STABILITY OF COMPLEX SOLUTIONS OF THE NONLINEAR SCHRÖDINGER EQUATION*

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The two dimensional stability of *complex* nonlinear wave and soliton solutions of the nonlinear Schrödinger equation is examined, thus largely extending recent results of Infeld and Rowlands for real solutions. All stationary entities are unstable to two dimensional perturbations. However, the solitons and a class of waves are one dimensionally stable (when perturbed in the direction of propagation only). These semi- stable entities could be of particular significance in narrow tube plasma experiments and in neurophysics.

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1. *Introduction*

The nonlinear Schrödinger equation is the basic equation of nonlinear wave propagation. If, instead of a plane wave given by $\Psi_0 e^{i(k_0x-\omega t)}$, we have a modulated plane wave Re $\Psi(x, t)e^{i(k_0x-\omega t)}$ we call $\Psi(x, t)$ the wave envelope. In various media $\Psi(x, t)$ can be slowly varying, thus describing deviations from simple plane waves such as propagate in a vacuum. One can show that in a variety of media Ψ satisfies the equation:

$$i\frac{\partial\Psi}{\partial t}+\frac{\partial^2\Psi}{\partial x^2}+c\frac{\partial^2\Psi}{\partial y^2}+a|\Psi|^2\Psi+b\Psi = 0, \tag{1.1}$$

(see [1–3] for $c = 0$ and Appendix A of this paper for general c). Although this equation with a and b *complex* has appeared in solid state physics [4], our interest here will be

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focused on a, b, c all real. For $c < 0$ we propose to call (1.1) the hyperbolic nonlinear Schrödinger equation (hNLS).

The form of (1.1) most often found in the literature is for $c = 1$ (unless the dynamics are one dimensional and $c = 0$). This form arises, among other fields, in nonlinear optics and in plasma physics [1, 5]. The hNLS describes the wave amplitude for deep water waves of small amplitude ($c = -2$; [6, 7]). In this case the equation is 2+1 dimensional, as the wave height in the z direction is linked to Ψ. The hNLS also appears in the context of electromagnetic plane waves in a plasma ($c = -1$, see [8]).

Weakly nonlinear wave solutions of (1.1) were investigated for stability with respect to one dimensional perturbations some time ago [9] (see also [1]); this problem is known as the modulational stability of the waves. However, the important step in a general stability analysis of (1.1) was taken in 1974, when Rowlands solved the problem of stability of all real stationary solutions when $c = 0$ [10]. These solutions include nonlinear waves (which were found to be stable or not depending on the sign of a), solitons and shocks (all stable). His results were then generalised to two dimensions by Infeld and Rowlands for $c = 1$ [11], and then for $c = -2$ by Infeld [12]. All real stationary entities were found to be unstable (previous two dimensional calculations had been limited to perpendicular perturbations on the soliton solution only; i.a. [13–15] for $c = 1$ and [16, 17] for $c < 0$). In this paper we propose to take the penultimate step in generality by looking at the stability of general complex stationary solutions to (1.1) for arbitrary real a, b, c. The results should yield those of [11, 12] when phase tends to zero (these are in fact some slight problems when taking this limit and they will be mentioned later).

By renormalisation of x, y, t and Ψ we limit a and b to the following:

$$a = -1 \qquad \text{or } +1$$

$$b = -1,\ 0, \qquad \text{or } +1,$$

($a = 0$ would no longer describe a nonlinear problem). This will considerably simplify our presentation.

2. *Form of waves, shocks and solitons*

We now propose to look at stationary solutions to (1.1) that do not depend on y. One can see by inspection that

$$\Psi(x, t) = \Psi_0(x-vt)e^{ik(x-vt)+i\Omega t} \tag{2.1}$$

and

$$\Psi(x) = \Psi_0(x) \tag{2.2}$$

obey essentially the same ordinary differential equation, though with different variable and constant b, so without loss of generality we choose $\Psi_0(x)$ and write it as

$$\Psi_0(x) = \Phi(x)e^{i\sigma(x)}; \qquad \Phi(x) \geqslant 0 \tag{2.3}$$

obtaining from (1.1)

$$\Phi_{xx}+a\Phi^3+b\Phi-\Phi\sigma_x^2 = 0, \tag{2.4}$$

$$\Phi^2\sigma_x = A = \text{const},$$
$$(\Phi_x = \partial\Phi/\partial x). \tag{2.5}$$

Dividing (2.5) by Φ^2 we obtain:

$$\sigma_x = \frac{A}{\Phi^2} + \lambda\delta(\Phi^2). \tag{2.6}$$

In what follows we take $\lambda = 0$, but the extra term would be needed if we wished to reconstruct the general real $\Psi(x)$ that are treated in [10] and [11], as for them $A = 0$ but Ψ can be negative. For $A \neq 0$, however, Φ^2 is never zero and the extra term will not appear.

Thus for $\lambda = 0$, from (2.4) and (2.6)

$$\Phi_{xx}+a\Phi^3+b\Phi-A^2\Phi^{-3} = 0 \tag{2.7}$$

and, upon multiplication by Φ_x and integration

$$\tfrac{1}{2}\Phi_x^2 = B-\frac{a}{4}\Phi^4-\frac{b}{2}\Phi^2-\tfrac{1}{2}A^2\Phi^{-2}, \tag{2.8}$$

where B is a constant of integration. Meaningful solutions will only exist if the left hand side of (2.8) is nonnegative between two zeros of Φ_x^2. This can happen for all values of $b(-1, 0, +1)$ when $a = 1$ and then

$$\Phi_1^2 < 0 < \Phi_2^2 \leqslant \Phi^2 \leqslant \Phi_3^2,$$

where Φ_k^2 are real zeros of the rhs of (2.8). When this is the case, (2.8) describes a nonlinear wave. The case $a = -1$ is much more interesting. Now b must be $+1$ and

$$0 < \Phi_1^2 \leqslant \Phi^2 \leqslant \Phi_2^2 \leqslant \Phi_3^2.$$

Solutions correspond to nonlinear waves and solitons. Permissable values of A and B are sandwiched between parameters such that:

$$\Phi_1^2 = \Phi_2^2 \quad \text{(linear wave limit)}$$

and such that

$$\Phi_2^2 = \Phi_3^2 \quad \text{(soliton limit)}.$$

A simple calculation shows A and B to be in a region, the boundary of which is given parametrically by (s is the linear value of Φ):

$$0 < A^2 < s^4(1-s^2),$$
$$0 < B < s^2(1-\tfrac{3}{4}s^2). \tag{2.9}$$

This region is shown in Figure 1 (for the moment disregard the little polar plots).

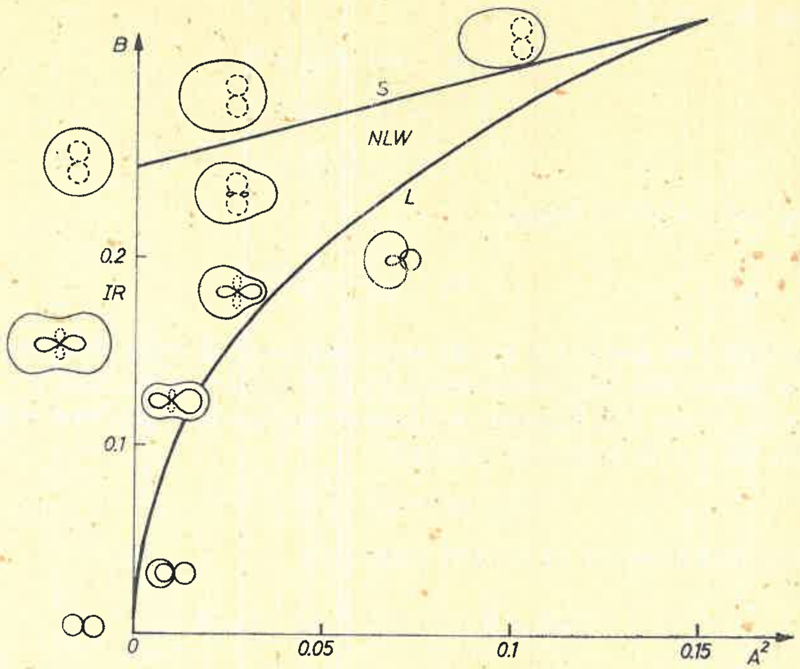


Fig. 1. Permitted region in A^2, B parameter space. The lower boundary corresponds to the linear or weak wave limit (L), the upper boundary to solitons (S), and the segment of the B axis between 0 and 1/4 to the calculations of Infeld and Rowlands [11] (IR). Parameters inside the region yield nonlinear waves. Phase diagrams of $\omega/k(\theta)$ are also shown: ——— real ω/k, $\cdots\cdots$ imaginary part of complex ω/k. Each little polar plot corresponds to one physical situation

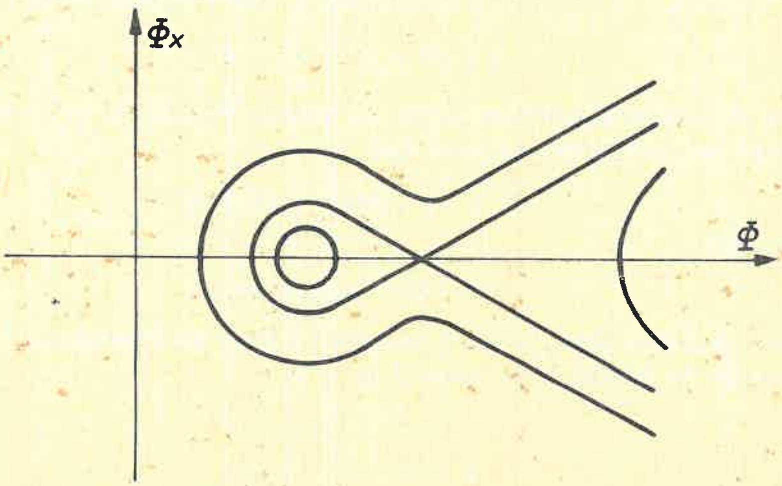


Fig. 2. Phase space diagram $\Phi_x(\Phi)$ for $a = -1$. Closed curves correspond to physically meaningful solutions

A phase diagram of $\Phi_x(\Phi)$ based on (2.8) is given in Figure 2 for $c = -1$, $b = +1$. The center of the closed region corresponds to Φ constant and the tiny circles surrounding this center to the weak wave limit. We will treat this case in some detail in the next Chapter. Interior closed curves correspond to nonlinear waves. The limiting curve depicts an (inverted) soliton solution. The general solution to (2.8) for nonlinear waves will not be given here (it would involve incomplete elliptic functions of the third kind). For the soliton case $\Phi_2^2 = \Phi_3^2$ and (2.8) becomes

$$(\Phi^2)_x = 2(\Phi^2-\Phi_2^2)(\Phi^2-\Phi_1^2) \tag{2.10}$$

solved by

$$\Phi^2 = \Phi_2^2\left[1-m\,\mathrm{sech}^2\left(\sqrt{\frac{m}{2}}\,\Phi_1 x\right)\right],$$

$$\sigma = A\int\frac{dx}{\Phi^2} = \frac{\sqrt{1-a^2}}{a}\,x-\tan^{-1}\left[\frac{\sqrt{1-a^2}}{a}\,\mathrm{cth}\,x\right]+\pi H(x)+\mathrm{const},$$

$$m = \frac{\Phi_2^2-\Phi_1^2}{\Phi_1^2},$$

$$H(x) = 0, \quad x < 0; \quad H(x) = 1, \quad x > 0. \tag{2.11}$$

This can be seen in the following way: if we differentiate the first two terms on the right hand side of (2.11) for any nonzero x we obtain $\dfrac{A}{\Phi^2}$. However, the second term is discontinuous at zero and jumps $-\pi$ when x passes through zero from the left. Since the left hand side of (2.11) must be continuous and monotone increasing, so must the left hand side be. Hence the Heavyside function must be introduced. In the $a \to 1$ limit (which corresponds to $A \to 0$) σ increases by π as x goes through zero from the left. This gives a change of sign in Ψ. Hence, if the constant is zero,

$$\lim_{a\to 1}\Psi = -\Phi_2\tanh x \tag{2.12}$$

and if it is π

$$\lim_{a\to 1}\Psi = \Phi_2\tanh x. \tag{2.13}$$

If we wished to obtain these results by taking $A \equiv 0$ from the beginning, we would need the $\delta(\Phi^2)$ contribution in (2.6), as $\Phi = |\Psi|$ would have a discontinuous derivative at zero and this would introduce a δ function contribution to Φ_{xx} in (2.7).

We conclude that both calculations, $A \to 0$ and $A \equiv 0$, lead to Ψ of the form (2.12) or (2.13).

Finally there is still another reason why $a = -1$ is the more interesting case. Previous calculations [10] led to stability for this case when $c = 0$, $A = 0$. It will be of interest to see if this is still true for $A^2 > 0$. After all, it is the stable wave and soliton solutions that we have most hope of observing in the laboratory.

3. *Weak wave limit*

So as to better understand the results, we will first look at the weak wave limit. For oscillations about an equilibrium we have

$$\Phi = \Phi_0(x)+\delta\Phi e^{i\omega t},$$
$$\sigma = \sigma_0(x)+\delta\sigma e^{i\omega t}, \tag{3.1}$$

where

$$\sigma_0 = A\int^x \frac{dx}{\Phi_0^2}+l.$$

If we substitute (3.1) into (1.1) and neglect terms like $\delta\Phi^2$, we obtain:

$$(L-A^2\Phi_0^{-4})\delta\Phi-2A\Phi_0^{-1}\delta\sigma_x = i\omega\Phi_0\delta\sigma,$$
$$2A(\Phi_0^{-1}\delta\Phi)_x+(\Phi_0^2\delta\sigma_x)_x+c(\Phi_0^2\delta\sigma_y)_y = -i\omega\Phi_0\delta\Phi, \tag{3.2}$$
$$L = \frac{\partial^2}{\partial x^2}+c\frac{\partial^2}{\partial y^2}+b+3a\Phi_0^2.$$

In the weak wave limit we assume Φ_0, $\delta\Phi$, $\delta\sigma$ constant and

$$\delta\Phi,\ \delta\sigma \sim e^{ikx},$$

and now (3.2) reduces to algebra. Solutions $\delta\Phi$, $\delta\sigma$ exist if

$$\begin{vmatrix} -\tilde{k}^2+b+3a\Phi_0^2-A^2\Phi_0^{-4}, & -2Aik_x\Phi_0^{-1}-i\omega\Phi_0 \\ 2Aik_x\Phi_0^{-1}+i\omega\Phi_0 \quad , & -\Phi_0^2\tilde{k}^2 \end{vmatrix} = 0, \tag{3.3}$$
$$\tilde{k}^2 = k_x^2+ck_y^2,$$
$$A^2 = a\Phi_0^6+b\Phi_0^4.$$

Thus, from the first and third component of (3.3)

$$(\omega+2Ak_x\Phi_0^{-2})^2 = -2a\Phi_0^2\tilde{k}^2+\tilde{k}^4. \tag{3.4}$$

As stated above, the $c = 0$, $a = -1$ case is of particular interest to us. For this case:

$$\omega = \pm k\sqrt{k^2+2\Phi_0^2}-2Ak\Phi_0^{-2} \tag{3.5}$$

(Figure 3). There are three values of $\dfrac{d\omega}{dk}$ for which ω is zero:

$$V_{g_1} = 2A\Phi_0^{-2}-\Phi_0^4A^{-1}, \quad \text{(double value)}$$
$$V_{g_2} = \sqrt{2}\,\Phi_0-2A\Phi_0^{-2},$$
$$V_{g_3} = -\sqrt{2}\,\Phi_0-2A\Phi_0^{-2}. \tag{3.6}$$

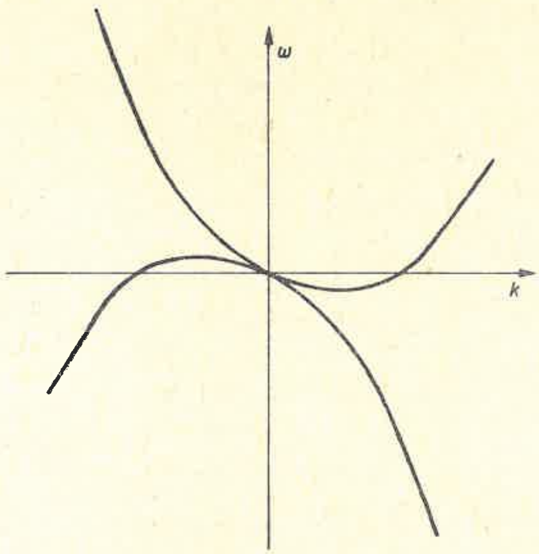


Fig. 3. Dispersion relation in the linear limit for $c = 0$, $a = -1$

If we divide the third component of (3.3) through by Φ_0^4 we obtain for $\Phi_0 \to 0$

$$\frac{A^2}{\Phi_0^4} \to b = 1$$

and

$$V_{g_1} \to 2; \quad V_{g_{2,3}} \to -2.$$

This agrees with Chapter 3 of [11]. In general when using the above for $a = -1$ we put $\Phi_1 = \Phi_2 = \Phi_0$.

4. *Small k stability analysis*

We will now look at small k stability. We will, however, keep the exact form of the nonlinear wave structure (2.8). This method (expansion in k but not in wave amplitude) was introduced into plasma physics in [18]. It is equivalent to considering slowly varying wavetrains (slowly varying in space and time) and yields the same results as Whitham's method. For more about the correspondence of the two methods see [19].

Return now to the general case, assuming $\Phi_0(x)$ to be any periodic solution to (2.8), thus for the moment excluding solitons. In this presentation we will take $c = 0$ in the hope that the rather heavy calculations will not be too obscure, reinstating c and simply quoting the general result at the end. We will look for $\delta\Phi$ and $\delta\sigma$ in the form of $P(x)e^{ikx}$, where P is periodic with the same period as Φ_0 (this is guaranteed by the Floquet theorem [20]). We also assume k to be small and look for ω, $\delta\Phi$, $\delta\sigma$ such that:

$$\begin{aligned} \omega &= k\omega_1 + k^2\omega_2 + \ldots, \\ \delta\Phi &= \delta\Phi_0 + k\delta\Phi_1 + \ldots, \\ \delta\sigma &= \delta\sigma_0 + k\delta\sigma_1 + \ldots . \end{aligned} \tag{4.1}$$

The zero order solutions to (3.2) are

$$\delta\Phi_0 = \frac{d\Phi_0}{dx}, \quad \delta\sigma_0 = \frac{A}{\Phi^2} + l$$

and in first order, dropping the zero in Φ_0:

$$(L - A^2\Phi^{-4})\delta\Phi_1 - 2A\Phi^{-1}\delta\sigma_{1x} = i\omega_1\Phi\delta\sigma_0 - 2i\Phi_{0x} + 2iA\Phi^{-1}\delta\sigma_0, \tag{4.2}$$

$$(2A\Phi^{-1}\delta\Phi_1)_x + (\Phi^2\delta\sigma_{1x})_x = -i\omega_1\Phi\Phi_x - 2iA\Phi_x\Phi^{-1} - 2il\Phi_x\Phi$$
$$-2iA\Phi_x\Phi^{-1} - 2iA\Phi^2\left(\frac{1}{\Phi^2}\right)_x. \tag{4.3}$$

The second equation (4.3) can be integrated to give:

$$\Phi^2\delta\sigma_{1x} = -2A\Phi^{-1}\delta\Phi_1 - i\left(\frac{\omega}{2} + l\right)\Phi^2 + D, \tag{4.4}$$

where D is a constant of integration. Upon substitution into (4.2) we obtain an equation for $\delta\Phi_1$ only:

$$\tilde{L}\delta\Phi_1 = i\omega l\Phi - 2i\Phi_{xx} + 2A(D + iA)\Phi^{-3},$$
$$\tilde{L} = L + 2A\Phi^{-4}. \tag{4.5}$$

Define Ψ_1, Ψ_2, Ψ_{-3} through:

$$\tilde{L}\begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_{-3}\end{pmatrix} = \begin{pmatrix}\Phi\\ \Phi_{xx}\\ \Phi^{-3}\end{pmatrix} \tag{4.6}$$

and the requirement that these functions Ψ_k be periodic. One can see by inspection that:

$$\Psi_1 = \tfrac{1}{2}\Phi_x \int^x \frac{\Phi^2 - \frac{\alpha_2}{\alpha}}{\Phi_x^2}\, dx,$$

$$\Psi_2 = -\frac{1}{2\alpha}\Phi_x \int^x \left(\frac{1}{\Phi^2} - \alpha\right) dx,$$

$$\Psi_{-3} = -\tfrac{1}{2}\Phi_x \int^x \frac{\Phi^{-2} - \frac{\alpha_{-2}}{\alpha}}{\Phi_x^2}\, dx, \tag{4.7}$$

(use the fact that $\tilde{L}\Phi_x = \tilde{L}\Phi_x \int \frac{dx}{\Phi_x^2} = 0$). Here α_k are defined through

$$\Phi_x \int^x \frac{dx}{\Phi_x^2}\Phi^n = \Phi_x\alpha_n x + \text{periodic function}$$

and $\alpha_0 = \alpha$. The values of the α_k in terms of A and B are given in Appendix B. Thus finally

$$\langle \delta\Phi_1 \rangle = i\omega l \Psi_1 - 2i\Psi_2 + (2AD + iA^2)\Psi_{-3}. \tag{4.8}$$

We are now in a position to determine D, so far unknown. Divide (4.4) through by Φ^2 and integrate over a period ($\langle\ \rangle$ is average over a period):

$$\langle \delta\sigma_{1x} \rangle = 0 = -2A\langle \delta\Phi_1 \Phi^{-3} \rangle - i\left(\frac{\omega}{2} + l\right) + D\gamma,$$

$$\gamma = \left\langle \frac{1}{\Phi^2} \right\rangle. \tag{4.9}$$

If we now substitute $\delta\Phi_1$ from (4.8) in (4.9) we obtain D:

$$\begin{aligned} D &= D_{00} + D_{0l}l + D_{0\omega}\omega + D_{l\omega}l\omega, \\ D_{00} &= [-4Ai\gamma^{-1}\langle \Psi_2 \Phi^{-3} \rangle + 4A^3 i\gamma^{-1}\langle \Psi_{-3}\Phi^{-3} \rangle]M^{-1}, \\ D_{0l} &= i\gamma^{-1}M^{-1} = 2D_{0\omega}, \\ D_{l\omega} &= 2Ai\gamma^{-1}\langle \Psi_1 \Phi^{-3} \rangle M^{-1}, \\ M &= 1 - 4A^2\gamma^{-1}\langle \Psi_{-3}\Phi^{-3} \rangle. \end{aligned} \tag{4.10}$$

From (4.4) and (4.9) $\delta\sigma_1$ is:

$$\delta\sigma_1 = -2i\Phi^{-1}\Psi_3 - 2A\int \frac{\delta\Phi_1}{\Phi^3}\,dx + 2A\gamma^{-1}\langle \delta\Phi_1 \Phi^{-3} \rangle,$$

$$\Psi_3 = -\frac{\Phi}{2\gamma}\int \left(\frac{1}{\Phi^2} - \gamma\right) dx. \tag{4.11}$$

It will prove convenient to write all first order quantities as polynomials in l and ω as in (4.10). This is done in Table I.

We now proceed to second order:

$$\begin{aligned} (L - A^2\Phi^{-4})\delta\Phi_2 - 2A\Phi^{-1}\delta\sigma_{2x} &= i\omega_1\Phi\delta\sigma_1 + i\omega_2\Phi\delta\sigma_0 - 2i\delta\Phi_{1x} \\ &\quad + \delta\Phi_0 + 2iA\Phi^{-1}\delta\sigma_1, \end{aligned} \tag{4.12}$$

$$(\Phi^2\delta\sigma_{2x})_x + 2A(\delta\Phi_2\Phi^{-1})_x = S + i\omega_2\Phi\Phi_x, \tag{4.13}$$

$$S \equiv -i\omega_1\Phi\delta\Phi_1 + l\Phi^2 + A - 2i\Phi^2\delta\sigma_{1x} - 2i\Phi\Phi_x\delta\sigma_1 - 2Ai\Phi^{-1}\delta\Phi_1.$$

Equation (4.13) can be integrated over a period, yielding a consistency condition:

$$\langle S \rangle = 0. \tag{4.14}$$

The second consistency condition is found by substituting $\delta\sigma_{2x}$ from the integrated form of (4.13) into (4.12). We then multiply on the left by Φ_x and use the self-adjoint property of $\tilde{L}$:

$$\langle \Phi_x \tilde{L}\delta\Phi_2 \rangle = \langle \delta\Phi_2 \tilde{L}\Phi_x \rangle = 0$$

TABLE I

F	F_{00}	F_{0l}	$F_{0\omega}$	$F_{l\omega}$	Others
$\bar{D} = -iD$	$-4A\gamma^{-1}\langle\Psi_2\Phi^{-3}\rangle M^{-1}$ $+4A^3\gamma^{-1}\langle\Psi_{-3}\Phi^{-3}\rangle M^{-1}$	$\gamma^{-1}M^{-1}$	$\frac{1}{2}\gamma^{-1}M^{-1}$	$2A\gamma^{-1}M^{-1}\langle\Psi_1\Phi^{-3}\rangle$	$M = 1$ $-\frac{4A^2}{\gamma}\langle\Psi_{-3}\Phi^{-3}\rangle$
$\delta\Phi_1$	$-2i\Psi_2+2iA^2\Psi_{-3}$ $+2iA\bar{D}_{00}\Psi_{-3}$	$2Ai\bar{D}_{0l}\Psi_{-3}$	$2Ai\bar{D}_{0\omega}\Psi_{-3}$	$i\Psi_1+2Ai\bar{D}_{l\omega}\Psi_{-3}$	
$\delta\sigma_1$	$-2A\int^x\Phi^{-3}\delta\Psi_{100}dx$ $+2A\gamma^{-1}\left\langle\frac{\delta\Phi_{100}}{\Phi^3}\right\rangle\int^x\frac{dx}{\Phi^2}$	$-2i\frac{\Psi_3}{\Phi}-2A\int^x\frac{\delta\Phi_{10l}}{\Phi^3}dx$ $+2A\gamma^{-1}\left\langle\frac{\delta\Phi_{10l}}{\Phi^3}\right\rangle\int^x\frac{dx}{\Phi^2}$	$-i\frac{\Psi_3}{\Phi}-2A\int^x\frac{\delta\Phi_{10\omega}dx}{\Phi^3}$ $+2A\gamma^{-1}\left\langle\frac{\delta\Phi_{10\omega}}{\Phi^3}\right\rangle\int^x\frac{dx}{\Phi^2}$	$-2A\int^x\frac{\delta\Phi_{1l\omega}}{\Phi^3}dx$ $+2A\gamma^{-1}\left\langle\frac{\delta\Phi_{1l\omega}}{\Phi^3}\right\rangle\int^x\frac{dx}{\Phi^2}$	
S	$A-2i\Phi^2\delta\sigma_{100x}$ $-2\Phi\Phi_x i\delta\sigma_{100}$ $-2A\Phi^{-1}i\delta\Phi_{100}$	$\Phi^2-2i\Phi^2\delta\sigma_{0lx}$ $-2i\Phi\Phi_x\delta\sigma_{10l}$ $-2Ai\Phi^{-1}\delta\Phi_{10l}$	$-i\Phi\delta\Phi_{100}$ $-2\Phi^2 i\delta\sigma_{10\omega x}$ $-2i\Phi\Phi_x\delta\sigma_{0\omega}$ $-2iA\Phi^{-1}\delta\Phi_{0\omega}$	$-i\Phi\delta\Phi_{0l}-2i\Phi^2\delta\sigma_{l\omega x}$ $-2i\Phi\Phi_x\delta\sigma_{l\omega}$ $-2Ai\Phi^{-1}\delta\Phi_{1\omega l}$	$S_{0\omega^2} = -i\Phi\delta\Phi_{0\omega}$ $S_{l\omega^2} = -i\Phi\delta\Phi_{l\omega}$

to obtain the second consistency condition:

$$\langle \Phi_x^2\rangle + i\omega_1\langle \Phi\Phi_x\delta\sigma_1\rangle + 2iA\langle \Phi^{-1}\Phi_x\delta\sigma_1\rangle$$
$$-2i\langle \Phi_x\delta\Phi_{1x}\rangle + A\langle S\Phi^{-2}\rangle = 0. \tag{4.15}$$

The last term has been integrated by parts:

$$\left\langle \frac{\Phi_x}{\Phi^3}\int^x S dx\right\rangle = \tfrac{1}{2}\langle S\Phi^{-2}\rangle.$$

We thus have two conditions for two unknowns, l and ω. Both are linear in l and we find this constant from (4.14):

$$l = L'(\omega)/M'(\omega),$$

where L' and M' are quadratics. The second condition (4.15) can be written as:

$$A'(\omega) + lB'(\omega) = 0,$$

so finally:

$$A'(\omega)M'(\omega) + B'(\omega)L'(\omega) = 0$$

is the dispersion relation. It is a quartic in ω_1. Writing ω/k in place of ω_1, and substituting from Table I we obtain:

$$\sum_{n=0}^{4} S_n\left(\frac{\omega}{k}\right)^n = 0,$$

$$S_n = \sum_{k=0} (A_k'M_{n-k}' + B_k'L_{n-k}'),$$

$$A' = \sum_{k=0}^{2} A_k'\omega^k \text{ etc.} \tag{4.16}$$

(With the convention that A_k' etc. are zero if $k > q$)

$$A_0' = -4a\langle \Psi_2\Phi^3\rangle - 4b\langle \Psi^2\Phi\rangle + a\langle \Phi^4\rangle + b\langle \Phi^2\rangle + 4A(A+\bar{D}_{00})\,\langle a\Phi^3 + b\Phi|\Psi_{-3}\rangle,$$
$$A_1' = -2A\langle \Psi_2\Phi^3\rangle\gamma^{-1} + 4A\bar{D}_{0\omega}\langle a\Phi^3 + b\Phi|\Psi_{-3}\rangle + 2A^2\gamma^{-1}(A+\bar{D}_{00})\,\langle \Psi_{-3}\Phi^{-3}\rangle,$$
$$A_2' = \langle \Phi_x\Psi_3\rangle + 2A^2\bar{D}_{0\omega}\gamma^{-1}\langle \Psi_{-3}\Phi^{-3}\rangle,$$
$$B_0' = A + 4A\bar{D}_{0l}\langle a\Phi^3 + b\Phi|\Psi_{-3}\rangle,$$
$$B_1' = 2\langle \Phi\Psi_3\rangle + 2\langle a\Phi^3 + b\Phi|\Psi_{-3}\rangle A\bar{D}_{l\omega} + 2A^2\bar{D}_{0l}\gamma^{-1}\langle \Psi_{-3}\Phi^{-3}\rangle,$$
$$B_2' = A\gamma^{-1}\langle \Psi_1\Phi^{-3}\rangle + 2A^2\gamma^{-1}\bar{D}_{l\omega}\langle \Psi_{-3}\Phi^{-3}\rangle,$$
$$L_0' = -A + 4A\gamma^{-1}\langle \Psi_2\Phi^{-3}\rangle - 4A^2\gamma^{-1}(A+\bar{D}_{00})\,\langle \Psi_{-3}\Phi^{-3}\rangle,$$
$$L_1' = -2A(A+\bar{D}_{00})\,\langle \Psi_{-3}\Phi\rangle + 2\langle \Psi_2\Phi\rangle - 2\langle \Phi_x\Psi_3\rangle - 4A^2\gamma^{-1}\bar{D}_{0\omega}\langle \Psi_{-3}\Phi^{-3}\rangle,$$

$$L_2' = -2A\bar{D}_{0\omega}\langle \Psi_{-3}\Phi\rangle,$$

$$M_0' = \langle \Phi^2\rangle - 4\langle \Psi_{3x}\Phi\rangle + 4A^2\gamma^{-1}\bar{D}_{0l}\langle \Psi_{-3}\Phi^{-3}\rangle,$$

$$M_1' = 2A(\bar{D}_{0l}+\gamma^{-1})\langle \Psi_{-3}\Phi\rangle + 4A^2\gamma^{-1}\bar{D}_{l\omega}\langle \Psi_{-3}\Phi^{-3}\rangle,$$

$$M_2' = \langle \Psi_1\Phi\rangle + 2A\bar{D}_{l\omega}\langle \Psi_{-3}\Phi\rangle.$$

Calculation of averaged products is also straightforward. For example:

$$\langle \Psi_1\Phi^n\rangle = \tfrac{1}{2}\left\langle \Phi^n\Phi_x \int^x \frac{\Phi^2 - \dfrac{\alpha_2}{\alpha}}{\Phi_x^2}\,dx\right\rangle \tag{4.17}$$

$$= -\frac{1}{2(n+1)}\left\langle \frac{\Phi^{n+1}\left(\Phi^2 - \dfrac{\alpha_2}{\alpha}\right)}{\Phi_x^2}\right\rangle = \frac{1}{2(n+1)}\left[\frac{\alpha_2\alpha_{n+1}}{\alpha} - \alpha_{n+3}\right].$$

Similarly:

$$\langle \Psi_2\Phi^n\rangle = \frac{1}{2\alpha(n+1)}\left[\alpha_{n+1} - \alpha\langle \Phi^{n+1}\rangle\right],$$

$$\langle \Psi_{-3}\Phi^n\rangle = \frac{1}{2(n+1)}\left[\alpha_{n-1} - \frac{\alpha_{-2}\alpha_{n+1}}{\alpha}\right]. \tag{4.18}$$

One can see by inspection that

$$\langle \Psi_{-3}\Phi\rangle = \langle \Psi_1\Phi^{-3}\rangle$$

and finally:

$$\langle \Psi_3\Phi_x\rangle = \tfrac{1}{4}(\gamma^{-1} - \langle \Phi^2\rangle).$$

Equation (4.16) can be simplified. The reduction of A_0', which is the most difficult, is given in Appendix C. We finally obtain:

$$\begin{aligned}
&A_0 = (\gamma + A^2\alpha_{-4})\alpha^{-1}, \quad B_0 = A\frac{\alpha_{-2}}{\alpha},\\
&A_1 = A\left(\frac{\alpha_{-2}}{\alpha} - M\gamma\right), \quad B_1 = \tfrac{1}{2}\left[1 + \frac{A^2N\alpha_{-1}}{\alpha} - \frac{M\alpha_2\gamma}{\alpha}\right]\\
&A_2 = \tfrac{1}{4}(1 - \gamma M\langle \Phi^2\rangle), \quad B_2 = \tfrac{1}{4}AN,\\
&L_k = -B_k,\\
&M_0 = 1, \quad M_1 = AN, \quad M_2 = \tfrac{1}{4}\left(A^2N^2 + M\gamma\left(\frac{\alpha_2^2}{\alpha} - \alpha_4\right)\right),
\end{aligned} \tag{4.19}$$

where:

$$M\gamma = \gamma + A^2\alpha_{-4} - A^2\frac{\alpha_{-2}^2}{\alpha}, \qquad A_k = M\gamma A_k' \text{ etc.}$$

$$N = \alpha - \frac{\alpha_2\alpha_{-2}}{\alpha}.$$

The first two coefficients S_0 and S_1 simplify:

$$S_0 = M\gamma\alpha^{-1},$$
$$S_1 = 0. \tag{4.20}$$

The first component of (4.20) is seen immediately. Also:

$$S_1 = A_0M_1 + A_1M_0 + B_0L_1 + B_1L_0$$
$$= (\gamma + A^2\alpha_{-4})\frac{AN}{\alpha} + A\left(\frac{\alpha_{-2}}{\alpha} - M\gamma\right) - A\frac{\alpha_{-2}}{\alpha}\left(1 + A^2N\frac{\alpha_{-2}}{\alpha} - M\gamma\frac{\alpha_2}{\alpha}\right) = 0. \tag{4.21}$$

Finally we write the generalisation to finite c, leaving the derivation to the interested reader:

$$S_0 \to \frac{M\gamma}{\alpha}\cos^4\theta + [\langle a\Phi^4 + b\Phi^2\rangle + A_0\langle\Phi^2\rangle + 2AL_0]c\sin^2\theta\cos^2\theta,$$
$$S_1 \to [A_1\langle\Phi^2\rangle + \langle a\Phi^4 + b\Phi^2\rangle M_1 + 2AL_1]c\sin^2\theta\cos\theta,$$
$$S_2 \to S_2\cos^2\theta + [\langle\Phi^2\rangle A_2 + \langle a\Phi^4 + b\Phi^2\rangle M_2 + 2AL_2]c\sin^2\theta,$$
$$S_3 \to S_3\cos\theta,$$
$$S_4 \to S_4,$$
$$\boldsymbol{k} = (k_x, k_y) = k(\cos\theta, \sin\theta).$$

We have thus obtained a quartic for $\omega(\theta, a, b, c, A, B)$. The integrals $\langle\Phi^2\rangle$, $\langle 1/\Phi^2\rangle$, $\langle\Phi^4\rangle$ are found in terms of complete elliptic functions in Appendix D to be:

$$\gamma = \langle 1/\Phi^2\rangle = \Pi(m, k)/\Phi_n^2K(k),$$
$$\langle\Phi^2\rangle = \Phi_p^2 - (\Phi_p^2 - \Phi_n^2)E(k)/K(k),$$
$$a\langle\Phi^4\rangle = \tfrac{4}{3}B - \tfrac{4}{3}b\langle\Phi^2\rangle,$$
$$m = \frac{\Phi_2^2 - \Phi_n^2}{\Phi_n^2}, \qquad k^2 = \frac{\Phi_2^2 - \Phi_n^2}{\Phi_p^2 - \Phi_n^2},$$
$$n = 1, \quad p = 3 \quad \text{if} \quad a = -1,$$
$$n = 3, \quad p = 1 \quad \text{if} \quad a = +1. \tag{4.22}$$

When we formally put $A = 0$ in (4.21) and redo the calculations of Appendix D we recover the formulas of [11] and [12]. One can see by inspection that terms containing $\Pi(m, k)$ cancel.

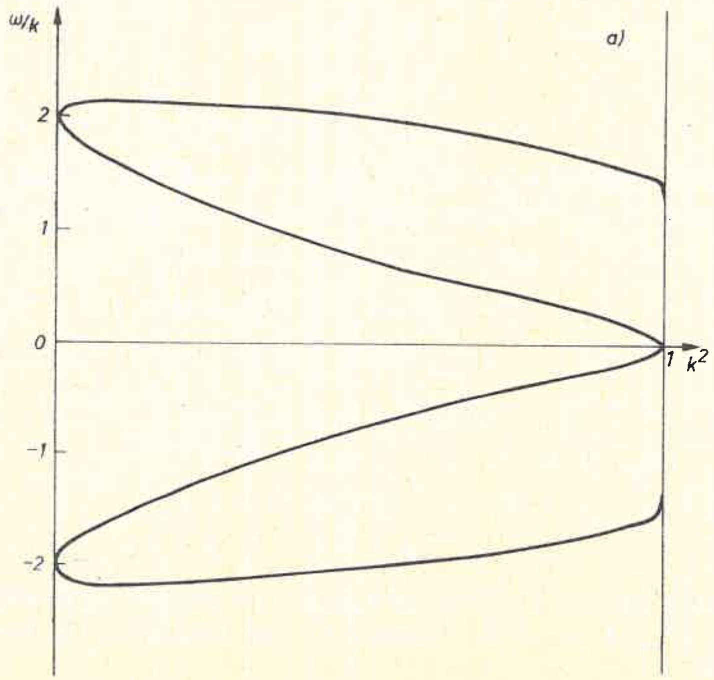
5. *Some special cases*

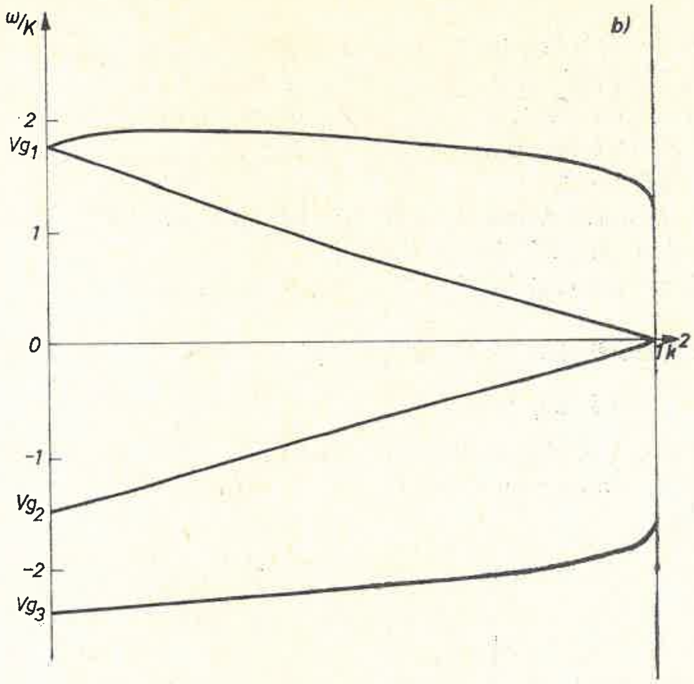
1. $c = 0,\ a = -1,\ b = +1$

When (4.22) is used in (4.19) with $a = -1$, $b = 1$ we can obtain ω/k as a function of B, say, for given A. This corresponds to a vertical cross section of the permitted region in Figure 1. So as to be able to compare these diagrams for different A^2 we use k^2 as defined in (4.22) in place of B, as this parameter always takes values from zero to one. Curves of $\omega/k(\theta)$ are given in Figure 4 for three chosen A^2. The linear limits ($k^2 = 0$) agree with those of (3.5) when $\Phi_0^2 = \Phi_1^2 = \Phi_2^2$. Continuity with the earlier results of Infeld and Rowlands [11] is found. Figure 4a depicts both ω/k for $A^2 = 10^{-14}$ and obtained from the formulas of [11], as differences between the two will not be discernable on this scale. All roots are seen to be real. The $k^2 \to 1$ limit corresponds to one of the solitons (2.11). Thus all nonlinear waves and solitons are one dimensionally stable for $a = -1$.

2. $c = +1,\ a = -1,\ b = +1$

The values of $\omega/k(\theta)$ for this case are shown in Figure 1 in the form of polar CMA diagrams [21] introduced into nonlinear physics in [19]. The whole diagram can be thought of as a huge pond in which, at any point of the permitted region, there is a nonlinear wave or soliton, corresponding to (A, B), that is not indicated but is understood. Each little polar plot corresponds to the wave or shock for values of (A, B) at its center. The distance from the center to the continuous line is proportional to real phase velocity $\omega/k(\theta)$ for given θ.





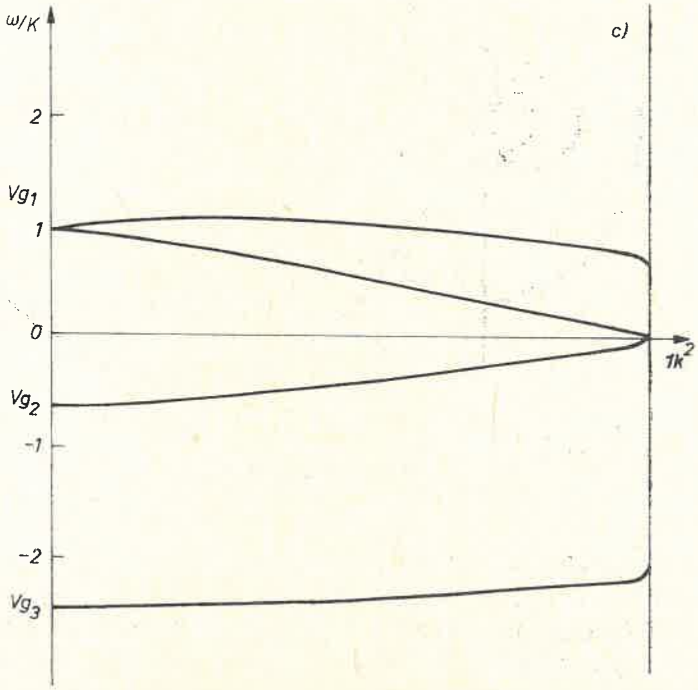


Fig. 4. ω/k for $c = 0$, $a = -1$ and three chosen values of A^2: a) 10^{-14} and 0 from [11], b) 10^{-2} and c) 10^{-1}. The parameter k^2 is essentially the amplitude of the basic nonlinear wave normalised to the amplitude in the soliton limit

In the case of complex roots the dotted line indicates the imaginary part of ω/k, the real parts having been omitted for clarity. We note that:

(a) For $\theta = 0$ all roots are real (stability of waves and solitons mentioned in point 1).

(b) In $A^2 \to 0$ limit we recover the plots of [11] (Figure 2 of reference, indicated here to the left of diagram).

(c) For general nonlinear waves (interior of region) instability sets in at a critical angle and all waves are unstable for $\theta \neq \pi/2$, $3\pi/2$.

(d) Solitons are unstable for all $\theta \neq 0$, π and the largest growth rates are at $\pi/2$, $3\pi/2$.

(e) The linear limit is described by (3.5) with $c = 1$.

3. $c = +1$, $a = +1$, any b

This case only includes nonlinear waves, so we will just give one phase diagram (Figure 5). All waves are unstable.

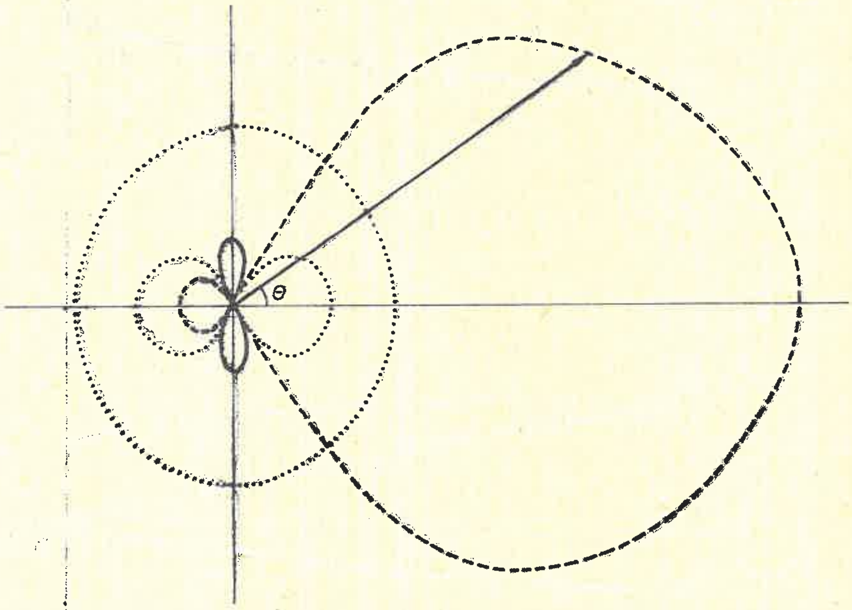


Fig. 5. Polar plot of $\omega/k(\theta)$ for $c = +1$, $a = +1$, $b = -1$, $A^2 = 0.05$, $B = -0.2$. Broken line is real part of complex root

4. $c < 0$

Now the L operator can introduce an instability. The $A^2 = 0$ case was treated in [12]. One can gain an idea of how general phase diagrams will be altered as compared with Figures 1 and 5 by using the schematics of Figure 6, which is drawn for $c = -2$, as this is the case of deep water waves. Roughly speaking, the waves and shocks are now subject to a larger number of instabilities when $a = -1$ due to the hyperbolic form of L (now instabilities appear even in the linear limit, (3.4) for $a = -1$ and $\theta > \tan^{-1}(1/\sqrt{c})$). For $a = +1$ on the other hand, unstable angles are diminished.

We can summarise our findings by stating that all nonlinear waves and solitons are unstable when $c \neq 0$. For the special case of $c = 0$, $a = -1$, $b = 1$, both nonlinear waves and solitons are stable. Continuity of results with these found previously for real waves and shocks [11, 12] is established.

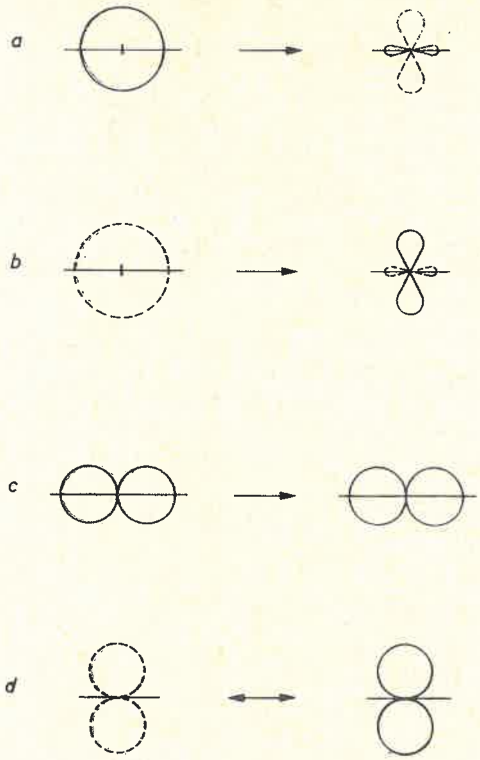


Fig. 6. Schematic of changes in Figures 1 and 5 when solving for hNLS

APPENDIX A

Suppose we have a wave packet with nonlinear dispersion such that most of the energy is in or near the wavenumber and frequency ω_0. The complex envelope will be Ψ and takes value Ψ_0 at $\boldsymbol{k}_0, \omega_0$:

$$\omega = \omega(\boldsymbol{k}, \omega_0, |\Psi|^2). \tag{A.1}$$

We may expand this relation around $\boldsymbol{k}_0, \omega_0, |\Psi|^2$:

$$\omega - \omega_0 = \frac{\partial\omega}{\partial \boldsymbol{k}_0}(\boldsymbol{k}-\boldsymbol{k}_0) + \tfrac{1}{2}\frac{\partial^2\omega}{\partial \boldsymbol{k}_0 \partial \boldsymbol{k}_0} : (\boldsymbol{k}-\boldsymbol{k}_0)(\boldsymbol{k}-\boldsymbol{k}_0) + \frac{\partial\omega}{\partial|\Psi|_0^2}(|\Psi|^2 - |\Psi|_0^2). \tag{A.2}$$

If we now substitute $i\partial/\partial t'$ for $(\omega-\omega_0)$ and $-\partial/\partial \boldsymbol{x}'$ for $\boldsymbol{k}-\boldsymbol{k}_0$ and operate on Ψ from the left, we obtain:

$$i\left(\frac{\partial\Psi}{\partial t'} + \frac{\partial\omega}{\partial \boldsymbol{k}_0}\frac{\partial\Psi}{\partial \boldsymbol{x}'}\right) + \tfrac{1}{2}\frac{\partial^2\omega}{\partial \boldsymbol{k}_0\partial \boldsymbol{k}_0} : \frac{\partial^2\Psi}{\partial \boldsymbol{x}'\partial \boldsymbol{x}'} - \frac{\partial\omega}{\partial|\Psi|_0^2}(|\Psi|^2 - |\Psi|_0^2)\Psi = 0. \tag{A.3}$$

We now introduce the transformation:

$$x = x' - \frac{\partial \omega}{\partial \boldsymbol{k}_0} t,$$

$$t = t' \tfrac{1}{2} \frac{\partial^2 \omega}{\partial k_x^2}, \tag{A.4}$$

assume the tensor $\partial^2\omega/\partial\boldsymbol{k}\partial\boldsymbol{k}$ to be diagonal in this coordinate system and introduce:

$$a = -2(\partial\omega/\partial|\Psi|_0^2)/(\partial^2\omega/\partial k_x^2),$$

$$b = 2(\partial\omega/\partial|\Psi|_0^2)/(\partial^2\omega/\partial k_x^2)\,|\Psi|_0^2,$$

$$c = (\partial^2\omega/\partial k_y^2)/(\partial^2\omega/\partial k_x^2), \tag{A.5}$$

to finally obtain

$$i\frac{\partial\Psi}{\partial t} + \frac{\partial^2\Psi}{\partial x^2} + c\frac{\partial^2\Psi}{\partial y^2} + b\Psi + a|\Psi|^2\Psi = 0, \tag{A.6}$$

(two space coordinates are assumed to describe the dynamics completely).

APPENDIX B

From (2.7)

$$\Phi\Phi_{xx} = -a\Phi^4 - b\Phi^2 + A^2\Phi^{-2}, \tag{B.1}$$

$$\tfrac{1}{2}\Phi_x^2 = B - \frac{a}{4}\Phi^4 - \frac{b}{2}\Phi^2 - \frac{A^2}{2}\Phi^{-2}. \tag{B.2}$$

1. Subtract $\frac{1}{4}$ times (B.2) from (B.1), divide result by Φ_x^2 and average over a period to obtain:

$$B\alpha - \tfrac{1}{4}b\alpha_2 - \tfrac{3}{4}A^2\alpha_{-2} = \tfrac{1}{4}. \tag{B.3}$$

2. Add $\frac{1}{2}$ times (B.2) to (B.1), divide result by $\Phi^2\Phi_x^2$ and average over a period:

$$-b\alpha - \tfrac{3}{4}a\alpha_2 + B\alpha_{-2} = 0. \tag{B.4}$$

3. Multiply (B.2) by Φ^2 and use Φ^4 calculated from (B.1) in result:

$$\tfrac{1}{2}\Phi_x^2\Phi^2 = B\Phi^2 - \tfrac{1}{4}\Phi^2(-\Phi\Phi_{xx} - b\Phi^2 + A^2\Phi^{-2}) - \frac{b}{2a}\left(-\Phi\Phi_{xx} - b\Phi^2 + \frac{A^2}{\Phi^2}\right) - \tfrac{1}{2}A^2, \tag{B.5}$$

$$\tfrac{1}{2}\Phi^2\Phi_x^2 - \tfrac{1}{4}\Phi^3\Phi_{xx} - \frac{b}{4a}\Phi\Phi_{xx} = \left(B + \frac{b^2}{4a}\right)\Phi^2 - \tfrac{3}{4}A^2 + \frac{b}{4a}\frac{A^2}{\Phi^2}. \tag{B.6}$$

Divide through by $\frac{1}{4}\Phi_x^2$ and take average over a period:

$$\left(4B + \frac{b^2}{a}\right)\alpha_2 - 3A^2\alpha - \frac{bA^2}{a}\alpha_{-2} = -\left(\frac{b}{a} + \langle\Phi^2\rangle\right). \tag{B.7}$$

Equation (B.4) can be used to remove α in (B.3) and (B.7), yielding:

$$\begin{bmatrix} \dfrac{4B^2}{b}-3A^2, & -\dfrac{3aB}{b}-b \\ A^2\left(-\dfrac{3B}{b}-\dfrac{b}{a}\right), & 4B+\dfrac{b^2}{a}+\dfrac{9a}{4b}A^2 \end{bmatrix} \begin{bmatrix} \alpha_{-2} \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -\dfrac{b}{a} \quad -\langle \Phi^2 \rangle \end{bmatrix}. \tag{B.8}$$

This allows us to calculate α_{-2} and α_2 in terms of known constants and $\langle \Phi^2 \rangle$ (one can show that the determinant is nonsingular when all Φ_k^2 are different and $b \neq 0$). The value of α follows from (B.4). Multiplication of (B.2) by $\Phi_x^{-2}\Phi^{-k}$ then yields the remaining α_n:

$$\alpha_4 = \frac{1}{a}(4B\alpha-2A^2\alpha_{-2}-2b\alpha_2-2), \tag{B.9}$$

$$\alpha_{-4} = \frac{1}{A^2}\left(2B\alpha_{-2}-\tfrac{1}{2}\,a\alpha_2-b\alpha-\left\langle\frac{1}{\Phi^2}\right\rangle\right), \tag{B.10}$$

$$\alpha_6 = \frac{1}{a}(4B\alpha_2-2A^2\alpha-2b\alpha_4-2\langle \Phi^2 \rangle). \tag{B.11}$$

Thus we have all relevant α_n in terms of A, B, $\langle \Phi^2 \rangle$ and $\gamma = \langle 1/\Phi^2 \rangle$.

APPENDIX C

Calculation of A_0. We will need two identities:

1. Add $\frac{1}{2}$ times (B.1) to (B.2) and take average over a period:

$$a\langle \Phi^4 \rangle = \tfrac{4}{3}B-\tfrac{4}{3}b\langle \Phi^2 \rangle. \tag{C.1}$$

2. Multiply (B.9) by α_{-2} and subtract it from 4α times (B.11):

$$-a\alpha_{-2}\alpha_4+2A^2\alpha_{-4}\alpha+2b\alpha^2+2\alpha\gamma-2b\alpha_2\alpha_{-2}-2A^2\alpha_{-2}^2-2\alpha_{-2}+a\alpha\alpha_2 = 0. \tag{C.2}$$

Now from (4.16), (4.17) and (4.18):

$$\begin{aligned} A_0' &= -\frac{a}{2\alpha}(\alpha_4-\alpha\langle \Phi^4 \rangle)-\frac{b}{\alpha}(\alpha_2-\alpha\langle \Phi^2 \rangle)+a\langle \Phi^4 \rangle+b\langle \Phi^2 \rangle \\ &\qquad +4a(A+\bar{D}_{00})\,\langle a\Phi^3+b\Phi|\Psi_{-3}\rangle \\ &= \frac{1}{2\alpha}(4B\alpha-a\alpha_4-2\alpha_2)+4A(A+\bar{D}_{00})\,\langle a\Phi^3+b\Phi|\Psi_{-3}\rangle, \end{aligned} \tag{C.3}$$

using (B.9) and (4.10):

$$A_0' = \frac{1+A^2\alpha_{-2}}{\alpha}+\frac{A^2\alpha_{-2}\left(b\alpha-\dfrac{b\alpha_{-2}\alpha_2}{\alpha}+\dfrac{a\alpha_2}{\alpha}-\dfrac{a\alpha_{-2}\alpha_4}{2\alpha}\right)}{\alpha\left(\gamma+A^2\alpha_{-4}-A^2\dfrac{\alpha_{-2}^2}{\alpha}\right)}$$

and finally, when we write this as one fraction and use (C.2)

$$A_0' = \frac{\gamma + A^2\alpha_{-4}}{\alpha\gamma M}$$

so

$$A_0 = \gamma M A_0' = \frac{\gamma + A^2\alpha_{-4}}{\alpha}.$$

APPENDIX D

Specify $a = -1$, $b = 1$,

$$0 < \Phi_1^2 \leqslant \Phi^2 \leqslant \Phi_2^2 \leqslant \Phi_3^2,$$

$$\langle 1/\Phi^2 \rangle = \oint \frac{dx}{\Phi^2} \Big/ \oint dx = \oint \frac{dx}{\Phi^2} \frac{d\Phi}{dx} \frac{dx}{d\Phi} \Big/ \oint d\Phi \frac{dx}{d\Phi}$$

$$= \int_{\Phi_1{}^2}^{\Phi_2{}^2} \frac{dt}{t\sqrt{(t-\Phi_1^2)(\Phi_2^2-t)(\Phi_3^2-t)}} \Big/ \int_{\Phi_1{}^2}^{\Phi_2{}^2} \frac{dt}{\sqrt{(t-\Phi_1^2)(\Phi_2^2-t)(\Phi_3^2-t)}}, \tag{D.1}$$

take $t-\Phi_1^2 = s^2$; $s_0^2 = \Phi_2^2-\Phi_1^2$; $s_1^2 = \Phi_3^2-\Phi_1^2$; $u = s/s_0$.

$$\text{Numerator} = \frac{1}{s_1} \int_0^{\sqrt{\Phi_2{}^2-\Phi_1{}^2}} \frac{2ds}{(\Phi_1^2+s)\sqrt{\left(1-\frac{s^2}{s_0^2}\right)\left(1-\frac{s^2}{s_1^2}\right)}}$$

$$= \int_0^1 \frac{2du}{s_1(\Phi_1^2+u^2s_0^2)\sqrt{(1-u^2)(1-k^2u^2)}} = \frac{2}{s_1\Phi_1^2}\prod(m, k);$$

$$m = \frac{\Phi_2^2-\Phi_1^2}{\Phi_1^2}; \quad k^2 = \frac{\Phi_2^2-\Phi_1^2}{\Phi_3^2-\Phi_1^2}. \tag{D.2}$$

$$\text{Denominator} = \frac{2}{s_1}K(k).$$

Similarly

$$\int_{\Phi_1{}^2}^{\Phi_2{}^2} \frac{tdt}{\sqrt{(t-\Phi_1^2)(\Phi_2^2-t)(\Phi_3^2-t)}} = \frac{2\Phi_3^2}{s_1}K(m) - 2s_1E(m).$$

Thus

$$\langle 1/\Phi^2 \rangle = \gamma = \prod(m, k)/\Phi_1^2 K(k),$$

$$\langle \Phi^2 \rangle = \Phi_3^2 - (\Phi_3^2-\Phi_1^2)E(k)/K(k). \tag{D.3}$$

The calculations for $a = +1$ are a bit different. Now

$$\Phi_1^2 \leqslant 0 \leqslant \Phi_2^2 \leqslant \Phi^2 \leqslant \Phi_3^2$$

and substitution $t - \Phi_3^2 = -s^2$ leads, after similar calculations, to:

$$\langle 1/\Phi^2 \rangle = \prod (m, k)/\Phi_3^2 K(k), \tag{D.4}$$

$$\langle \Phi^2 \rangle = \Phi_1^2 + (\Phi_3^2 - \Phi_1^2) E(k)/K(k),$$

$$m = \frac{\Phi_2^2 - \Phi_3^2}{\Phi_3^2}, \qquad k^2 = \frac{\Phi_3^2 - \Phi_2^2}{\Phi_3^2 - \Phi_1^2}.$$

Finally $\langle \Phi^4 \rangle$ is given in (C.1):

$$a\langle \Phi^4 \rangle = \tfrac{4}{3} B - \tfrac{4}{3} b \langle \Phi^2 \rangle. \tag{D.5}$$

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