Effect of Loss/Gain of Energy on Standing Oscillations in a Planar Resonator

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The dynamics of small-magnitude acoustic perturbations in a planar resonator is considered. Fluid flow is affected by a heating–cooling function, which may disturb its adiabaticity. This concerns open flows with an external inflow of energy and flows with relaxation of thermodynamic processes, such as exothermic chemical reaction and excitation of vibrational degrees of freedom of a molecule. These processes make the flow acoustically active under some conditions. The heating–cooling function is supposed to depend on the thermodynamic parameters of the fluid. The dynamics of perturbations in the volume of a resonator is described analytically by the separation of variables in the wave equation with an account of proper boundary conditions. Some particular cases of the heating–cooling function allow us to describe a considerable deviation of the adiabaticity of the flow analytically.

topics: standing waves, acoustically active gas, resonator, linear propagation of sound

1. Introduction

It is well understood that the boundaries of a vessel over which sound spreads have a key impact on the distribution of acoustic perturbations in its volume. The spectrum of perturbations becomes discrete in order to match boundary conditions. Physically founded conditions at the boundaries of a resonator determine the magnitude and phase of the reflected waves. The attention has mostly been paid to the dynamics of perturbations in resonators that contain Newtonian fluids [1–4]. Nonlinearity is important for large-magnitude perturbations; it has been discussed, e.g., by Kaner, Rudenko, and Khokhlov [5] and Rudenko and Soluyan [6]. Waves with discontinuities in the resonators filled with Newtonian fluids have been considered by Keller [2] and Biwa and Yazaki [4]. In recent decades, the interest in acoustically active flows has been constantly growing [7–9]. This is connected with the expansion of the technical applications of open flows, the quick development of non-equilibrium molecular physics due to the laser revolution, and the astrophysical plasma applications [10]. Finite-magnitude perturbations in standing waves in acoustically active media with losses due to thermal conduction have been studied by Kumar, Nakariakov, and Moon [11] using the method of successive approximations. Special attention is paid to the magnetohydrodynamic (MHD) oscillations in the solar atmosphere since they play a key role in plasma heating and because of the diagnostic potential of these waves [12]. The standing waves form in the coronal loops caused by the structuring of plasma across the magnetic field with footpoints as effective reflectors. Standing longitudinal oscillations are readily excited by heat deposition or localized variation of pressure in a volume of the coronal loop. They are experimentally detected with confidence by the Doppler spectrometry [13]. The example considered by Ofman and Wang [14] refers to the coronal loops of 400 Mm length. About half of the oscillations are associated with flares. As usual, necessary but not understood physical processes, such as coronal heating, are included in the model in the general form that could be determined empirically from observation [15]. The radiative loss, which is mainly controlled by the composition of a plasma, also contributes to the heating–cooling function. The conclusions, starting from the first theoretical studies (e.g., by Kumar, Nakariakov, and Moon [11]), are that standing oscillations of slow MHD waves are highly sensitive to the loss/gain mechanism and reveal dramatically variable evolution depending on the kind of heating–cooling function.

Apart from standing oscillations of slow MHD in the coronal loops, there are a number of applications for acoustic oscillations with unspecified inflow of energy and/or internal relaxation processes. The total field of small-magnitude perturbations in the resonator represents a sum of incident and reflected waves that satisfy the boundary conditions. In the flow with normal dispersion and attenuation, magnitudes of acoustic perturbations in the resonator get smaller, but they increase in the acoustically active regime. In this study, we consider dynamics of small magnitude perturbations of pressure and velocity in the bounded planar flow affected by some heating–cooling function exclusively without account of irreversible damping due to viscosity and thermal conduction. In the case of a damped field due to loss in energy determined by a loss/gain function, attenuation differs from the Newtonian one (Sect. 2). This allows the application of the method of separation of variables to the total field in the volume of a resonator. The analytical description of standing waves is approximate for small deviation from adiabaticity and is exact (for weak, medium, or strong deviation) if the heating–cooling function depends only on pressure (Sect. 3). Standing waves in the Newtonian flow are discussed shortly in Sect. 4.

2. Small-magnitude perturbations

The starting point of studies are conservation equations of mass (the continuity equation), momentum (the second Newtonian law in the absence of external forces tangential to the surface of the fluid element), and energy balance of the fluid element in the differential form

$$\frac{\mathrm{D}\rho}{\mathrm{D}t}+\rho\boldsymbol{\nabla}\cdot\boldsymbol{U}=0,$$

$$\rho\frac{\mathrm{D}\boldsymbol{U}}{\mathrm{D}t}=-\nabla P,$$

$$T\frac{\mathrm{D}s}{\mathrm{D}t}=\frac{\mathrm{D}e}{\mathrm{D}t}+P\,\frac{\mathrm{D}\rho^{-1}}{\mathrm{D}t}=\frac{\delta Q}{\mathrm{D}t}=L(P,\rho), \qquad (1)$$

where $P,\rho,\boldsymbol{U},e,s,T$ are hydrostatic pressure and density of the gas, its velocity, internal energy (energy per unit mass of gas), entropy per unit mass, and temperature of a fluid element, respectively. The Del operator is denoted as ∇. The material derivative $\frac{\mathrm{D}}{\mathrm{D}t}$ equals the sum of the partial derivative with respect to time and the convective term, i.e.,

$$\frac{\mathrm{D}}{\mathrm{D}t}=\frac{\partial}{\partial t}+(\boldsymbol{U}\cdot\nabla). \qquad (2)$$

The energy balance takes into account some heating–cooling function $L(P,\rho)$, which reflects the deviation of adiabaticity of the flow. Actually, it represents the first law of thermodynamics, which confirms that the sum of variations in internal energy and work of a fluid element along its trajectory in time $\mathrm{D}t$ equals incoming energy δQ over this temporal domain.

2.1. Perturbations in an ideal gas

The internal energy of an ideal gas depends only on its temperature. Taking into account the thermodynamic law for an ideal gas, it equals [6, 9]

$$e=C_VT=\frac{RT}{(\gamma-1)\mu}=\frac{P}{(\gamma-1)\rho}, \qquad (3)$$

where R and γ denote the universal gas constant and the ratio of specific heats under constant pressure and constant gas density, i.e., $\gamma=C_P/C_V$. The molar mass of the gas is denoted by μ. The approximation of an ideal gas is valid within reasonable accuracy for the majority of gases over a considerable parameter range around standard temperature and pressure. In general, a gas behaves as an ideal gas at higher temperature and lower pressure. The model of an ideal gas does not describe phase transitions, and it fails for heavy gases and for gases with strong intermolecular interactions, such as water vapor [9]. The equation of state for an ideal gas is almost always used in astrophysical applications related to weakly coupled plasma (solar corona and atmosphere, interstellar plasma, etc.). Using the energy balance and the continuity equation, after some algebraic calculations, we arrive at the following dynamic equation

$$\frac{\mathrm{D}e}{\mathrm{D}t}+P\,\frac{\mathrm{D}\rho^{-1}}{\mathrm{D}t}=\frac{1}{(\gamma-1)\rho}\left(\frac{\mathrm{D}P}{\mathrm{D}t}-\frac{\gamma P}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}\right)=$$
$$\frac{1}{(\gamma-1)\rho}\left(\frac{\partial P}{\partial t}+(\boldsymbol{U}\cdot\nabla)P+\gamma P(\nabla\cdot\boldsymbol{U})\right)=L(P,\rho). \qquad (4)$$

The first two equations of (1) together with (4) form the initial system for unknown fields, $\rho,\boldsymbol{U},P$. In planar one-dimensional flow, the velocity field has one component U, and the number of dynamic equations is reduced from five to three. The flow in terms of perturbations $\rho'=\rho-\rho_0, U, P'=P-P_0$ is described by the leading-order equations valid up to quadratically nonlinear terms

$$\frac{\partial\rho'}{\partial t}+\rho_0\frac{\partial U}{\partial x}=-\rho'\frac{\partial U}{\partial x}-U\frac{\partial\rho'}{\partial x},$$

$$\frac{\partial U}{\partial t}+\frac{1}{\rho_0}\frac{\partial P'}{\partial x}=-U\frac{\partial U}{\partial x}+\frac{\rho'}{\rho_0^2}\frac{\partial P'}{\partial x},$$

$$\frac{\partial P'}{\partial t}+c^2\rho_0\frac{\partial U}{\partial x}-(\gamma-1)\rho_0\left(L_PP'+L_\rho\rho'\right)=$$
$$-\gamma P'\frac{\partial U}{\partial x}-U\frac{\partial P'}{\partial x}+(\gamma-1)\rho'\left(L_PP'+L_\rho\rho'\right)$$
$$+(\gamma-1)\rho_0\left(0.5L_{PP}P'^2+0.5L_{\rho\rho}\rho'^2+L_{\rho P}\rho'P'\right), \qquad (5)$$

where partial derivatives $L_P=(\frac{\partial L}{\partial P})_\rho$, $L_\rho=(\frac{\partial L}{\partial\rho})_P$, $L_{PP}=(\frac{\partial^2L}{\partial P^2})_\rho$, $L_{\rho\rho}=(\frac{\partial^2L}{\partial\rho^2})_P$, $L_{\rho P}=\frac{\partial^2L}{\partial P\partial\rho}$ are evaluated in the unperturbed state (P_0,ρ_0). We refer to constant unperturbed hydrodynamic perturbations and hence require zero $L(P_0,\rho_0)$, so that $L(P,\rho) \approx L_P\,P' + L_\rho\,\rho' + 0.5L_{PP}P'^2 + 0.5L_{\rho\rho}\rho'^2 + L_{P\rho}P\rho'$. Limiting ourselves to consideration of small-magnitude perturbations allows us to neglect the quadratic terms on the right-hand sides of (5). This is a reasonable approximation for small-magnitude flows with low Mach numbers. The Mach number is a ratio of the amplitude velocity to the sound speed. In the planar flow along axis x proportional to $\exp(\mathrm{i}\omega t - \mathrm{i}kx)$ (ω, k designate, respectively, the frequency and the wave number), one arrives at the dispersion relation in the frames of the linear theory [16, 17]

$$\omega^2\big(\mathrm{i}\omega-(\gamma-1)\rho_0 L_P\big) - k^2\big(\mathrm{i}c^2\omega-(\gamma-1)\rho_0\,L_\rho\big) = 0 \qquad (6)$$

with the leading-order solutions corresponding to two acoustic branches and the entropy non-wave mode

$$\omega_1 = c\,k-\mathrm{i}cB, \quad \omega_2 = -c\,k-\mathrm{i}cB,$$
$$\omega_{ent} = \mathrm{i}\,(\gamma-1)\rho_0 L_\rho/c^2. \qquad (7)$$

In the above expressions, $c = \sqrt{\gamma\frac{P_0}{\rho_0}}$ denotes the sound speed at equilibrium pressure and density, and B is the decrement/increment coefficient of the wave process given as

$$B = \frac{(\gamma-1)\,\rho_0}{2c^3}\,(c^2 L_P + L_\rho). \qquad (8)$$

In the general case of non-zero L_ρ, the dispersion relations (7) satisfy (6) with accuracy up to the terms of order L_P^1 and L_ρ^1. This means that deviation from adiabaticity is small over the characteristic wavelength of a perturbation (or during its characteristic period), $|B| \ll k$ ($|B| \ll \omega/c$). Frequency in both acoustic branches satisfies the algebraic equation

$$\omega^2 - c^2k^2 + 2\mathrm{i}\,c\,B\omega = 0. \qquad (9)$$

This coincides with (7) and yields a dynamic equation for any perturbation ϕ in the planar one-dimensional flow

$$\frac{\partial^2\phi}{\partial t^2} - c^2\frac{\partial^2\phi}{\partial x^2} - 2c\,B\frac{\partial\phi}{\partial t} = 0. \qquad (10)$$

If L is a function of pressure only, $\omega_{ent} = 0$, and (9) and (10) are exact, i.e., they do not represent series in powers of B and may be applied to moderate and strong deviations from adiabaticity. In other words, the dynamic equations for the acoustic and entropy fields exactly decouple if $L_\rho = 0$.

2.2. Perturbations in a fluid obeying an arbitrary caloric equation of state

The conclusions may be readily generalized in the case of the fluids (including liquids) obeying any caloric equation of state in the form $e(P,\rho)$, so that the first terms in the Taylor series expansion take the form

$$e(P,\rho) \approx e(P_0,\rho_0) + \left(\frac{\partial e}{\partial P}\right)_\rho P' + \left(\frac{\partial e}{\partial \rho}\right)_P \rho'. \qquad (11)$$

In this case, the linear form of the third equation from (5) reads as follows

$$\frac{\partial P'}{\partial t} + c^2\rho_0\frac{\partial U}{\partial x} = \left(\frac{\partial e}{\partial P}\right)_\rho^{-1} (L_P P' + L_\rho\rho') =$$
$$\frac{1}{T}\left(\frac{\partial s}{\partial P}\right)_\rho^{-1} (L_P P' + L_\rho\rho') \qquad (12)$$

with

$$c^2 = \left(\frac{\partial e}{\partial P}\right)_\rho^{-1}\left[\frac{P}{\rho^2} - \left(\frac{\partial e}{\partial \rho}\right)_P\right] = \left(\frac{\partial P}{\partial \rho}\right)_s. \qquad (13)$$

All partial derivatives with respect to thermodynamic parameters P and ρ are evaluated in the equilibrium state (P_0,ρ_0). In the general case of any fluid flow,

$$\omega_{ent} = \mathrm{i}\left(\frac{\partial e}{\partial P}\right)_\rho^{-1}\frac{L_\rho}{c^2},$$
$$B = \frac{1}{2c^3}\left(\frac{\partial e}{\partial P}\right)_\rho^{-1}(c^2L_P + L_\rho), \qquad (14)$$

and the dispersion relation and the dynamic equation for sound take the forms (9) and (10) with c and B given by (13) and (14). The van der Waals gas example refers to the caloric and thermal equations of state

$$e = \frac{RT}{(\gamma-1)\mu} - \frac{\rho}{\mu^2}a, \quad \frac{P}{\rho} = \frac{RT}{\mu}\left[1+\frac{\rho}{\mu}\left(b-\frac{a}{RT}\right)\right], \qquad (15)$$

where a and b are the van der Waals constants. That yields the leading-order expressions [18, 19]

$$e = \frac{P}{(\gamma-1)\rho} - \frac{(\gamma-2)\,\rho}{(\gamma-1)\,\mu^2}\,a - \frac{P}{(\gamma-1)\,\mu}\,b,$$
$$\left(\frac{\partial e}{\partial P}\right)_\rho = \frac{1}{(\gamma-1)\,\rho} - \frac{b}{(\gamma-1)\,\mu},$$
$$\left(\frac{\partial e}{\partial \rho}\right)_P = -\frac{P}{(\gamma-1)\,\rho^2} - \frac{\gamma-2}{(\gamma-1)\,\mu^2}\,a, \qquad (16)$$

and

$$c^2 = \frac{\gamma P_0}{\rho_0} + \frac{(\gamma-2)\,\rho_0}{\mu^2}\,a + \frac{\gamma P_0}{\mu}\,b,$$
$$\omega_{ent} = \mathrm{i}\left[\frac{(\gamma-1)\rho_0^2}{\gamma P_0} - \frac{(\gamma-1)(\gamma-2)\rho_0^4}{\gamma^2\mu^2P_0^2}a\right]L_\rho. \qquad (17)$$

3. Standing waves in a one-dimensional resonator

Let us consider velocity in the form

$$U(x,t) = \overline{X}(x)\,\overline{T}(t). \qquad (18)$$

Substituting (18) in (10) and separating functions of different variables, one arrives at

$$\frac{\overline{T}'' - 2c\,B\,\overline{T}'}{c^2\overline{T}} = \frac{\overline{X}''}{\overline{X}}, \qquad (19)$$

where the apostrophe denotes the derivative of the function with respect to its variable. Function X is responsible for the boundary conditions. Zero conditions at the boundaries of a resonator $x = 0$ and $x = L$ yield the differential equation and corresponding set of solutions

$$\frac{\overline{X}''_n}{\overline{X}_n} = -\lambda_n^2, \quad \overline{X}_n = C_{1,n}\sin(\lambda_n x) + C_{2,n}\cos(\lambda_n x), \tag{20}$$

where $C_{1,n}$, $C_{2,n}$ are constants, and

$$\lambda_n = \frac{n\pi}{L}, \quad n = 1, 2, \ldots \tag{21}$$

A set of functions $\overline{T}_n$ is determined by the differential equation

$$\overline{T}''_n - 2\,c\,B\,\overline{T}'_n + c^2\,\lambda_n^2\,\overline{T}_n = 0 \tag{22}$$

and depends on the ratio of $|B|$ and λ_n, i.e., on the spectrum of initial perturbation

$$\overline{T}_n = \begin{cases} \mathrm{e}^{cBt}\Big[D_{1,n}\exp\big(\sqrt{B^2-\lambda_n^2}\,ct\big) \\ \quad +D_{2,n}\exp\big(-\sqrt{B^2-\lambda_n^2}\,ct\big)\Big], & \text{if } |B| > \lambda_n, \\ \mathrm{e}^{cBt}\big(D_{1,n} + D_{2,n}t\big), & \text{if } |B| = \lambda_n, \\ \mathrm{e}^{cBt}\Big[D_{1,n}\sin\big(\sqrt{\lambda_n^2-B^2}\,ct\big) \\ \quad +D_{2,n}\cos\big(\sqrt{\lambda_n^2-B^2}\,ct\big)\Big], & \text{if } |B| < \lambda_n. \end{cases} \tag{23}$$

In (23), $D_{1,n}$, $D_{2,n}$ are constants determined by initial perturbations. Equations (9) and (10) impose weak effects by the heating–cooling function over the characteristic wavelength in the general case $L_\rho \neq 0$, so that $|B|$ should be smaller than any λ_n that contributes to the spectrum. This condition is satisfied for all natural n if $|B| < \frac{\pi}{L}$. We start with a discussion of this case and the borderline case $|B| = \frac{\pi}{L}$ if the spectrum contains λ_1 (see Sect. 3.1). The most complex case of weak, moderate, and strong damping of different harmonics in the initial spectrum will be considered in Sect. 3.2. The dynamics of perturbations depends on the sign of B. In the case of positive B, the inflow of energy ensures acoustical activity, i.e., enlargement of their magnitudes in time. Negative B corresponds to damping (different from the Newtonian one).

3.1. Particular cases of dynamics

3.1.1. Case $|B| < \frac{\pi}{L}$

This case reflects weak attenuation/amplification over the largest wavelength $2L$ and, hence, over all possible smaller wavelengths $2L/n$ that may contribute to the field in the resonator. The general solution satisfying boundary conditions

$$U(x = 0, t) = U(x = L, t) = 0 \tag{24}$$

at any time takes the form

$$U(x,t) = \exp(cBt)\sum_{n=1}^{\infty}\Big[M_n\sin\big(\sqrt{\lambda_n^2 - B^2}\,ct\big) + N_n\cos\big(\sqrt{\lambda_n^2 - B^2}\,ct\big)\Big]\,\sin(\lambda_n x). \tag{25}$$

The pressure field follows from the conservation of momentum

$$\frac{\partial P(x,t)}{\partial x} = -\rho_0\frac{\partial U(x,t)}{\partial t}, \tag{26}$$

so that

$$P(x,t) = \mathrm{e}^{cBt}\rho_0 c\sum_{n=1}^{\infty}\Big[\big(BM_n - N_n\sqrt{\lambda_n^2-B^2}\big) \times \sin\big(\sqrt{\lambda_n^2-B^2}\,ct\big) + \big(BN_n + M_n\sqrt{\lambda_n^2-B^2}\big) \times \cos\big(\sqrt{\lambda_n^2-B^2}\,ct\big)\Big]\frac{\cos(\lambda_n x)}{\lambda_n}. \tag{27}$$

The particular solution satisfies the initial conditions

$$U(x,0) \equiv \tilde{U}(x) = \sum_{n=1}^{\infty} N_n\sin(\lambda_n x),$$

$$P(x,0) \equiv \tilde{P}(x) = \rho_0 c\sum_{n=1}^{\infty}\frac{\big(BN_n + M_n\sqrt{\lambda_n^2-B^2}\big)}{\lambda_n} \times \cos(\lambda_n x). \tag{28}$$

The coefficients in the series are determined by the Fourier transforms of the initial waveforms as

$$N_n = \frac{2}{L}\int_0^L \mathrm{d}z\ \tilde{U}(z)\sin(\lambda_n z),$$

$$M_n = \frac{2\lambda_n\int_0^L \mathrm{d}z\ \tilde{P}(z)\cos(\lambda_n z)}{\rho_0\,cL\sqrt{\lambda_n^2-B^2}} - \frac{BN_n}{\sqrt{\lambda_n^2-B^2}}. \tag{29}$$

For example, let us consider the evolution of only one lowest harmonics with $\lambda_1 = \frac{\pi}{L}$. The constants ensuring maximum velocity U_0 and pressure perturbation P_0 over the length of a resonator at $t = 0$ are

$$N_1 = U_0, \quad M_1 = \frac{\frac{\pi P_0}{L} - B\,c\,\rho_0\,U_0}{c\rho_0\sqrt{\frac{\pi^2}{L^2} - B^2}}. \tag{30}$$

In the dimensionless quantities

$$X = \frac{x}{L}, \qquad T = \frac{\pi ct}{L}, \qquad b = \frac{BL}{\pi}, \tag{31}$$

the field corresponding to $P_0 = 0$ is

$$\frac{U}{U_0} = \exp(bT)\,\cos\big(\sqrt{1-b^2}\,T\big)\,\sin(\pi X),$$

$$\frac{P}{\rho_0\,c\,U_0} = -\exp(bT)\,\frac{\sin\big(\sqrt{1-b^2}\,T\big)}{\sqrt{1-b^2}}\cos(\pi X). \tag{32}$$

The dynamics of U and P in antinodes ($X = 0.5$ and $X = 0$, respectively) for different b are shown in Fig. 1.

The period of oscillations in the standing wave enlarges $1/\sqrt{1-b^2}$ times compared to the case $b = 0$. Magnitudes of perturbations increase in

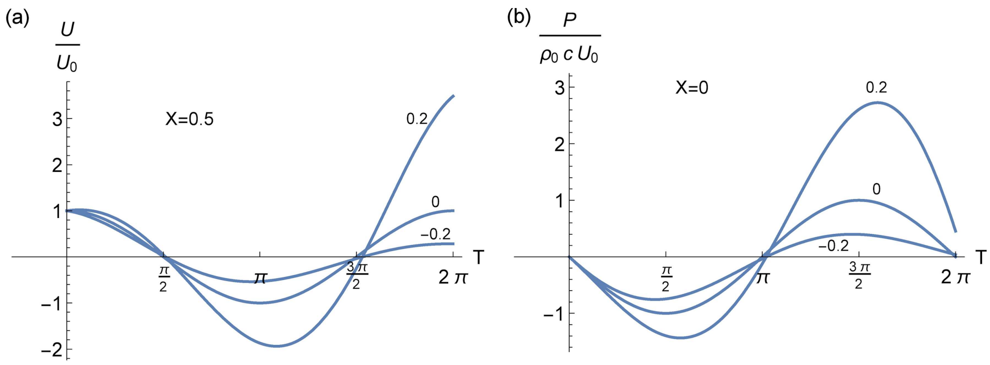

Fig. 1. The dimensionless perturbations of velocity (a) and pressure (b) as functions of T in antinodes ($X = 0.5$ and $X = 0$, respectively). The case of weak damping and only first harmonics in the initial spectrum.

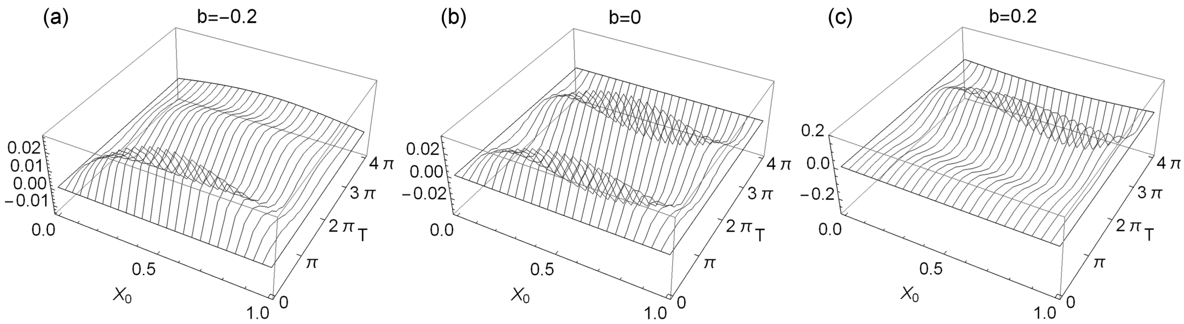

Fig. 2. The dimensionless coordinate $X - X_0$ as a function of X_0 and T. Cases $b = -0.2$ (a), $b = 0$ (b), and $b = 0.2$ (c). All evaluations refer to $U_0/c = 0.1$.

acoustically active flow with $b > 0$, remain constant in the neutral case $b = 0$, and get smaller if $b < 0$. The analysis is performed using the Eulerian variables. The velocity field and initial coordinate $X(T = 0) = X_0$ determine the trajectory, which is a particular solution to

$$\frac{\mathrm{d}X}{\mathrm{d}T} = \frac{U(X,T)}{\pi c}. \tag{33}$$

In turn, the trajectory determines acoustic fields in the Lagrangian variables. For example, the Eulerian velocity in the form (32) leads to the separation of variables X and T in the differential equation (33), and the resulting trajectory

$$X = \frac{2}{\pi}\arctan\left(\tan\left(\frac{\pi X_0}{2}\right)\exp\left(\frac{U_0}{c}\sin(T)\right)\right). \tag{34}$$

The trajectories for different b are shown in Fig. 2.

Once the trajectory is determined, any acoustic perturbation in the Lagrangian variables takes the form $\phi(X_0, T) = \phi(X(X_0, T), T)$.

3.1.2. Case $|B| = \frac{\pi}{L}$

This particular case corresponds to the moderate effects relating to non-adiabaticity of the lowest harmonics, if it contributes to the initial spectrum. If magnitudes of the harmonics with numbers larger than N are much smaller than magnitudes of all harmonics with numbers lower than N in the initial spectrum, the deviation from adiabaticity is weak if $|B| \le \frac{N\pi}{L}$ and the theory is still valid. Let us consider as an example only one lowest harmonics, zero boundary conditions (24), and $B = \pm\frac{\pi}{L}$ ($b = \pm 1$). The field in the resonator is given by

$$U(x,t) = \mathrm{e}^{\pm\frac{\pi ct}{L}}(M_1 + N_1 t)\sin\left(\frac{\pi x}{L}\right),$$

$$P(x,t) = \frac{\rho_0 L \mathrm{e}^{\pm\frac{\pi ct}{L}}}{\pi}\left(N_1 \pm \frac{\pi c(M_1+N_1 t)}{L}\right)\cos\left(\frac{\pi x}{L}\right). \tag{35}$$

In particular, coefficients $M_1 = U_0$ and $N_1 = \pi(P_0 \mp c\rho_0 U_0)/(L\rho_0)$ ensure maximal initial velocity U_0 and pressure P_0 over the length of a resonator. Initial zero perturbation of pressure yields

$$M_1 = U_0, \quad N_1 = \mp\frac{\pi c U_0}{L}. \tag{36}$$

In dimensionless quantities (31), (35) takes the form

$$\frac{U(X,T)}{U_0} = \exp\left(\pm T\right)(1 \mp T)\sin(\pi X),$$

$$\frac{P(X,T)}{\rho_0 c U_0} = \mp T\,\exp\left(\pm T\right)\cos(\pi X). \tag{37}$$

The absolute value of pressure reaches a maximum at $T=1$ if $b=-1$ and then quickly decreases, in contrast to velocity, which decreases continuously. In the case of $b=1$, the absolute values of both the pressure and the velocity increase with time.

3.1.3. Case $|B|>\frac{\pi}{L}$

This case can also be applied to $|B|>\frac{N\pi}{L}$, if the initial spectrum contains the highest harmonic number N. For lower harmonics, the ratio of $|B|$ and the wave number $n<N$ determines the form of T_n described by (23). Equations (9) and (10) are assumed to be exact. This means that strong attenuation is justified (strictly speaking) if $L_\rho=0$. The case of only the lowest harmonics available in the spectrum, $|B|>\frac{\pi}{L}$ and boundary conditions (24) leads to the perturbations in the resonator in the form

$$U(X,T)=\Big[M_1\,\mathrm{e}^{(b+\sqrt{b^2-1})\,T}+N_1\,\mathrm{e}^{(b-\sqrt{b^2-1})\,T}\Big]\times\sin(\pi X), \tag{38}$$

$$P(X,T)=c\rho_0\,\mathrm{e}^{(b-\sqrt{b^2-1})\,T}\Big[\exp\Big(2\sqrt{b^2-1}\,T\Big)\times(b+\sqrt{b^2-1})M_1+(b-\sqrt{b^2-1})N_1\Big]\cos(\pi X). \tag{39}$$

In particular,

$$M_1=\frac{P_0-(b-\sqrt{b^2-1})\,c\,U_0\,\rho_0}{2c\rho_0\sqrt{b^2-1}},\qquad N_1=\frac{-P_0+(b+\sqrt{b^2-1})\,c\,U_0\,\rho_0}{2c\rho_0\sqrt{b^2-1}} \tag{40}$$

ensure initial maximum velocity U_0 and pressure P_0 over the length of resonator. In the case of $P_0=0$, the dimensionless solutions are as follows

$$\frac{U(X,T)}{U_0}=\frac{\exp\big((b-\sqrt{b^2-1})T\big)\ \Big[b(1-\exp(2\sqrt{b^2-1}\,T))+\sqrt{b^2-1}(1+\exp(2\sqrt{b^2-1}\,T))U_0\Big]}{2\sqrt{b^2-1}}\sin(\pi X),$$
$$\frac{P(X,T)}{\rho_0 cU_0}=\frac{\exp\big((b-\sqrt{b^2-1})T\big)\ \big(1-\exp\big(2\sqrt{b^2-1}\,T\big)\big)}{2\sqrt{b^2-1}}\cos(\pi X). \tag{41}$$

3.2. Weak, moderate, and strong attenuation of different harmonics of the initial spectrum

If $L_\rho=0$, we may reasonably consider strong and moderate damping over higher harmonics along with weak attenuation over lower harmonics of the initial spectrum. Velocity satisfying boundary condition (24) takes the form

$$U=U_0\exp(bT)\Bigg[\sum_{n=1}^{n_1-1}\Big(A_{1,n}\sin(\sqrt{n^2-b^2}\,T)+A_{2,n}\cos(\sqrt{n^2-b^2}\,T)\Big)\sin(n\pi X)+(C_1{+}C_2\,T)\sin(n_1\pi X)$$
$$+\sum_{n=n_1+1}^{N}\Big(D_{1,n}\exp(-\sqrt{b^2-n^2}\,T)+D_{2,n}\exp(\sqrt{b^2-n^2}\,T)\Big)\sin(n\pi X)\Bigg], \tag{42}$$

if $|b|=n_1$. The corresponding pressure field is

$$P=c\rho_0U_0\exp(bT)\Bigg[\sum_{n=1}^{n_1-1}\frac{\cos(n\pi X)}{n}\Big((bA_{1,n}-\sqrt{n^2-b^2}A_{2,n})\sin(\sqrt{n^2-b^2}\,T)+(A_{1,n}\sqrt{n^2-b^2}+bA_{2,n})$$
$$\times\cos(\sqrt{n^2-b^2}\,T)\Big)+\frac{C_2+b(C_1{+}C_2\,T)}{n_1}\cos(n_1\pi X)+\sum_{n=n_1+1}^{N}\frac{\cos(n\pi X)}{n}\Big(\exp((b-\sqrt{b^2-n^2})T)$$
$$\times\big(-\sqrt{b^2-n^2}(D_{1,n}-\exp(2\sqrt{b^2-n^2}\,T)D_{2,n})+b\big(D_{1,n}+\exp(2\sqrt{b^2-n^2}\,T)D_{2,n}\big)\Big)\Bigg]. \tag{43}$$

If $|b|$ is not equal to some natural n_1, the fields consist of sums over intervals where $|b|<n_1$ and $|b|>n_1$. The initial conditions

$$U(X,0)\equiv\tilde{U}(X),\quad P(X,0)\equiv\tilde{P}(X) \tag{44}$$

determine the system for unknown coefficients $A_{1,n}$, $A_{2,n}$, C_1, C_2, $D_{1,n}$, $D_{2,n}$ as follows:

- $1 \le n < n_1$,

$$A_{2,n} = 2\int_0^1 \mathrm{d}Z\, \frac{\tilde{U}(Z)}{U_0}\sin(n\pi Z),$$

$$A_{1,n}\sqrt{n^2-b^2}+bA_{2,n} = 2n\int_0^1 \mathrm{d}Z\, \frac{\tilde{P}(Z)}{c\rho_0 U_0}\sin(n\pi Z), \tag{45}$$

- $n = n_1$,

$$C_1 = 2\int_0^1 \mathrm{d}Z\, \frac{\tilde{U}(Z)}{U_0}\sin(n_1\pi Z),$$

$$C_2 + bC_1 = 2n_1\int_0^1 \mathrm{d}Z\, \frac{\tilde{P}(Z)}{c\rho_0 U_0}\sin(n_1\pi Z), \tag{46}$$

- $n_1 < n \le N$,

$$D_{1,n} + D_{2,n} = 2\int_0^1 \mathrm{d}Z\, \frac{\tilde{U}(Z)}{U_0}\sin(n\pi Z),$$

$$-\sqrt{b^2-n^2}(K_n-D_{2,n}) + b(D_{1,n}+D_{2,n}) = 2n\int_0^1 \mathrm{d}Z\, \frac{\tilde{P}(Z)}{c\rho_0 U_0}\sin(n\pi Z). \tag{47}$$

The system (45)–(47) determines the set definitely. The magnitude of perturbations increases if the flow is acoustically active ($b > 0$).

4. Dynamics of perturbations in Newtonian flow

Comparative analysis of the small-magnitude Newtonian flow in a planar resonator would be interesting. In the case of the Newtonian flow, dispersion relations for acoustic and entropy modes do not decouple exactly, apart from the case with zero thermal conduction when the entropy mode is stationary, $\omega_{ent} = 0$. The small-magnitude acoustic perturbations in the resonator are described by the approximate equation

$$\frac{\partial^2\phi}{\partial t^2} - c^2\frac{\partial^2\phi}{\partial x^2} - \beta\frac{\partial^3\phi}{\partial t\partial x^2} = 0 \tag{48}$$

valid in the case of weak attenuation, i.e., up to the terms of order $\beta^1 = \left(\frac{4\mu}{3\rho_0} + \frac{\chi}{\rho_0}\left(\frac{1}{C_v}-\frac{1}{C_P}\right)\right)$, where μ is the viscosity and χ is the thermal conductivity of the gas. Equation (48) cannot be solved by the Fourier method of separation of variables. Considering that for both acoustic branches, the zero-order equation reads $\frac{\partial^2\phi}{\partial t^2} - c^2\frac{\partial^2\phi}{\partial x^2} = 0$, (48) can be rearranged readily to the approximate equation

$$\frac{\partial^2\phi}{\partial t^2} - c^2\frac{\partial^2\phi}{\partial x^2} - \frac{\beta}{c^2}\frac{\partial^3\phi}{\partial t^3} = 0, \tag{49}$$

which allows for the separation of variables. Associating φ with velocity and assuming the solution in the form (18), one arrives at

$$\frac{\overline{T}'' - \frac{\beta}{c^2}\overline{T}'''}{c^2\overline{T}} = \frac{\overline{X}''}{\overline{X}}. \tag{50}$$

Zero boundary conditions at $x = 0$ and $x = L$ yield a set of λ_n and $\overline{X}_n$ (see (20)–(21)). A set of functions T_n is determined by the differential equation

$$\overline{T}_n'' - \frac{\beta}{c^2}\overline{T}_n''' + c^2\lambda_n^2\overline{T}_n = 0. \tag{51}$$

The approximate solution to it, i.e.,

$$\overline{T}_n = \mathrm{e}^{-\lambda_n^2\beta t/2}\left[D_{1,n}\sin(\lambda_n c t)+D_{2,n}\cos(\lambda_n c t)\right], \tag{52}$$

is valid for $\beta \ll c/\lambda_n$ for all characteristic wavelengths λ_n available in the initial spectrum. The temporal behavior (23) is significantly different from the one determined by (52). Velocity in the resonator takes the form

$$U(x,t) = \sum_{n=1}^{n_1}\mathrm{e}^{-\lambda_n^2\beta t/2}\sin(\lambda_n x) \times\Big(M_n\sin(\lambda_n ct) + N_n\cos(\lambda_n ct)\Big), \tag{53}$$

if n_1 is the largest natural quantity that satisfies the relation $n_1 \ll cL/\pi\beta$. The pressure field follows from the conservation of momentum, so that

$$P(x,t) = -\frac{\rho_0}{2}\sum_{n=1}^{n_1}\mathrm{e}^{-\lambda_n^2\beta t/2}\cos(\lambda_n x) \Big[\sin(\lambda_n ct)(\beta\lambda_n M_n+2cN_n) + \cos(\lambda_n ct)(\beta\lambda_n N_n-2cM_n)\Big]. \tag{54}$$

The set of coefficients M_n, N_n may be determined from the initial conditions for velocity and pressure.

The description of the relatively large magnitude field in the resonator requires taking nonlinear effects into consideration and using special mathematical methods demanding the periodicity of oscillations [6]. The second-order solution, which concerns slow-standing MHD waves in coronal thermal conducting magnetic loops, can be found in [15]. It overlaps with the one derived by Kaner, Rudenko, and Khokhlov [5]. The authors of [5, 15] concluded that the leading-order nonlinear standing wave is a sum of two identical nonlinear waves propagating in opposite directions. The evolution of each wave is governed by the Burgers equation. The velocity in the resonator represents a sum of two parts, U_1 and U_2, satisfying equations

$$\frac{\partial U_1}{\partial x} + \frac{1}{c}\frac{\partial U_1}{\partial t} - \frac{\gamma+1}{2c^2}U_1\frac{\partial U_1}{\partial t} - \frac{\beta}{c^2}\frac{\partial^2 U_1}{\partial t^2} = 0,$$

$$\frac{\partial U_2}{\partial x} - \frac{1}{c}\frac{\partial U_2}{\partial t} - \frac{\gamma+1}{2c^2}U_2\frac{\partial U_2}{\partial t} - \frac{\beta}{c^2}\frac{\partial^2 U_2}{\partial t^2} = 0, \tag{55}$$

which are readily rearranged into the linear diffusion equations by the Hopf–Cole transformation and can be solved analytically. The Newtonian attenuation always leads to a decrease in wave energy and a decrease in the perturbation magnitudes.

5. Conclusions

The case of the impact of the heating–cooling function imposes a separation of variables and an analytical solution. The dynamics of perturbations in a flow with the Newtonian attenuation considered in Sect. 4 resembles that in the flow with weak deviation from adiabaticity supported by the heating–cooling function (Sect. 3.1.1), but with important additional remarks. Magnitudes of acoustic perturbations never increase in the Newtonian flow but may increase due to the impact of the heating–cooling function. Equation (49) is approximately valid for weak attenuation of all wavelengths available in the spectrum (even if dynamic equations for the entropy and sound modes decouple exactly in the case of $\chi = 0$), while (10) is precise in the case of decoupling (if $L_\rho = 0$) and imposes exact analytical solution for medium and strong deviation from adiabaticity as well. As an example of the heating–cooling function $L(P)$, we may mention the heating of a high-temperature atomic astrophysical plasma by coronal current dissipation ($L \sim P$) and heating by Alfvén mode/mode conversion ($L \sim P^{7/6}$) [20, 21]. The details of heating–cooling mechanisms are continuously updated following novel calculations of atomic data and transition rates. That makes authors treat a loss/gain function as a free parameter and discuss only possible regimes and seismological applications. Most astrophysical applications refer to plasma as an ideal gas, with the exception of planetary and stellar interiors. This concerns weakly coupled plasmas such as the solar corona, solar atmosphere, interstellar gas, and thermonuclear reactor plasma. Standing waves in plasma are very difficult to describe theoretically due to many factors, such as the complex geometry of a loop, nonlinear effects, thermal conduction of plasma, variable background parameters [22–24], and the coexistence of slow and fast magnetosound modes. Ruderman [15] made use of the model of dynamics of thermoconducting gas in a one-dimensional resonator and concluded that for realistic coronal loop parameters, the non-linearity effect must be taken into account only when the Mach number of the initial velocity perturbation is sufficiently large, of the order of or larger than 0.2. In astrophysical applications, thermal conduction is insignificant. Also, the huge lengths of astrophysical loops (dozens of Mm) make the thermal damping insignificant for fundamental harmonics. In cold loops with a temperature below 1 MK, the damping due to thermal conduction is very weak. In the absence of thermal conduction and nonlinear effects, the theoretical conclusions of this study may be applied to plasma applications.

In acoustically active flows, the magnitudes of perturbations grow and, starting from a certain point, get too large to be described within the framework of the linear theory. Viscous, thermal, and nonlinear damping oppose the growth of magnitudes in real flows. Temporal decay of harmonics in the course of the Newtonian attenuation depends on the number of harmonics (proportional to $\exp(-\lambda_n^2\beta t/2)$), while the impact of the heating–cooling function leads to the various decay or growth of all harmonics magnitudes depending on the ratio of $|B|$ and λ_n in accordance to (23). The frequency of harmonics does not depend on the Newtonian attenuation. The ratio of $|B|$ and λ_n also determines periodic or aperiodic behavior of individual harmonics (see (23)). The idea comes to mind to evaluate B by varying the length of a resonator L based on this peculiarity in temporal behavior. For example, if $|B| = \pi/L$, perturbations over the length of a resonator vary as $(C_1+C_2t)\exp(cBt)$ (Sect. 3.1.2). Estimations may be done at the early stage of evolution, even in the case of strong acoustic activity. The small-magnitude perturbations in a resonator with some loss/gain of energy may be described analytically. They are determined by the initial spectrum and heating–cooling function. Boundary conditions and two initial conditions determine the field at any moment definitely. Apart from standard initial conditions $U(x,0)$ and $\frac{\partial U}{\partial t}(x,0)$, the initial conditions $U(x,0)$ and $P(x,0)$ (see (28)) are used. Optionally, both kinds of initial conditions may be used since they are equivalent.

This study concerns perturbations in a resonator in the presence of some loss/gain of energy. It would not hurt to mention some examples of flows apart from astrophysical applications, namely flows of gases with excited vibrational degrees of freedom of molecules and flows of gases with chemical reactions. The first example relates to a gas with energy pumping into the vibrational degrees of freedom of the molecules [8, 9]. The parameter responsible for the deviation of adiabaticity, i.e.,

$$B = -\frac{(\gamma-1)^2 T}{2c^3}\left(\frac{C_v}{\tau_V} + \frac{\varepsilon-\varepsilon_{eq}}{\tau_V^2}\frac{\mathrm{d}\tau}{\mathrm{d}T}\right), \tag{56}$$

is evaluated at unperturbed pressure and temperature of a gas, $C_v = \mathrm{d}\varepsilon_{eq}/\mathrm{d}T$ designates the equilibrium specific heat at constant volume (ε_{eq} is the equilibrium value of vibrational energy, ε), and τ_V denotes the vibrational relaxation time. The non-equilibrium excitation is in principle possible due to negative $\mathrm{d}\tau_V/\mathrm{d}T$. In (56), B is the decrement (or increment, if positive) of acoustic magnitudes in the high-frequency oscillations if $\omega\tau_V \gg 1$, where ω designates the characteristic frequency of sound.

In the case of gases in which an exothermic chemical reaction occurs [25, 26],

$$B = \frac{Q(\gamma-1)\big(Q_\rho + (\gamma-1)Q_T\big)}{2c^2 m} \tag{57}$$

is a quantity evaluated at unperturbed pressure, temperature, and mass fraction Y of the reagent A^* in the $A^* \to B^*$ exothermic reactions. Here, Q is the heat produced in the chemical reaction, and m is the mass of the molecule. The dimensionless quantities Q_T and Q_ρ are determined by means of partial derivatives of Q with respect to temperature and density of the mixture, respectively, as

$$Q_T = \frac{T}{Q}\left(\frac{\partial Q}{\partial T}\right) \quad \text{and} \quad Q_\rho = \frac{\rho}{Q}\left(\frac{\partial Q}{\partial \rho}\right). \qquad (58)$$

The characteristic time of chemical reaction is

$$\tau_C = \frac{H\, m\, Y_0}{Q_0 Q_Y}, \qquad (59)$$

where H is the reaction enthalpy per unit mass of reagent A^*, $Q_Y = \frac{Y}{Q}(\frac{\partial Q}{\partial Y})$. Equation (57) is valid in the course of the high-frequency excitations if $\omega \tau_C \gg 1$. The isobaric entropy mode may be present in both examples and have a weak impact on the total field and boundary conditions in the resonator [27, 28].

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