Vol. A58 (1980) ACTA PHYSICA POLONICA No 6

SOLITON LIKE SOLUTION OF THE MASSIVE THIRRING MODEL IN A SINGULAR PERTURBATION APPROACH

By R. S. Banerjee, A. Roy Chowdhury and T. Roy

Department of Physics, Jadavpur University, Calcutta*

(Received March 23, 1979; final version received March 21, 1980)

A singular perturbation technique has been applied to the massive Thirring model, as a first step for exploring the possibility of applying the technique of reductive perturbation to extract soliton like structure from field equations of quantum field theory, for which no other method of solution is known to exist. The results obtained have all the features of the exact solution and are essentially non perturbative with respect to the coupling constant, and can be used as a model for fundamental particles and quark confinement. The only unusual feature, not found previously, is the connection between the mass and coupling constant, which becomes essential for the existence of the solution. The physical implication of such a constraint is not yet clear, and perhaps requires further study of other field equations by this technique. The structure of the computations suggest that the method will be usefull in obtaining the multi-soliton structure for the ϕ^4, ϕ^3, ϕ^6 theory for which there is no Bäcklund transformation or inverse scattering technique. Also the method can be quite successfully applied for obtaining a multisolution structure of coupled field equations of the quantum field theory.

PACS numbers: 03.40.Kf, 03.65.Ge

1. *Introduction*

Recently there have been vigorous attempts to explore the soliton like solutions of non-linear equations. The packet structure of such solutions is usefull both for the description of extended solutions of elementary particles [1] and interactions of nonlinear waves [2]. Incidentally, it should be mentioned that such solutions could not be constructed unless some technique for the exact solution of the equations or some perturbation procedure could be devised which does not utilise the expansion in the coupling constant. It can be easily demonstrated that the usual perturbation in the coupling constant does not yield soliton like behaviour. The method for the exact solution was introduced by Geardner et al. [3] and also by Lax [4]. But one should not be over enthusiastic about such mathematical recipes as they hold only for a class of nonlinear equations, and many of the equations of quantum field theory do not fall in this category. Furthermore the above mentioned

* Address: Department of Physics, Jadavpur Unicersity, Calcutta-700032, India.

method is valid only in the space-one time domain, and in general not extensible to many space variables. But reductive perturbation has been already applied in three dimensions and yield satisfactory results [5]. At present the existence and behaviour of classical lumps for many equations have been demonstrated only by vigorous computer experiments though in many cases a need of analytical treatment was hardly felt. A few years back Oikawa and Yajima [6] formulated a version of singular perturbation known as the reductive perturbation technique for the solution of highly dispersive and nonlinear equations of plasma physics. The merit of the procedure lies in the fact that it does not rely upon an expansion in the coupling parameter, and so there is no need for weak coupling [7]. Also the small parameter (ε) is not related in any way to the dynamics of the particular equation under consideration and so it has a generalised character. One really proceeds by scaling the space and time variables by ε and by eliminating secular terms in each order of ε. In the wake of the search for soliton like solutions and bag like structure for the equations of quantum field theory, we have tried to visualise the effect of a singular perturbation and a well known, exactly solvable equation — that of the Thirring model with mass. In the past it has been seen that the equations of Thirring model always served as a laboratory for the test of any particular theory. So, such is our motivation in the following. As the formalism is not very popular to particle physicists we have dealt with the details of philosophy and methodology and now describe our results in the case of the massive Thirring model.

2. *Basic formulation*

It has been observed that in many classical and quantum mechanical equations the chief trouble lies with the non existence of a small parameter, for effecting the usual form of perturbation theory. Quite often, it may so happen that the coupling parameters occuring in the theory (e.g. the strong interaction coupling in hadron physics, the parameters of mode-mode coupling, or those occurring in the governing equations of plasma physics) are quite large and usual perturbation approach is meaningless. Furthermore the trouble with the nonlinear equations is that they possess many types of solutions of which only the soliton like solution is of importance to us. One straightforward approach to these types of solution is by the method of exact solution. Unfortunately, only a few equations admit such solutions. So what is essential is a perturbation theory which is not in any dynamical parameter leading directly to the soliton like structures. This is the reductive perturbative approach.

In this technique the small parameter ε is the scaling length of the space and time variable, introduced to separate the slow and rapid variations of the dynamical quantities over a wide range. All the quantities are considered to be expansible in powers of ε, that is, $\sum \varepsilon^n \psi_n(\xi, \tau)$ where (ξ, τ) are the usual shifted and stretched variables written in terms of (x and t). Usually one sets

$$\xi_j = \varepsilon^a [x - \lambda_j t - \varepsilon^{1-a} \phi_j(x, t)],$$
$$\tau = \varepsilon^{a+1} t, \tag{1}$$

and the nonlinear field $U(x, t)$ is supposed to be a combination of n "quasi-simple" waves in every order of ε. That is one sets

$$U = U_0+\varepsilon U_1+\varepsilon^2 U_2+ \dots \tag{2}$$

and each U_j is assumed to be of the above form. The explicite functions $\phi_j(x, t)$ are introduced in ξ_j to account for the variation of the velocity of the waves over space-time, due to interaction. Lastly, we will see that these yields the usual expressions for the phase shift and time delay for the scattering of soliton, as obtained from the exact solutions. Suppose for simplicity that we consider a two-soliton situation. Then one can explain the origin of the factor ε^{1-a} as follows. The variation in the wave velocity due to the two wave interaction is expected to be proportional to the product of wave amplitude and the interaction time. The former is of the order ε. The latter is considered to be the time during which the two waves pass through each other and then estimated by dividing the width of the wave $(\sim O(\varepsilon^{-a}))$ with their relative velocity $(\sim O(1))$ i.e. being of the order ε^{-a}. Therefore the variation in the wave velocity is of the order ε, $\varepsilon^{-a} = \varepsilon^{1-a}$. A similar consideration holds for many wave interactions. Once the parameter ε is fixed one has to set the equation under consideration in the form;

$$\frac{\partial U}{\partial t}+A\frac{\partial U}{\partial x}+B(U) = 0, \tag{3}$$

where U is column vector and A and B metric functions of U. The search for the soliton solution starts by finding some U_0 such that

$$B(U_0) = 0, \tag{4}$$

where U_0 is a constant vector. Then one sets

$$U = U_0+\sum_{\alpha=1}^{\alpha}\varepsilon^{\alpha}\sum_{l,n=-\alpha}^{\alpha}U^{\alpha}_{l,n}(\xi_1, \xi_2, \tau)Z_{l,n}, \tag{5}$$

for the determination of the space-time structure of the functions $U^{(\alpha)}_{l,n}$. The ansatz (5) is valid in the case of two "quasi-simple waves with

$$\xi_s = \varepsilon[x-\lambda_s t-\sum_{r=0}^{\alpha}\varepsilon^r\psi^{(r)}_s(\xi_1, \xi_2, \tau)-\gamma_s],$$
$$\tau = \varepsilon^2 t, \tag{6}$$

along with

$$Z_{ln} \equiv \exp[ilx_1+inx_2],$$
$$x_s = k_s x-\omega_s t+\sum_{r=1}^{\alpha}\varepsilon^r\Omega^{(r)}_s(\xi_1, \xi_2, \tau),$$
$$\lambda_s = \left(\frac{\partial\omega}{\partial k}\right)_{k=k_s}, \quad s = 1, 2, \tag{7}$$

γ_1, γ_2 are arbitrary constants and $\Omega^{(r)}_s$, $\psi^{(r)}_s$ are again introduced to account for the frequency shifts and the orbit modification due to nonlinear interaction. Substitution of the form (5) along with (6) and (7) in the above equation (3) yields in each order of ε matrix equations for the determination of $U_{l,n}^{(\alpha)}$. These matrix equations can yield a nonlinear solution only if the determinant of the coefficients vanish. In the zero order case this essentially yields the linearised dispersion relation. It is quite easy to observe that such a method can be generalised to study the interaction of more than two quasi-simple waves in the case of both non-linear and highly dispersive equations. It will turn out later that these "quasi-simple" waves are nothing but solitons. One of the most interesting properties of the quasi-simple waves is that they are always localized in space, that is the amplitudes go to zero as $x \to \pm\alpha$. Furthermore, a peculiar feature that has become prominent is that almost all the non-integrable (in the sense that they are not amenable to the inverse scattering analysis) non-linear partial differential equations are reducible to a non-linear Schroedinger equation in some order of reductive perturbation, whose exact multisoliton solutions are known. In quantum field theory, which can represent the physical situation, we work in a four dimensional world, and up till now there is no known method of solution of the field equations which will lead to soliton like structures and hence quark confinement. Up till now the most exhaustively studied field theoretical model is the Sine–Gordon (SG) equation [8], but with one space and one time domain. At this point one important aspect of the SG equation is worth mentioning. It has been found that though the SG equation is completely integrable but the corresponding ϕ^4 theory

$$(\partial_x^2 - \partial_t^2)\phi = g\left(\phi - \frac{\phi^3}{3}\right),$$

is not, though it can be considered as an approximate version of the SG system. Reductive perturbation may yield some clue to these important and crucial properties of the nonlinear equations. So that in the absence of any inverse scattering formalism for the nonlinear equations the reductive perturbation technique seems to be the only tool for the introduction of solitons for such equations.

3. *Application to Thirring model*

The equations of the massive Thirring model [9]

$$(\gamma_\mu \partial_\mu + m)\psi + g(\bar{\psi}\gamma_\mu \psi)\gamma_\mu \psi = 0, \tag{8}$$

which can be put into the form (3) with;

$$\psi = U(x, t); \qquad U = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \qquad B = (2g\psi_1\psi_2 + m)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{9}$$

where we have used the usual representation of γ_μ in two dimensional space time. The determinant condition yeilds

$$U^{(0)} = \begin{pmatrix} 1 \\ \dfrac{-m}{2g} \end{pmatrix}; \qquad U_{0,0}^{(1)} = \begin{pmatrix} 1 \\ \mp \dfrac{(2g \mp 1)}{4g} \end{pmatrix}, \tag{10}$$

along with the non-trival condition

$$g = m \pm \tfrac{1}{2}. \tag{11}$$

This condition which is nothing but a restriction on the allowed value of the parameters, is the only unexpected feature of the application of the singular perturbation to the Thirring model. If one then proceeds to the second order in ε one obtains;

$$W_{0,0}U^{(2)}_{0,0} - (\lambda_1 I - A_0)\frac{\partial U^{(1)}_{0,0}}{\partial \xi_1} - (\lambda_2 I - A_0)\frac{\partial U^{(1)}_{0,0}}{\partial \xi_2}$$

$$+\tfrac{1}{2}\sum_{l',n'} \nabla\nabla B_0 \; U^{(1)}_{0-l',0-n'} U^{(1)}_{l',n'} = 0,$$

$$W_{l,n} = -i(l\omega_1 + n\omega_2)I + i(lk_1 + nk_2)A_0 + \nabla B_0. \tag{12}$$

But the equation gets simplified as $U^{(1)}_{0,0}$ is constant. Also it is important to notice that Det $(W_{0,0}) = 0$. So proceeding as in reference [4] we can obtain

$$U^{(2)}_{0,0} = \begin{pmatrix} 1 \\ \dfrac{2m}{g} \end{pmatrix}, \quad R_s = \begin{pmatrix} \dfrac{m/a_s}{1} \end{pmatrix},$$

with

$$a_s = i(k_s - \omega_s) + \frac{m^2}{2g}, \qquad s = 1, 2,$$

where R_1 and R_2 are the eigenvectors of $W_{1,0}$ and $W_{0,1}$ required for proceeding with higher order calculation. It is rather important to notice that none of the results derived above are defined for $g = 0$, suggesting that it is impossible to deduce the above results in the usual perturbation analysis about $g = 0$. Now proceeding to the third order in ε we obtain;

$$W_{1,0}U^{(3)}_{1,0} - (\lambda_1 I - A_0)\frac{\partial U^{(2)}_{1,0}}{\partial \xi_1} - (\lambda_2 I - A_0)\frac{\partial U^{(2)}_{1,0}}{\partial \xi_2} + \frac{\partial U^{(1)}_{1,0}}{\partial \tau}$$

$$+\left[(\lambda_1 I - A_0)\frac{\partial \psi^{(0)}_1}{\partial \xi_1} + (\lambda_2 I - A_0)\frac{\partial \psi^{(0)}_1}{\partial \xi_2}\right]\frac{\partial U^{(1)}_{1,0}}{\partial \xi_1}$$

$$-i\left[(\lambda_1 I - A_0)\frac{\partial \Omega^{(1)}_1}{\partial \xi_1} + (\lambda_2 I - A_0)\frac{\partial \Omega^{(1)}_1}{\partial \xi_2}\right]U^{(1)}_{1,0} + \sum_{l,n'} \nabla\nabla B_0 \colon U^{(2)}_{1-l',n'} U^{(1)}_{l',n'}$$

$$+\tfrac{1}{6}\sum_{l',n',l'',n''} \nabla\nabla B_0 \colon U^{(1)}_{1-l'-l'',-n'-n''} U^{(1)}_{l',n'} U^{(1)}_{l'',n''} = 0. \tag{13}$$

Multiplying equation (13) by L_1 (row vector corresponding to (R_1) from the left and using

$$U^{(1)}_{1,0} = \phi_1(\xi_1, \xi_2, \tau)R_1,$$

$$U^{(1)}_{0,1} = \phi_2(\xi_1, \xi_2, \tau)R_2,$$

$$U_{l,n}^{(1)} = 0 \quad \text{for} \quad |l|+|n| \neq 1,$$

$$L_1 W_{1,0} = 0, \quad (\lambda_1-\lambda_2)\frac{\partial \phi_1}{\partial \xi_2} = 0,$$

$$U_{1,0}^{(2)} = \phi_1^{(2)}(\xi_1, \xi_2, \tau)R_1 - i\frac{\partial \varphi_1}{\partial \xi_1}\cdot\frac{dR_1}{dk_1},$$

$$(\lambda_1-\lambda_2)\frac{\partial \phi_1^{(2)}}{\partial \xi_2} + A_1\phi_1^{(2)} + \left[\frac{\partial \phi_1}{\partial \tau} - \frac{i}{2}\frac{d^2\omega_1}{\partial k_1^2}\frac{\partial^2 \phi_1}{\partial \xi_1} + i\alpha_1|\phi_1|^2\phi_1\right]$$

$$+i\left[(\lambda_1-\lambda_2)\frac{\partial \Omega_1^{(1)}}{\partial \xi_2} + \beta_1|\phi_2|^2 + A_2\right]\phi_1 + \left[(\lambda_1-\lambda_2)\frac{\partial \psi_1^{(0)}}{\partial \xi_2} + A_3\right]\frac{\partial \phi_1}{\partial \xi_1} = 0, \tag{14}$$

where the coefficients occurring in equation (14) are calculated from equation (8, 12) of reference [7]. As the calculations are quite straightforward and lengthy we just quote some results for A_1, A_2, A_3

$$A_1 = \left(1+\frac{m^2}{a_1^2}\right)^{-1}\left[4g - \frac{4m^2}{a_1^2} \mp \frac{2m}{a_1}(2g\mp 1) + \frac{m^3(2g\mp 1)}{2a_1^2 g}\right],$$

$$A_2 = \left(1+\frac{m^2}{a_1^2}\right)^{-1}\left[6g + \frac{12m^2}{a_1} - \frac{4m^4}{a_1^2 g} \mp \frac{m}{a_1}(2g\mp 1) + \frac{m^2}{8a_1^2 g}(2g\mp 1)^2\right],$$

$$A_3 = \left(1+\frac{m^2}{a_1^2}\right)^{-1}\left[\mp\frac{m^3}{2a_1^3 g}(2g\mp 1) - \frac{2m^2}{a_1^2} \mp \frac{m^2}{a_1^2}(2g\mp 1)\right]. \tag{15}$$

After the coefficients are known, the most important step follows, that of imposing non-secularity on the solution of equation (14), which yields

$$\frac{\partial \phi_1}{\partial \tau} - \frac{i}{2}\frac{d^2\omega_1}{dk_1^2}\frac{\partial^2 \phi_1}{\partial \xi_1^2} + i\alpha_1|\phi_1|^2\phi_1 = 0, \tag{16a}$$

$$(\lambda_1-\lambda_2)\frac{\partial \Omega_1^{(1)}}{\partial \xi_2} + \beta_1|\phi_2|^2 + A_2 = 0, \tag{16b}$$

$$(\lambda_1-\lambda_2)\frac{\partial \psi_1^{(0)}}{\partial \xi_2} + A_3 = 0, \tag{16c}$$

$$(\lambda_1-\lambda_2)\frac{\partial \phi_1^{(2)}}{\partial \xi_2} + A_1\phi_1^{(2)} = 0, \tag{16d}$$

determining $\phi_1^{(2)}$, $\psi_1^{(0)}$, $\Omega_1^{(1)}$ and ϕ_1. Similar equations could be set up for ϕ_2 and its associated quantities. Equation (16a) is the non-linear Schrödinger equation having the well known soliton solution;

$$\phi_1(x, t) = \varrho_0 e^{i\bar{\phi}(x,t)}\operatorname{sech}^2[x-\bar{\eta}t],$$

where $\overline{\phi}$ is the instantaneous phase ϱ_0, $\overline{\eta}$ all are obtainable in terms of $\frac{1}{2}\frac{d^2W_1}{dk_1^2}$ and α_1.

Similar results are also obtained for ϕ_2. Corresponding to these solitons, our original equation (8) will have solutions with similar properties.

4. Discussion

Some pertinent questions can be raised about the results obtained in the preceding paragraphs. The most important point is the relation between mass and coupling constant obtained in equation (11) in the course of computation. As in the usual case the self interacting Fermi field can have any value of mass and coupling constant, one can raise the question of the significance of such a result. However, one should remember that in general there does not exist any criterion, for judging the applicability of the reductive perturbation technique to a nonlinear equation. So that equation (11) yields really the sector in the values of m and g in which the method is applicable to this particular equation. But one should keep in mind that this type of restriction may not occur in other cases. Except for this point the solutions have all the features of exact solutions obtained via the inverse scattering transform, (IST). Though for a long time IST for the Thirring model was unknown, it has been recently discovered by Kaup et al. [10]. Furthermore, it has been found, to the surprise of all the investigators that the reductive perturbation reduces the original nonlinear equation to a nonlinear Schrödinger equation, whose soliton solutions are quite well known. Lastly, one can see from our above discussions that our method will be quite suitable for exploring the many-soliton sectors of coupled equations of the quantum field involving a scaler field ϕ and a complex scalar field ψ, interacting via $g_1\phi^2\psi\psi^* + g_2\phi^4 + g_3(\psi^*\psi)^2$ for which uptill now no method of solution exists for even one of many soliton sectors. Investigations of values of g_1, g_2, g_3 and masses of ϕ and ψ, if any constant of the form of equation (11) arises, are at present under consideration.

Editorial note. This article was proofread by the editors only, not by the authors.

REFERENCES

[1] H. Joos, DESY preprint 1976.
[2] M. J. Ablowitz, *Stud. Appl. Math.* **58**, 17 (1978).
[3] C. S. Gardner, J. M. Greene, M. D. Kruskal, R. M. Miura, *Phys. Rev. Lett.* **19**, 1095 (1967).
[4] P. D. Lax, *Commun. Pure and Appl. Math.* **21**, 467 (1968).
[5] S. G. Tagore, *Phys. Lett.* **56A**, 379 (1976).
[6] Oikawa, Yajema, *Proc. Phys. Soc. Japan* **37**, 486 (1974).
[7] A. H. Nayafeh, *Perturbation Theory*, Academic Press 1972.
[8] R. K. Bullough, R. K. Dodd, *Synergetics*, Ed. H. Haken Springer, Berlin 1977, pp. 92–119.
[9] S. Coleman, *Phys. Rev.* **D11**, 2088 (1975).
[10] D. J. Kaup et al., *Lett. Nuovo Cimento* **20**, 325 (1977).