\pi)^3} A^2(\boldsymbol{q}^2) = \frac{8m_e}{16\gamma^4 A^2(0)} \tag{19}$$

where the usual normalization $\int \mathrm{d}^3 q \, A^2(\boldsymbol{q}^2)/(2\pi)^3 = 1$ has been used. Using (9),

$$g_{\mathrm{P}} = \frac{\sqrt{2}\gamma^2 A(0)}{\sqrt{m_e}} = \frac{2\gamma^2 A(0)}{\sqrt{m_{\mathrm{P}}}}, \tag{20}$$

which all leads to the non-relativistic result

$$f_{\mathrm{P}} \simeq \frac{2}{\sqrt{m_{\mathrm{P}}}} \psi(0). \tag{21}$$

This expression is the Van Royen–Weisskopf formula [32] for the positronium. Now, for a quark–antiquark state, (21) is as follows

$$f_{\mathrm{P}}^Q \simeq \frac{2\sqrt{N_c}}{\sqrt{m_Q}} \Psi_Q(0), \tag{22}$$

where $N_c = 3$ is the number of color, m_Q is the quarkonium mass, and $\Psi_Q(0)$ is the quarkonium wave function evaluated at the origin (for the large-N_c scaling, see the lectures [24]).

For the positronium case, the wave function is (see e.g. [1])

$$\psi(r) = \frac{\exp(-\frac{r}{a_{\mathrm{P}}})}{\sqrt{\pi}\, a_{\mathrm{P}}^{3/2}} \quad \text{with} \quad a_{\mathrm{P}} = \frac{2}{m_e\, \alpha}, \tag{23}$$

leading to

$$f_{\mathrm{P}} \simeq \frac{m_e}{2\sqrt{\pi}} \alpha^{3/2} = 89.8593 \ \ \text{eV} \tag{24}$$

— the value thus only slightly larger than the one previously quoted. This is expected, since the non-relativistic limit is very accurate for positronium.

5. Conclusions

In this work, the weak decay constant of positronium has been defined and evaluated within the framework of a composite quantum field theoretical model. The positronium field, treated as a pseudoscalar bound state of the electron and the positron, was coupled to the weak axial current in analogy with the treatment of pseudoscalar mesons in QCD. The resulting expression for f_{P} is strongly suppressed and scales as $f_{\mathrm{P}} \simeq \alpha^{3/2} m_e/(2\sqrt{\pi})$.

Although such a quantity is strongly suppressed in positronium phenomenology, it provides a consistent formal bridge between lepton and hadron composite systems. In turn, the momentum vertex function is connected to the wave function as $\phi(\boldsymbol{q}^2) \sim A(\boldsymbol{q})(\boldsymbol{q}^2 + m_e^2 - m_{\mathrm{P}}^2)$. This result may be used in future studies linking positronia and quarkonia.

The approach developed here may also serve also as a starting point for further studies of parity-violating effects and excited positronium states within the same QFT framework. Moreover, even if extremely small, the process may offer a transition to particles beyond the Standard Model of particles, such as the putative state $X(17)$ [33]. The effective interaction is of the type $c_X f_{\mathrm{P}} \partial_\mu P X^\mu$, where the dimensionless constant c_X describes the coupling of $X(17)$ to electrons. Notably, this interaction implies that $X(17)$ is an axial-vector state and resembles the mixing $a_1(1260)$–π in effective hadronic models, e.g. [34, 35]. Interestingly, the composite model may also be used to test the case in which the $X(17)$ is a pseudoscalar object. In such a case, the effective interaction is $c_X \lambda_{PX} PX$, where λ_{PX} is the PX transition loop. Both models are worth studying in the future.

Acknowledgments

This work was supported by the Minister of Science (Poland) under the 'Regional Excellence Initiative' program (project no.: RID/SP/0015/2024/01, with sub-projects RID/2024/LIDER/08 and RID/2025/LIDER/02).

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