INEQUALITIES FOR AUTOCORRELATION FUNCTIONS AND THE STABILITY OF BALIAN-WETHAMER SYSTEMS*

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Some exact inequalities for thermal autocorrelation functions have been established and discussed. These inequalities cause the zeroes of autocorrelation functions, i.e. antiresonances, to appear if this function has at least two poles, which is typical for ^{3}He(B). It has been shown that the autocorrelation functions of density and transversal current for Balian-Werthamer systems fulfil these inequalities, at least in the quasihomogeneous regime.

1

The negative sign of static autocorrelation functions at the vanishing temperature has been exploited by Leggett [1] in his microscopic proof of Pomeranchuk inequalities [2] for normal Landau Fermi liquids [3, 4]. In paper [5] it has been found that the expansion coefficients of such functions, with respect to the inverse squared frequency, if they exist, should be positive. Moreover, from the last conditions, applied to a normal Fermi liquid, the Pomeranchuk inequalities have been obtained [5]. Both types of inequalities result from the spectral representation of those functions but, nevertheless, the fact that they lead to the same restrictions on Landau parameters [3] is not obvious.

In the paper [5] there appeared the possibility that both types of inequalities lead to the same restrictions on Landau parameters for paramagnetic and normal Fermi liquids only. This has been confirmed by Kołodziejczak [6], for the simplest model of ferromagnetic Fermi liquid, with spherical Fermi surfaces for up and down spins and with a broken spin-rotation symmetry in the effective interaction of quasiparticles. It was stated, that the inequalities discussed in [5] give richer restrictions on Landau parameters than the static inequalities.

The density and transversal current autocorrelation functions for Balian-Werthamer systems [7], i.e. for ^{3}He(B), were investigated by the author [8] in the quasihomogeneous

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regime, i.e. for $kv \ll |\omega|$ ($\boldsymbol{k}$ — the wave vector, v — the Fermi velocity, $\hbar = 1$). We restricted ourselves to the terms of the order of $(kv/\omega)^2$. This did not allow us to check if the density autocorrelation function fulfils all proper inequalities though so did the autocorrelation function of transversal current. Our present task consists in examining of the properties of the density autocorrelation function. It is necessary to take into account also the terms of the order of $(kv/\omega)^4$, [8]. They correspond to the fourth-order perturbation term in our perturbation procedure and lead to very complicated calculations. In order to facilitate them, let us equate all Landau parameters to zero. On the other hand, one can show that the very big parameter A_0 will not influence our calculations, [9]. Note also, that our analysis [8] was performed without any restrictions imposed on Landau parameters.

In paper [8], it was observed that the inequalities for autocorrelation functions, mentioned here, can be extended to nonzero temperatures. On the other hand, the proof of this statement has not been completed in [8]. Hence, to start with, that proof opens further calculations.

2

We restrict ourselves to consideration of retarded autocorrelation functions, i.e.

$$K_\xi(\boldsymbol{x}-\boldsymbol{x}', t-t') = -i\theta(t-t')\langle[\xi(\boldsymbol{x}, t), \xi(\boldsymbol{x}', t')]\rangle, \tag{1}$$

where θ is Heaviside's step function, ξ — one-particle Hermitian second-quantized operator in the Heisenberg picture, $\langle\ldots\rangle$ denotes averaging over grand-canonical ensemble and $[\ldots, \ldots]$ — the commutator. Let us confine ourselves to such ξ, that $P\xi(\boldsymbol{x}, t) = \pm\xi(-\boldsymbol{x}, t)P$, where P is the operator of spatial inversion. In fact, our confinement is physically meaningless, because we usually consider autocorrelation functions of even operators, such as particle density, or odd operators, such as components of current. On the other hand, everybody can combine in ξ the particle density and the component of current. For our class of operators ξ, K_ξ is an even function of the variable $\boldsymbol{x}-\boldsymbol{x}'$, for translationally invariant systems, provided that the density matrix is inversionally invariant. This last fact may not hold for liquid and solid crystals. On the other hand, for both phases of the superfluid ^{3}He, all the vectors determining their possible anisotropy has an axial character. Hence, one can conclude that the equilibrium density matrices of ^{3}He(A) and B commute with the operator P. Hence, for such systems as well as for ordinary liquids, the Fourier transforms of the functions (1) are even functions of the wave vector $\boldsymbol{k}$.

The space-time Fourier transforms of the function (1) has the following spectral representation (cf. [10], formula (3))

$$K_\xi(\boldsymbol{k}, \omega) = \varrho^{-1}\sum_{nm} e^{-\beta E_m}|\xi_{\boldsymbol{k}nm}|^2 \frac{2\omega_{nm}}{(\omega+i\delta)^2-\omega_{nm}^2}. \tag{2}$$

Here $\beta \equiv 1/k_B T$, ϱ is the grand-statistical sum, the summation over n, m goes over all states of the system, with energies E_n, E_m, $\delta = 0^+$, $\omega_{nm} \equiv E_n - E_m$, and $\xi_{\boldsymbol{k}nm}$ denote the *nm* matrix element of the spatial Fourier transform of the Hermitian operator ξ. Taking into account that $K_\xi(-\boldsymbol{k}, \omega) = K_\xi(\boldsymbol{k}, \omega)$ and that $\xi_{-knm} = \xi^*_{knm}$, replacing the summation variable n by m and vice versa, one can rewrite (2) in the form

$$K_\xi(\boldsymbol{k}, \omega) = -\varrho^{-1} \sum_{nm} e^{-\beta E_n} |\xi_{knm}|^2 \frac{2\omega_{nm}}{(\omega+i\delta)^2 - \omega_{nm}^2}. \tag{3}$$

Adding formulae (2) and (3), one finds

$$K_\xi(\boldsymbol{k}, \omega) = \varrho^{-1} \sum_{nm} (e^{-\beta E_m} - e^{-\beta E_n}) |\xi_{knm}|^2 \frac{\omega_{nm}}{(\omega+i\delta)^2 - \omega_{nm}^2}. \tag{4}$$

This formula has been used in the paper [8]. It is easy to see, that only the symmetric part of the matrix $|\xi_{knm}|^2$ i.e. $|\xi^s_{knm}|^2 \equiv \frac{1}{2}[|\xi_{kmn}|^2 + |\xi_{kmn}|^2]$, gives the contribution to the sum (4). The formula (4) can be also rewritten in the form

$$K_\xi(\boldsymbol{k}, \omega) = 2\varrho^{-1} \sum_{nm}{}' (e^{-\beta E_m} - e^{-\beta E_n}) |\xi^s_{knm}|^2 \frac{\omega_{nm}}{(\omega+i\delta)^2 - \omega_{nm}^2}, \tag{5}$$

where the primmed sum is restricted to $\omega_{nm} > 0$. As the direct result of the formulae (4) and (5), one finds that

(*i*) $K_\xi(\boldsymbol{k}, 0) < 0$,

(*ii*) the expansion coefficients of $K_\xi(\boldsymbol{k}, \omega)$ with respect to inverse squared ω are positive, if they exist,

(*iii*) the sign of residues of $K_\xi(\boldsymbol{k}, \omega)$ coincides with the sign of ω.

Such properties of autocorrelation functions were used by the author [8] in the discussion of autocorrelation functions for ^{3}He(B). These inequalities result from an absence of the occupancy inversion in the system; the detailed form of the grand-partition function was not important in the proof. The inequalities of (*i*), with the help of the Ward identities, cf. e.g. [11], for appropriate ξ-operators, reproduce the conditions of thermodynamic stability.

The inequalities (*i*)–(*iii*) allow one to make some conclusions about mutual positions of zeroes and poles of the functions $K_\xi(\boldsymbol{k}, \omega)$. According to (*iii*), $K_\xi(\boldsymbol{k}, \omega)$ tends to minus infinity if ω tends to the positive pole ω_p for $\omega \leqslant \omega_p$ and is coming back from plus infinity for $\omega \geqslant \omega_p$. Hence, and from (*i*), one finds that $K_\xi(\boldsymbol{k}, \omega)$ does not have any zeroes on the interval between $\omega = 0$ and the first positive pole, or has there an even number of zeroes. The same reasons cause the $K_\xi(\boldsymbol{k}, \omega)$ to have one zero, or an odd number of zeroes, on the interval between two adjacent positive poles or negative poles. Moreover, for ω exceeding the greatest pole of $K_\xi(\boldsymbol{k}, \omega)$, this function does not have any zeroes or has an even number of them, cf. (*ii*). Note, that analogous rules for the negative semiaxis ω could be also established due to the obvious ω-symmetry of the function (5).

3

For Balian-Werthamer systems, i.e. $^3He(B)$, the autocorrelation functions of density, longitudinal spin and transversal spin should have their poles in the acoustic region, i.e. for ω, $kv \ll \Delta$ (v — the Fermi velocity, Δ — the energy gap) and also the poles with a gap, for $kv \ll \Delta$, cf. [12, 8, 13]. Hence, each of these functions should have one or an odd number of zeroes lying between the acoustic pole and the pole with a gap. Let us add, that for the autocorrelation function of transversal current the acoustic pole disappears, at least at sufficiently low temperatures [8, 14].

According to our result [8], the density autocorrelation function has only one zero outside the acoustic region. This function, taken with an accuracy up to the terms of the order of $(kv/\omega)^2$, does not allow one to determine why its zero lies in the proper interval i.e. between $\omega = 0$ and the pole with a gap. This fact is not affected by values of Landau parameters, provided that we restrict ourselves to only one interaction harmonic in the pairing channel. Now, we are going to discuss the above function, $S^{00}(\boldsymbol{k}, \omega)$, with an accuracy up to $(kv/\omega)^4$, but for vanishing Landau amplitudes. According to our results, [8, 15], $S^{00}(\boldsymbol{k}, \omega)$ at vanishing Landau amplitudes can be expressed as follows

$$S^{00}(\boldsymbol{k}, \omega) = \nu\{\langle Q1\rangle + t^2[D\langle g\rangle^2 + B\langle g(1-w^2)\rangle^2 - 2C\langle g\rangle \langle g(1-w^2)\rangle](BD-C^2)^{-1}\}. \tag{6}$$

Let us explain the symbols used. ν is the density of states on the Fermi surface per unit volume, $\langle ... \rangle$ denotes the average over spherical angles, g — the Maki-Ebisawa function [16] multiplied by $2\Delta^2$. The function g depends on dimensionless variables $t = \omega/2\Delta$ and $u = kv/2\Delta$; its angular dependence is connected with the variable uw, where $w = \cos\theta$ and θ — the spherical angle. The remaining symbols are defined as follows

$$B = \langle (t^2-u^2w^2)g\rangle, \quad C = \langle (1-w^2)(t^2-u^2w^2)g\rangle,$$
$$D = C - \langle gw^2(1-w^2)\rangle, \quad \langle Q1\rangle = \langle uw(t-uw)^{-1}(1-g)-g\rangle. \tag{7}$$

In order to perform our task we have to expand the function g on the power series with respect to uw. With an accuracy sufficient for us, we have

$$g = g_0 + g_1u^2w^2 + g^2u^4w^4 \equiv g_0(1+\Lambda_1u^2w^2+\Lambda_2u^4w^4), \tag{8}$$

where

$$g_0 = \int_1^\infty [th(\alpha f)(f^2-1)^{-1/2}h]df, \quad \alpha \equiv \Delta/2T, \quad h = (f^2-t^2)^{-1},$$

$$g_1 = \int_1^\infty \{th(\alpha f)(f^2-1)^{-1/2}[2t^{-2}h+(5t^2-3)t^{-2}h^2-4(1-t^2)h^3]\}df,$$

$$g_2 = \int_1^\infty \{th(\alpha f)(f^2-1)^{-1/2}[2t^{-4}h-(7-6t^2)t^{-4}h^2+(17t^4-26t^2+5)t^{-4}h^3 + 4(1-t^2)(5-7t^2)t^{-2}h^4+16(1-t^2)^2h^5]\}df. \tag{9}$$

In the immediate vicinity of T_C, we have

$$g_0 = \pi\Delta/4T\,(1-t^2)^{1/2}, \quad \Lambda_1 = -(2t^2-1)/2t^2(1-t^2) \to -5/12,$$
$$\Lambda_2 = (8t^4-8t^2+3)/8t^4(1-t^2)^2 \to 75/32, \tag{10}$$

where the limits of Λ_1 and Λ_2 are taken at $t^2 = 3/5$. On the other hand, at $T = 0$ we have

$$g_0 = (\arcsin t)/t(1-t^2)^{1/2}, \quad \Lambda_1 = -(g_0^{-1}+2t^2-1)/2t^2(1-t^2),$$
$$\Lambda_2 = [8t^4-8t^2+3+3(2t^2-1)g_0^{-1}]/8t^4(1-t^2)^2. \tag{11}$$

Substituting (8) into the elements of (6), performing the average over spherical angles and collecting the terms up to the fourth order, we get

$$B = g_0[t^2+\tfrac{1}{3}u^2(t^2\Lambda_1-1)+\tfrac{1}{5}u^4(\tfrac{3}{5}\Lambda_2-\Lambda_1)],$$
$$C = \tfrac{2}{3}g_0[t^2+\tfrac{1}{5}u^2(t^2\Lambda_1-1)+\tfrac{3}{35}u^4(\tfrac{3}{5}\Lambda_2-\Lambda_1)],$$
$$D = \tfrac{2}{3}g_0[t^2-\tfrac{1}{5}+\tfrac{1}{5}u^2(t^2\Lambda_1-\tfrac{3}{7}\Lambda_1-1)+\tfrac{3}{35}u^4(\tfrac{2}{45}\Lambda_2-\Lambda_1)], \tag{12}$$

and

$$\langle g\rangle = g_0(1+\tfrac{1}{3}u^2\Lambda_1+\tfrac{1}{5}u^4\Lambda_2), \quad \langle g(1-w^2)\rangle = \tfrac{2}{3}g_0\left(1+\frac{u^2}{5}\Lambda_1+\tfrac{3}{35}\Lambda_2u^4\right),$$

$$\langle Q1\rangle = -g_0\left[1-\tfrac{1}{3}u^2\left(\frac{g_0^{-1}-1}{t^2}-\Lambda_1\right)-\frac{u^4}{3}\left(\frac{5}{3g_0}-\tfrac{5}{3}-\Lambda_1-\tfrac{3}{5}\Lambda_2\right)\right]. \tag{13}$$

Note, that in the formula (12) and (13) $t^2 = 3/5$ was put in the fourth order terms, since these formulae will be used in the vicinity of that point, for investigation of the mutual position of the zero and pole of S^{00}. Taking into account (12), we find the denominator of S^{00}. With the appropriate accuracy we have

$$BD-C^2 = \tfrac{4}{15}g_0t^2\{\tfrac{5}{6}(t^2-\tfrac{3}{5})-u^2\delta_1+u^4[-\tfrac{2}{175}\Lambda_1^2-\tfrac{12}{35}\Lambda_1+\tfrac{1}{6}+\tfrac{4}{105}\Lambda_2]\}$$
$$\equiv \tfrac{4}{15}g_0^2t^2Z, \tag{14}$$

where

$$\delta_1 = -\tfrac{1}{6}(\Lambda_1-t^{-2})(4t^2-1)+\tfrac{3}{14}\Lambda_1. \tag{15}$$

The function δ_1 at $t^2 = 3/5$ describes the dispersion of the longitudinal excitations with a gap, for $A_l = 0$ if $l \geqslant 1$, [8]. This fact becomes clear also via expression (14). Calculating the numerator of the second term in the curly bracket of (6), with the help of (12) and (13), we obtain a very complicated expression which will not be reproduced here. On the other hand, if we transform the curly bracket in (6) to the form of the fraction with the denominator $BD-C^2$, then the numerator of this fraction becomes much simpler. In particular, all terms proportional to Λ_1^2 and Λ_2 are cancelled in this numerator. Finally, the function (6) taken with our accuracy, has the following form

$$S^{00}(k,\omega)/\nu = \tfrac{1}{3}t^{-2}u^2[\tfrac{5}{6}(t^2-\tfrac{3}{5})-u^2(\delta_1-\tfrac{4}{45}g_0)]Z^{-1}, \tag{16}$$

cf. (14). Looking for zeroes of (16), with the accuracy up to u^2-order terms, one finds that S^{00} vanishes at

$$t = (3/5)^{1/2}[1+u^2(\delta_1 - \tfrac{4}{45} g_0)|_{t^2=0.6}]. \tag{17}$$

Moreover, S^{00} has a pole at $t = (3/5)^{1/2}$ $(1+u^2\delta_1|_{t^2=0.6})$. Since $g_0 > 0$ for $t^2 < 1$, zero of S^{00} lies in the proper ω-interval, from the point of view of stability of the system. Let us repeat that the inclusion of Landau's parameter A_0 will not affect this statement [9]. For the additional inclusion of the Landau parameter A_1, it is necessary to handle a few hundred terms in calculations analogous to those reported here. If we include also A_2, then the corresponding number of terms grows to a few thousands. Hence, it is rather impossible to give such a discussion stability of properties in an analytic way.

4

Formula (14) allows us to find the dispersion of the longitudinal excitations with a gap, with the accuracy up to u^4-order terms. Substituting there $t = (3/5)^{1/2}(1+\delta_1 u^2+\delta_2 u^4)$ and looking for zeroes of (14), one finds that

$$\delta_2 = -[\tfrac{1}{2}\delta_1^2 - (3/5)^{1/2}\delta_1'\delta_1 - \tfrac{2}{175}A_1^2 - \tfrac{12}{35}A_1 + \tfrac{1}{6} + \tfrac{4}{105}A_2]|_{t^2=0.6}, \tag{18}$$

where δ_1' is the derivative of δ_1 (15) with respect to t. At $T = T_C$, δ_2 is equal to -0.0819, at $T = 0$ $\delta_2 = -0.3502$ and it seems that δ_2 remains negative in the whole interval $(0, T_C)$. The formula (16) allows one to compute the residuum of the function $S^{00}(k, \omega)/v$ at the pole at $t = (3/5)^{1/2}(1+\delta_1 u^2)$. This residuum is given by $4(3/5)^{1/2}g_0u^4/81$, cf. (*iii*).

Our final results, such as (14), (16) and (17) are surprisingly much simpler than the intermediate ones, which have not been reproduced here. This is a rather typical situation for Balian-Werthamer systems, cf. e.g. [8], manifesting some hidden symmetry of such systems.

Zeroes of the magnetic susceptibility of ferromagnetic metals were theoretically predicted and called antiresonances by Kaganov [17]. They have become objects of the experimental investigations [18]. In this paper we have proved, that the appearance of, at least, two poles of the autocorrelation function leads to the appearance of, at least, one antiresonance. It seems that necessary antiresonances of longitudinal and transversal spin susceptibility of $^3He(B)$ should also be good objects of experimental investigations.

Note added in proof. Irrespective of estimations of the number of terms made at the end of the third part of the paper, one can prove by a slightly tricky method that the residuum of S^{00} is always positive for $\omega \gg kv$ and any A_l fulfilling Pomeranchuk inequalities.

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