ON APPLICABILITY OF THE SELF-CONSISTENTLY RENORMALIZED SPIN WAVE THEORY FOR HEISENBERG FERROMAGNET IN THE CRITICAL REGION*

By J. Szaniecki

Institute of Molecular Physics of the Polish Academy of Sciences, Poznań**

(Received December 7, 1979; revised version received January 29, 1980)

The self-consistently renormalized spin wave theory for cubic Heisenberg ferromagnet is extended by turning to account a selected series of dynamical diagrams containing energy denominators. The term representing a sum of this series can be obtained only with the proviso that the wave vectors are small. On doing it, this sum is subsequently assumed to be valid for all spin wave vectors and thus it enables computation of the transition temperature which proves to be by 3.5 per cent larger than the one derived with the aid of the high temperature series expansion method. As to the magnetization series, it is found to exhibit a temperature dependence similar to that due to molecular field theory and the like. On keeping in view, however, that in the vicinity to the Curie temperature the free energy of a ferromagnet is for the most part contributed to by the magnons endowed with large momenta and with large average population numbers, the magnetization critical exponent is shown to be preferably 1/3 than 1/2.

PACS numbers: 75.30.Ds, 75.40.Fa

1. Introduction

Throughout this paper, we shall investigate the problem of applicability of the self-consistently renormalized spin wave theory (SCR) for cubic Heisenberg ferromagnet close to the critical point. Such analysis for the classical limit $S \to \infty$, with S being the resultant atomic spin quantum number, was given by Loly [1]. Jeżewski [2] achieved a more comprehensive examination of that problem. Both papers have made use of usual Bloch theory [3].

Here, we shall aim at extending the SCR by inserting therein a series of ladder diagrams being composed of energy denominators [4]. Since carrying out the exact summation of

* This work was supported by the Polish Academy of Sciences under project MR. I. 9.

** Address: Instytut Fizyki Molekularnej PAN, Smoluchowskiego 17/19, 60-179 Poznań, Poland.

this series is clearly unfeasible, we shall do it by allowing for small spin wave vectors, because in this case the summation procedure is performable. The sum obtained will be assumed to hold for all wave vectors. In that way, we will be able to establish the expression for the free energy of a ferromagnet and therefrom compute the transition temperature proving by 10 per cent smaller and the magnetization in the critical point twice as much as in the usual Bloch theory. Moreover, the magnetization critical exponent derived along these lines turns out to be equal to 1/2.

On the other hand, owing to the thermal excitation close to the Curie temperature the spin waves with very large momenta and very large mean population numbers become predominant. Therefore, as it will be shown in Section 5 all integrations can be confined to a small interval from zero to a wave vector λ and these integrations entail reducing the critical magnetization exponent from 1/2 to 1/3. To a certain degree, this argumentation is connected with Kadanoff's considerations concerning the Renormalization Group and the Scaling Theory [5] (see also [6] and [7]).

An interesting theory using T-matrix method and touching on the subject of thermodynamics of Heisenberg ferromagnet within the entire range of temperatures from absolute zero to the Curie temperature was formulated in papers [8], [9] and [10].

2. *The Hamiltonian and the partition function of cubic Heisenberg ferromagnet*

Here, we adopt Dyson's [11] scheme of spin wave theory. As for the Hamiltonian, we confine ourselves to Heisenberg exchange- and Zeeman spin operator terms which have to describe the cubic isotropic ferromagnet. We obtain

$$\tilde{H} = E_0 + H_0 + H_I, \tag{2.1}$$

$$E_0 = -LSN - \tfrac{1}{2} JNS^2 \gamma_0, \tag{2.2}$$

$$H_0 = \sum_{\lambda} (L + \varepsilon_\lambda) a_\lambda^* a_\lambda, \tag{2.3}$$

$$\varepsilon_\lambda = JS(\gamma_0 - \gamma_\lambda), \tag{2.4}$$

$$\gamma_\lambda = \sum_{\delta} \exp i\boldsymbol{\lambda} \cdot \boldsymbol{\delta}, \tag{2.5}$$

$$H_I = -\tfrac{1}{4} JN^{-1} \sum_{\lambda\varrho\sigma} \Gamma^{\lambda}_{\varrho,\sigma} a^*_{\sigma+\lambda} a^*_{\varrho-\lambda} a_\varrho a_\sigma, \tag{2.6}$$

$$\Gamma^{\lambda}_{\varrho,\sigma} = \gamma_\lambda + \gamma_{\lambda+\sigma-\varrho} - \gamma_{\lambda+\sigma} - \gamma_{\lambda-\varrho}, \tag{2.7}$$

where L stands for the magnetic field strength multiplied by Bohr magneton and Landé's isotropic factor, S is the quantum number of resultant atom spin, N determines the number of lattice sites in the crystal under consideration, J denotes the exchange integral between nearest neighbours, ε_λ is the energy of independent spin waves, $\boldsymbol{\lambda}, \boldsymbol{\varrho}, \boldsymbol{\sigma}$ are reciprocal lattice vectors, $\boldsymbol{\delta}$ point from one lattice site to all its nearest neighbours and a_λ^*, a_λ represent the creation and annihilation Bose-operators of ideal spin waves, respectively.

On neglecting kinematic interaction of spin waves, we get for the partition function (see [12], [13], [14])

$$Z = \mathrm{Tr}\,(e^{-\beta\tilde{H}}) = e^{-\beta E_0}\,\mathrm{Tr}\,(e^{-\beta H_0})\,\frac{\mathrm{Tr}\,(e^{-\beta H_0}\hat{S}(\beta))}{\mathrm{Tr}\,e^{-\beta H_0}}$$

$$= \exp\,[\beta E_0 + \sum_{\lambda} \ln\,(1+\bar{n}_{\lambda}) + \sum_{p=1}^{\infty} D_p], \quad \beta = 1/kT, \tag{2.8}$$

where

$$\bar{n}_{\lambda} = [\exp \beta(L+\varepsilon_{\lambda})-1]^{-1} \tag{2.9}$$

is the average population number of independent spin waves, and

$$\hat{S}(\beta) = e^{\beta H_0}e^{-\beta(H_0+H_I)} = \hat{T}\exp\,[-\int_0^{\beta} d\tau H_I(\tau)], \tag{2.10}$$

$$H_I(\tau) = e^{\tau H_0}H_I e^{-\tau H_0} \tag{2.11}$$

with $\hat{T}$ being Wick's ordering symbol [15].

The quantities

$$D_p = \frac{(-1)^p}{p}\int_0^{\beta} d\tau_1 \int_0^{\beta} d\tau_2 \ldots \int_0^{\beta} d\tau_p \langle \hat{T}[H_I(\tau_1)H_I(\tau_2)\ldots H_I(\tau_p)]\rangle_c,$$

$$p = 1, 2, 3, \ldots, \tag{2.12}$$

where the letter c denotes that exclusively connected graphs i.e. those not divisible into several independent parts are allowed for, and

$$\langle \hat{A} \rangle = \frac{\mathrm{Tr}\,(e^{-\beta H_0}\hat{A})}{\mathrm{Tr}\,e^{-\beta H_0}}, \tag{2.13}$$

may be called dynamical graphs (diagrams). They are owing to the energy operators H_I responsible for dynamic interaction of spin waves. The graphs D_p can be figured out with the aid of the quantum field theory method developed by Matsubara [12] and by having recourse to Thouless [13] and Wick [15] theorems. Along these lines, the average value of $\hat{T}$-product of operators is equal to the sum of products of all possible contractions (or propagation functions) of pairs of them. The contractions (propagators) have the form:

$$a_{\varrho}^*(\tau_1)^{\cdot}a_{\sigma}(\tau_2)^{\cdot} = \delta_{\varrho,\sigma}e^{(L+\varepsilon_{\varrho})(\tau_1-\tau_2)}[\theta_{1,2}\bar{n}_{\varrho}+\theta_{2,1}(\bar{n}_{\varrho}+1)], \tag{2.14}$$

$$a_{\varrho}(\tau_1)^{\cdot}a_{\sigma}^*(\tau_2)^{\cdot} = \delta_{\varrho,\sigma}e^{-(L+\varepsilon_{\varrho})(\tau_1-\tau_2)}[\theta_{1,2}(\bar{n}_{\varrho}+1)+\theta_{2,1}\bar{n}_{\varrho}], \tag{2.15}$$

$$a_{\varrho}^*(\tau_1)^{\cdot}a_{\sigma}^*(\tau_2)^{\cdot} = a_{\varrho}(\tau_1)^{\cdot}a_{\sigma}(\tau_2)^{\cdot} = 0, \tag{2.16}$$

$$a_{\varrho}^*(\tau)^{\cdot}a_{\sigma}(\tau)^{\cdot} = \delta_{\varrho,\sigma}\bar{n}_{\varrho}, \tag{2.17}$$

$$a_{\varrho}(\tau)^{\cdot}a_{\sigma}^*(\tau)^{\cdot} = \delta_{\varrho,\sigma}(\bar{n}_{\varrho}+1), \tag{2.18}$$

$$\theta_{i,k} \equiv \theta(\tau_i - \tau_k) = \begin{cases} 1, & \tau_i \geqslant \tau_k, \\ 0, & \tau_i < \tau_k, \end{cases} \tag{2.19}$$

$$\delta_{\varrho,\sigma} = \begin{cases} 1, & \varrho = \sigma, \\ 0, & \varrho \neq \sigma. \end{cases} \tag{2.20}$$

The dots in above relations mark which pair of operators has to be contracted. The propagators (2.14) and (2.15) can be graphically represented in the form of a finite line with the arrow showing the propagation direction of a spin wave (magnon). In the case of (2.14) the magnon endowed with the wave vector ϱ is created in the point τ_1 and propagates towards the point τ_2 wherein it annihilates. As to the equation (2.15), the magnon moves in the opposite direction i.e. from τ_2 to τ_1. The contractions (2.17), (2.18) are picture das rings and they contribute to the self-energy of a system of spin waves.

3. The dynamic diagrams comprising energy denominators

On availing ourselves of Eqs (2.6), (2.12), (2.14) and (2.15), we obtain

$$\begin{aligned}
D_2 = \frac{1}{2!}\frac{1}{4^2} J^2 N^{-2} \sum_{\substack{\lambda\varrho\sigma \\ \kappa\mu\nu}} \Gamma^{\lambda}_{\varrho,\sigma}\Gamma^{\kappa}_{\mu,\nu} \int_0^{\beta} d\tau_1 \int_0^{\beta} d\tau_2 [a^*_{\sigma+\lambda}(\tau_1)^{\cdot} a^*_{\varrho-\lambda}(\tau_1)^{\cdot\cdot} a_{\varrho}(\tau_1)^{\cdot\cdot\cdot} a_{\sigma}(\tau_1)^{\cdot\cdot\cdot\cdot} \\
\times a^*_{\nu+\kappa}(\tau_2)^{\cdot\cdot\cdot} a^*_{\mu-\kappa}(\tau_2) a_{\mu}(\tau_2)^{\cdot} a_{\nu}(\tau_2)^{\cdot\cdot} + a^*_{\sigma+\lambda}(\tau_1)^{\cdot} a^*_{\varrho-\lambda}(\tau_1)^{\cdot\cdot} \\
\times a_{\varrho}(\tau_1)^{\cdot\cdot\cdot} a_{\sigma}(\tau_1)^{\cdot\cdot\cdot\cdot} a^*_{\nu+\kappa}(\tau_2)^{\cdot\cdot\cdot} a^*_{\mu-\kappa}(\tau_2)^{\cdot\cdot\cdot\cdot} a_{\mu}(\tau_2)^{\cdot\cdot} a_{\nu}(\tau_2)^{\cdot} \\
+ a^*_{\sigma+\lambda}(\tau_1)^{\cdot} a^*_{\varrho-\lambda}(\tau_1)^{\cdot\cdot} a_{\varrho}(\tau_1)^{\cdot\cdot\cdot} a_{\sigma}(\tau_1)^{\cdot\cdot\cdot\cdot} a^*_{\nu+\kappa}(\tau_2)^{\cdot\cdot\cdot\cdot} a^*_{\mu-\kappa}(\tau_2)^{\cdot\cdot\cdot} \\
\times a_{\mu}(\tau_2)^{\cdot} a_{\nu}(\tau_2)^{\cdot\cdot} + a^*_{\sigma+\lambda}(\tau_1)^{\cdot} a^*_{\varrho-\lambda}(\tau_1)^{\cdot\cdot} a_{\varrho}(\tau_1)^{\cdot\cdot\cdot} a_{\sigma}(\tau_1)^{\cdot\cdot\cdot\cdot} \\
\times a^*_{\nu+\kappa}(\tau_2)^{\cdot\cdot\cdot\cdot} a^*_{\mu-\kappa}(\tau_2)^{\cdot\cdot\cdot} a_{\mu}(\tau_2)^{\cdot\cdot} a_{\nu}(\tau_2)^{\cdot}] = \tfrac{1}{2}\frac{1}{4^2} J^2 N^{-2} \sum_{\substack{\lambda\varrho\sigma \\ \kappa\mu\nu}} \Gamma^{\lambda}_{\varrho,\sigma}\Gamma^{\kappa}_{\mu,\nu} \\
\times \int_0^{\beta} d\tau_1 \int_0^{\beta} d\tau_2 4 a^*_{\sigma+\lambda}(\tau_1)^{\cdot} a^*_{\varrho-\lambda}(\tau_1)^{\cdot\cdot} a_{\varrho}(\tau_1)^{\cdot\cdot\cdot} a_{\sigma}(\tau_1)^{\cdot\cdot\cdot\cdot} a^*_{\nu+\kappa}(\tau_2)^{\cdot\cdot\cdot\cdot} \\
\times a^*_{\mu-\kappa}(\tau_2)^{\cdot\cdot\cdot} a_{\mu}(\tau_2)^{\cdot\cdot} a_{\nu}(\tau_2)^{\cdot} = \tfrac{1}{8} \beta J^2 N^{-2} \sum_{\lambda\varrho\sigma} \Gamma^{\lambda}_{\varrho,\sigma}\Gamma^{\lambda}_{\sigma+\lambda,\varrho-\lambda} \\
\times (\varepsilon_{\sigma+\lambda}+\varepsilon_{\varrho-\lambda}-\varepsilon_{\varrho}-\varepsilon_{\sigma})^{-1}[(\bar{n}_{\sigma+\lambda}+1)(\bar{n}_{\varrho-\lambda}+1)\bar{n}_{\varrho}\bar{n}_{\sigma} - \bar{n}_{\sigma+\lambda}\bar{n}_{\varrho-\lambda}(\bar{n}_{\varrho}+1)(\bar{n}_{\sigma}+1)] \\
= \tfrac{1}{4} \beta J^2 N^{-2} \sum_{\lambda\varrho\sigma} \frac{\Gamma^{\lambda}_{\varrho,\sigma}\Gamma^{\lambda}_{\sigma+\lambda,\varrho-\lambda}}{\varepsilon_{\sigma+\lambda}+\varepsilon_{\varrho-\lambda}-\varepsilon_{\varrho}-\varepsilon_{\sigma}} \bar{n}_{\varrho}\bar{n}_{\sigma} \\
+ \tfrac{1}{2} \beta J^2 N^{-2} \sum_{\lambda\varrho\sigma} \frac{\Gamma^{\lambda-\sigma}_{\varrho,\sigma}\Gamma^{\lambda-\sigma}_{\lambda,\varrho+\sigma-\lambda}}{\varepsilon_{\lambda}+\varepsilon_{\lambda-\varrho-\sigma}-\varepsilon_{\varrho}-\varepsilon_{\sigma}} \bar{n}_{\lambda}\bar{n}_{\varrho}\bar{n}_{\sigma}.
\end{aligned} \tag{3.1}$$

In the above expressions four diagram parts were reduced to one term, as they all are topologically equivalent.

In arbitrary order, the needful graph assumes the form (see [4]):

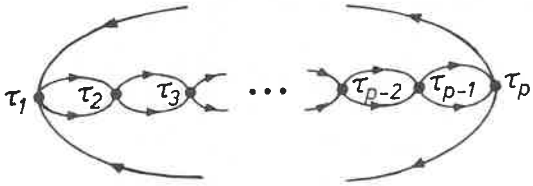

Fig. 1. The $(p+1)$-th order ladder diagram containing energy denominators

In approximation of bilinear product of average population numbers there is

$$D_{p+1} = \frac{1}{2^{p+1}} (JN^{-1})^{p+1} \sum_{\lambda_1, \lambda_2, \ldots, \lambda_p} \sum_{\varrho\sigma} \Gamma^{\lambda_1}_{\varrho,\sigma} \Gamma^{-\lambda_1+\lambda_2}_{\varrho-\lambda_1,\sigma+\lambda_1}$$

$$\times \Gamma^{-\lambda_2+\lambda_3}_{\varrho-\lambda_2,\sigma+\lambda_2} \ldots \Gamma^{-\lambda_{p-1}+\lambda_p}_{\varrho-\lambda_{p-1},\sigma+\lambda_{p-1}} \Gamma^{-\lambda_p}_{\varrho-\lambda_p,\sigma+\lambda_p} (\varepsilon_{\sigma+\lambda_1}+\varepsilon_{\varrho-\lambda_1}-\varepsilon_\varrho-\varepsilon_\sigma)^{-1} (\varepsilon_{\sigma+\lambda_2}+\varepsilon_{\varrho-\lambda_2}-\varepsilon_\varrho-\varepsilon_\sigma)^{-1}$$

$$\times \ldots (\varepsilon_{\sigma+\lambda_{p-1}}+\varepsilon_{\varrho-\lambda_{p-1}}-\varepsilon_\varrho-\varepsilon_\sigma)^{-1} (\varepsilon_{\sigma+\lambda_p}+\varepsilon_{\varrho-\lambda_p}-\varepsilon_\varrho-\varepsilon_\sigma)^{-1} \bar{n}_\varrho \bar{n}_\sigma, \quad p = 1, 2, 3, \ldots \tag{3.2}$$

In similar approximation D_2 becomes

$$D_2 \approx \tfrac{1}{4} \beta J^2 N^{-2} \sum_{\lambda\varrho\sigma} \frac{\Gamma^{\lambda}_{\varrho,\sigma} \Gamma^{\lambda}_{\sigma+\lambda,\varrho-\lambda}}{\varepsilon_{\sigma+\lambda}+\varepsilon_{\varrho-\lambda}-\varepsilon_\varrho-\varepsilon_\sigma} \bar{n}_\varrho \bar{n}_\sigma. \tag{3.3}$$

For small ϱ and σ

$$\frac{\Gamma^{\lambda}_{\varrho,\sigma} \Gamma^{\lambda}_{\sigma+\lambda,\varrho-\lambda}}{\gamma_\varrho+\gamma_\sigma-\gamma_{\sigma+\lambda}-\gamma_{\varrho-\lambda}} \to \frac{\gamma_\lambda(2\gamma_\lambda-\gamma_\varrho-\gamma_\sigma)(1-\gamma_\varrho/\gamma_0)(1-\gamma_\sigma/\gamma_0)}{\gamma_\varrho+\gamma_\sigma-\gamma_{\sigma+\lambda}-\gamma_{\varrho-\lambda}} \to$$

$$\to \frac{x_\lambda(2x_\lambda-x_\varrho-x_\sigma)(1-x_\varrho)(1-x_\sigma)}{2(1-x_\lambda)} \gamma_0 \tag{3.4}$$

with

$$x_\lambda = \gamma_\lambda/\gamma_0. \tag{3.5}$$

Thus,

$$D_2 \approx \frac{1}{8S} \beta J \gamma_0 N^{-2} \sum_{\lambda\varrho\sigma} \frac{x_\lambda(2x_\lambda-x_\varrho-x_\sigma)(1-x_\varrho)(1-x_\sigma)}{1-x_\lambda} \bar{n}_\varrho \bar{n}_\sigma. \tag{3.6}$$

Recurring to [4], [11] and [16], we have

$$D_3 = \frac{\Gamma}{2S} D_2, \tag{3.7}$$

$$D_4 = \frac{\Gamma}{2S} D_3 = \left(\frac{\Gamma}{2S}\right)^2 D_2, \tag{3.8}$$

and so forth. Finally, we get

$$\sum_{p=2}^{\infty} D_p = \sum_{p=2}^{\infty} \left(\frac{\Gamma}{2S}\right)^{p-2} D_2 = \frac{D_2}{1-\Gamma/2S}, \tag{3.9}$$

where for the simple cubic lattice

$$\Gamma = \frac{1}{(2\pi)^3} \int\int\int_0^{2\pi} dxdydz \, \frac{\cos x(1-\cos y)}{1-1/3(\cos x+\cos y+\cos z)} = 0.2110. \tag{3.10}$$

4. *The free energy of cubic Heisenberg ferromagnet*

Taking into account that

$$N^{-1}\sum_{\lambda} \frac{x_\lambda^2}{1-x_\lambda} = N^{-1}\sum_{\lambda} \frac{(1-x_\lambda-1)^2}{1-x_\lambda} = N^{-1}\sum_{\lambda}(1-x_\lambda) - 2N^{-1}\sum_{\lambda}$$

$$+N^{-1}\sum_{\lambda}(1-x_\lambda)^{-1} = -1+N^{-1}\sum_{\lambda}(1-x_\lambda)^{-1} = N^{-1}\sum_{\lambda}\frac{x_\alpha}{1-x_\lambda} = \alpha, \tag{4.1}$$

we bring the free energy of cubic Heisenberg ferromagnet into the form

$$F = E_0 + \sum_{\lambda}(L+\varepsilon_\lambda)\bar{n}_\lambda - \frac{1}{2JS^2\gamma_0} N^{-1} \sum_{\varrho\sigma} \varepsilon_\varrho \varepsilon_\sigma \bar{n}_\varrho \bar{n}_\sigma - \frac{1}{4JS^2\gamma_0}\frac{\alpha}{2S-\Gamma} N^{-1}$$

$$\times \sum_{\varrho\sigma}(1-x_\varrho+1-x_\sigma)\varepsilon_\varrho\varepsilon_\sigma\bar{n}_\varrho\bar{n}_\sigma + \beta^{-1}\sum_{\lambda}[\bar{n}_\lambda \ln \bar{n}_\lambda + (\bar{n}_\lambda+1)\ln(\bar{n}_\lambda+1)], \tag{4.2}$$

where the first, second, third and fifth terms are due to Bloch's theory [3] and the fourth term is the sum (3.9) of ladder diagrams which, as it was hinted at in Introduction, will be assumed henceforth valid for all wave vectors. We treat $\bar{n}_\lambda$ as variational parameters i.e. we put

$$\frac{\partial F}{\partial \bar{n}_\lambda} = 0, \tag{4.3}$$

and obtain

$$L+\varepsilon_\lambda\left\{1-\frac{1}{S}\left(1+\frac{\alpha}{2}\frac{1-x_\lambda}{2S-\Gamma}\right)N^{-1}\sum_{\varrho}(1-x_\varrho)\bar{n}_\varrho - \frac{\alpha/2}{S(2S-\Gamma)}N^{-1}\sum_{\varrho}(1-x_\varrho)^2\bar{n}_\varrho\right\}$$

$$= \beta^{-1}\ln\frac{1+\bar{n}_\lambda}{\bar{n}_\lambda}. \tag{4.4}$$

By introducing notations

$$Y = N^{-1} \sum_{\lambda} (1-x_\lambda)\tilde{n}_\lambda, \tag{4.5}$$

$$V = N^{-1} \sum_{\lambda} (1-x_\lambda)^2\tilde{n}_\lambda, \tag{4.6}$$

the renormalized average spin wave population number becomes

$$\tilde{n}_\lambda = \frac{1}{\exp \beta \left\{ L+\varepsilon_\lambda \left[1-\left(1+\frac{\alpha}{2}\frac{1-x_\lambda}{2S-\Gamma}\right)\frac{Y}{S} - \frac{\alpha/2}{2S-\Gamma}\frac{V}{S}\right]\right\} -1}. \tag{4.7}$$

The spontaneous magnetization is given by

$$\mu(T) = -\frac{1}{NS}\left(\frac{\partial F}{\partial L}\right)_{L=0}, \tag{4.8}$$

hence

$$\mu(T) = 1-\frac{1}{NS}\sum_{\lambda} \tilde{n}_\lambda. \tag{4.9}$$

The quantity V gives rise to a small decrease of the transition temperature and an increase in the magnetization at the critical point. To facilitate subsequent calculations, we put

$$V = 0, \tag{4.10}$$

and finally have

$$\tilde{n}_\lambda = \frac{1}{\exp \beta \left\{ L+\varepsilon_\lambda \left[1-\left(1+\frac{\alpha}{2}\frac{1-x_\lambda}{2S-\Gamma}\right)\frac{Y}{S}\right]\right\} -1}. \tag{4.11}$$

Thus, the renormalization factor

$$Y'_\lambda = \left[1+\frac{\alpha/2}{2S-\Gamma}(1-x_\lambda)\right] Y \tag{4.12}$$

depends on the wave vector λ and correctly describes at least the long-wave-length spin waves. Since we supposed the factor Y to hold for all wave vectors, we should determine the critical point by the condition

$$\left(\frac{\partial Y}{\partial T}\right)_{T=T_m} = \infty, \tag{4.13}$$

with T_m being the transition temperature. The relation (4.13) is necessitated by infinite gradient of a magnetization curve at the critical point i.e. by

$$\left[\frac{\partial \mu(T)}{\partial T}\right]_{T=T_m} = -\infty, \tag{4.14}$$

where T_m is related to x_m by the equation

$$x_m = \lim_{T\to T_m} x = \lim_{T\to T_m} \beta J\gamma_0 = \beta_m J\gamma_0 = J\gamma_0/kT_m. \tag{4.15}$$

The values of x_m and T_m are listed in Table I.

TABLE I

Simple cubic lattice, $\gamma_0 = 6$

S	1/2	1
x_m	6.933 6.144 [3] 7.140+0.036 [17]	2.293 2.128 [3]
Y_m	0.116 0.164 [3]	0.302 0.368 [3]
$\mu(T_m)$	0.430 0.220 [3]	0.534 0.265 [3]

The data in Table I are obtained by the computer "Odra 1204". Making use of (4.15), we get the ratio of Rushbrooke et al. [17] to our transition temperatures

$$\frac{x_R}{x_S} = \frac{T_S}{T_R} = 1.035.$$

Let us now concentrate our attention on the problem of applicability of SCR to cubic Heisenberg ferromagnet in the critical region. In actual fact, SCR fails to give reliable information thereof, except of the Curie temperature. In particular, the spontaneous magnetization derived by SCR does not vanish in the critical point and its temperature dependence proves incorrect. Indeed, putting

$$t = 1 - T/T_m, \tag{4.16}$$

it can be shown after little algebra that the renormalization function Y, Eq. (4.5), is (see also [2])

$$Y \equiv Y(T) \equiv Y(t) = Y_m - a_0 t^{1/2}(1 + a_1 t^{1/2} + a_2 t + \ldots), \tag{4.17}$$

where

$$Y_m = \lim_{T\to T_m} Y = N^{-1} \sum_{\lambda} \frac{1 - x_\lambda}{e^{A_m(1-x\lambda) - B_m(1-x\lambda)^2} - 1}, \tag{4.18}$$

$$A_m = x_m(S - Y_m), \tag{4.19}$$

$$B_m = x_m c Y_m, \tag{4.20}$$

$$c = \frac{\alpha/2}{2S - \Gamma}, \tag{4.21}$$

where for the simple cubic lattice $c = 0.5164$ and

$$a_0 = \left(\frac{SN^{-1} \sum_\lambda (1-x_\lambda)^2 [\tilde{n}_\lambda(\tilde{n}_\lambda+1)]_{T=T_m} - Y/x_m}{N^{-1} \sum_\lambda [1-x_\lambda+c(1-x_\lambda)^2]\,(1-x_\lambda)\,[\tilde{n}_\lambda(\tilde{n}_\lambda+1)\,(2\tilde{n}_\lambda+1)]_{T=T_m}} \right)^{1/2} . \tag{4.22}$$

Since the spontaneous magnetization reveals the identical type of the temperature dependence, the self-consistently renormalized spin wave theory yields incorrect critical magnetization exponent inasmuch as it has rather to be 1/3 than 1/2.

5. The asymptotic form of SCR in the critical region

The SCR proved to be inadequate to describing Heisenberg ferromagnet at temperatures close to the critical point and we should then modify it. As mentioned in the Introduction, in the vicinity to the transition temperature only small spin wave vectors play important role. Indeed, at low temperature small spin deviation propagating through a crystal lattice induce in the reciprocal lattice long-wave-length spin waves. Due to thermal excitation close to the critical point spin deviations become large and thus spin waves having large momenta get predominant. We can show it as follows. Let us take into consideration that according to (2.4), (2.5) the spin wave energy for the simple cubic lattice assumes the form

$$\varepsilon_\lambda = 6JS[1-\tfrac{1}{3}(\cos\lambda_1+\cos\lambda_2+\cos\lambda_3)], \quad 0 \leqslant \lambda_i < 2\pi, \quad i = 1, 2, 3, \tag{5.1}$$

where (for $\hbar = 1$) λ_i are the spin waves momenta. By (2.9) ($L = 0$)

$$\bar{n}_\lambda = (\exp\beta\varepsilon_\lambda - 1)^{-1}, \tag{5.2}$$

and for λ_i approaching 2π, $\bar{n}_\lambda$ tends to infinity. Therefore, all integrations in SCR over population numbers have to be reduced to the interval $\langle 2\pi-\lambda, 2\pi\rangle$ for every component of the wave vector. But owing to the spin wave energy periodicity with the periods 2π the integration limits must be confined to $\langle 0, \lambda\rangle$, where λ is the largest spin wave vector. Now, it can be easily verified that the renormalization factor Y should take the form

$$Y = c' - d'\lambda^3 + O(\lambda^5), \tag{5.3}$$

where c' and d' are positive constants, and

$$\lambda \sim t^a, \quad a > 0. \tag{5.4}$$

Indeed,

$$Y = N^{-1} \sum_{\varrho \leqslant \lambda} \frac{1-x_\varrho}{e^{A(1-x_\varrho)-B(1-x_\varrho)^2}-1} = N^{-1} \sum_{\varrho \leqslant \lambda} (1-x_\varrho)$$

$$\times\left\{\frac{1}{A(1-x_\varrho)\left[1-\frac{B}{A}(1-x_\varrho)\right]}-\tfrac{1}{2}+\ldots\right\}=\frac{1}{A}N^{-1}\sum_{\varrho\leqslant\lambda}+\ldots$$

$$=\frac{1}{x(S-Y)}N^{-1}\sum_{\varrho\leqslant\lambda}+\ldots\approx\frac{1}{x(S-Y)}\lambda^3+\ldots \tag{5.5}$$

wherein

$$A = x(S-Y), \tag{5.6}$$

$$B = xcY. \tag{5.7}$$

After little computation work we get approximative solution

$$Y \approx S-\frac{\lambda^3}{xS}+O(\lambda^5). \tag{5.8}$$

In virtue of (4.13) and (5.6)

$$\left(\frac{dY}{dT}\right)_{T=T_m}=-\frac{1}{T_m}\left(\frac{dY}{dt}\right)_{T=T_m}\approx\left(\frac{3\lambda^2}{xST_m}\frac{d\lambda}{dt}\right)_{T=T_m}\approx\left(\frac{3ak}{J\gamma_0 S}t^{3a-1}\right)_{T=T_m} \tag{5.9}$$

i.e.

$$3a-1<0 \tag{5.10}$$

and

$$a<1/3. \tag{5.11}$$

The critical exponent equal to 1/3 is more consistent with experimental data than the 1/2 one (e.g. see [6]).

As to the spontaneous magnetization, it can be computed as follows:

$$\mu(T)=1-\frac{1}{NS}\sum_{\varrho\leqslant\lambda}\frac{1}{e^{A(1-x_\varrho)-B(1-x_\varrho)^2}-1}=1-\frac{1}{AS}N^{-1}\sum_{\varrho\leqslant\lambda}\frac{1}{1-x_\varrho}+\ldots$$

$$=1-\frac{1-t}{x_m(S-Y)}N^{-1}\sum_{\varrho\leqslant\lambda}\frac{1}{\varrho^2}+\ldots$$

$$=1-\frac{1}{x_m(S-Y)}N^{-1}\sum_{\varrho\leqslant\lambda}\frac{1}{\varrho^2}+\frac{t}{x_m(S-Y)}N^{-1}\sum_{\varrho\leqslant\lambda}\frac{1}{\varrho^2}+\ldots \tag{5.12}$$

In an exact theory the first two terms in the last line of (5.10) should cancel out, whence

$$\mu(T)=\frac{t}{x_m(S-Y)}N^{-1}\sum_{\varrho\leqslant\lambda}\frac{1}{\varrho^2}\sim S\lambda\sim t^{1/3}, \tag{5.13}$$

instead of

$$\mu(T) \sim t^{1/2} \tag{5.14}$$

resulting from (4.17).

Obviously, a correct theory ought to express the Kadanoff model in terms of graphs suppressing diagrams due to short-range spin interactions, but setting up such theory is extremely difficult problem.

6. *Conclusions*

In this paper, we extended the SCR theory by adding to it a series of ladder diagrams involving in energy denominators. This series enabled to improve the transition temperature although it entailed increasing the spontaneous magnetization close to and at the critical point.

On the other hand, we inquired into the question of the applicability of SCR in the critical region. The Bloch theory [3] in its usual form does not serve this purpose. Therefore, we employed Kadanoff's model [5] discriminating in favour of long range interactions between localized spins at temperatures close to the transition point and thus permitting only long spin waves. As a result of this restriction, the spontaneous magnetization turned out to be proportional to $(1-T/T_m)^{1/3}$ instead of $(1-T/T_m)^{1/2}$ according to Bloch theory [3].

REFERENCES

[1] P. D. Loly, *Ann. Phys.* (N.Y.) **56**, 40 (1970).
[2] W. Jeżewski, *Acta Phys. Pol.* **A55**, 863 (1979).
[3] M. Bloch, *Phys. Rev. Lett.* **9**, 286 (1962).
[4] J. Szaniecki, *Acta Phys. Pol.* **A22**, 379, 389 (1962).
[5] L. P. Kadanoff, Proceedings of the International School of Physics "Enrico Fermi", Academic Press, New York and London 1971.
[6] A. I. Larkin, D. E. Khmelnitskii, *Sov. Phys. JETP* **56**, 2087 (1969).
[7] K. K. Singh, *Phys. Rev.* **12**, 2810 (1975).
[8] J. F. Cooke, H. H. Hahn, *Phys. Rev.* **184**, 509 (1969).
[9] J. F. Cooke, H. H. Hahn, *Phys. Rev.* **B1**, 1243 (1970).
[10] J. F. Cooke, *Phys. Rev.* **B2**, 220 (1970).
[11] F. J. Dyson, *Phys. Rev.* **102**, 1217, 1230 (1956).
[12] T. Matsubara, *Prog. Theor. Phys.* **14**, 351 (1955).
[13] D. J. Thouless, *Phys. Rev.* **107**, 1162 (1957).
[14] J. Szaniecki, *Acta Phys. Pol.* **A52**, 81 (1977).
[15] G. C. Wick, *Phys. Rev.* **80**, 268 (1950).
[16] S. H. Charap, P. R. Weiss, *Phys. Rev.* **116**, 1372 (1959).
[17] G. S. Rushbrooke, G. A. Baker Jr, P. J. Wood, *Phase Transitions and Critical Phenomena*, Academic Press, London — New York 1974.