# SOME SPECIAL MULTI-SOLITON SOLUTIONS OF NONLINEAR EQUATIONS

BY R. S. BANERJEE, A. ROY CHOWDHURY AND T. ROY

High Energy Physics Division, Department of Physics, Jadavpur University, Calcutta*

(Received September 14, 1979)

It is seen that straightforward quadrature of nonlinear equations may yield many soliton solutions by proper choice of the integration constants. The results have been seen in some specific models: Thirring model, two wave interaction equation, and equations of harmonic generation. Explicite numerical study has been made in the case of two wave interactions, and the results are interesting because no inverse scattering or Backlund transformation exists for this equation.

Inverse scattering technique [1] has become quite successful and favourite method in tackling problems related to nonlinear partial differential equations. It has been found that the only straightforward method for obtaining $N$-soliton solution is the method of I.S.T. But one peculiar situation occurs in connection with the equation of two wave interaction, where no Backlund transformation or I.S.T. occurs but the $N$-soliton solution is known to exist [2]. We can see in this note that the proper handling of the constants of integration can leed to multi-soliton states and one does not have to use other techniques. The equation under consideration is

$$\frac{\partial v_0}{\partial \xi} = v_0 v_1; \quad \frac{\partial v_1}{\partial n} = v_0 v_1, \tag{1}$$

where

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x} + \vartheta_1 \frac{\partial}{\partial t}$$

and

$$\frac{\partial}{\partial \eta} = \frac{\partial}{\partial x} + \vartheta_2 \frac{\partial}{\partial t}.$$

* Address: Department of Physics, Jadavpur University, Calcutta-700032, India.

Equations (1) imply that

$$\frac{\partial v_0}{\partial \xi} = \frac{\partial v_1}{\partial \eta},$$

that is $v_0 = \dfrac{\partial S}{\partial \eta}$, $v_1 = \dfrac{\partial S}{\partial \xi}$ for some function $S$ of $(\xi, \eta)$.
Substituting (1) yields

$$\frac{\partial^2 S}{\partial \xi \partial \eta} = \frac{\partial S}{\partial \xi} \frac{\partial S}{\partial \eta}. \tag{2}$$

This can be integrated to yield

$$S = -\ln\left[\int e^{\lambda(\xi)} d\xi + \mu(\eta)\right],$$

where $\lambda(\xi)$, $\mu(\eta)$ are arbitrary functions really arising from the quadratures with respect to $\xi, \eta$. From the above equations we get immediately

$$v_1 = f(x, t) = e^{\lambda(\xi)} / \left[\int e^{\lambda(\xi)} d\xi + \mu(\eta)\right],$$

$$v_0 = g(x, t) = \frac{d\mu}{d\eta} / \left[\int e^{\lambda(\xi)} d\xi + \mu(\eta)\right]. \tag{3}$$

We now make simple choice of $\lambda$ and $\mu$

$$\lambda(\xi) = \ln \xi, \quad \mu(\eta) = \frac{a}{2} \eta \tag{4}$$

which yields

$$v_1 = -\frac{2\xi}{\xi^2 + a\eta}, \quad v_0 = -\frac{a}{\xi^2 + a\eta}. \tag{5}$$

These equations immediately show that both $v_0$ and $v_1$ have two peaked solitary waves near the zeros of the denominator $\xi^2 + a\eta = 0$. We have plotted the functions with $x$ for different times which show very clearly that both $v_0$ and $v_1$ essentially are two soliton solutions which with time decay in two one-solitons or sometimes the two one-solitons comes together to create the new two soliton. Figures 1 and 2 clearly depict this situation. The above situation also occurs in equations of harmonic generate as

$$\frac{\partial q_1}{\partial x} + \vartheta_1 \frac{\partial q_1}{\partial t} = q_1 q_2,$$

$$\frac{\partial q_2}{\partial x} + \vartheta_2 \frac{\partial q_2}{\partial t} = -2q_1^2 \tag{6}$$

which can be transformed to

$$q_{1\xi} = q_1 q_2, \quad q_{2\eta} = -2q_1^2.$$

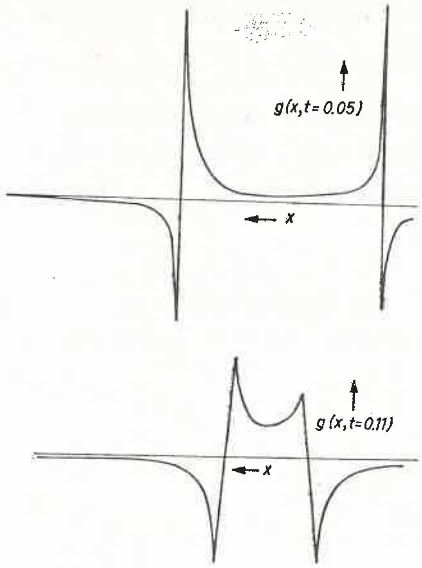


Fig. 1

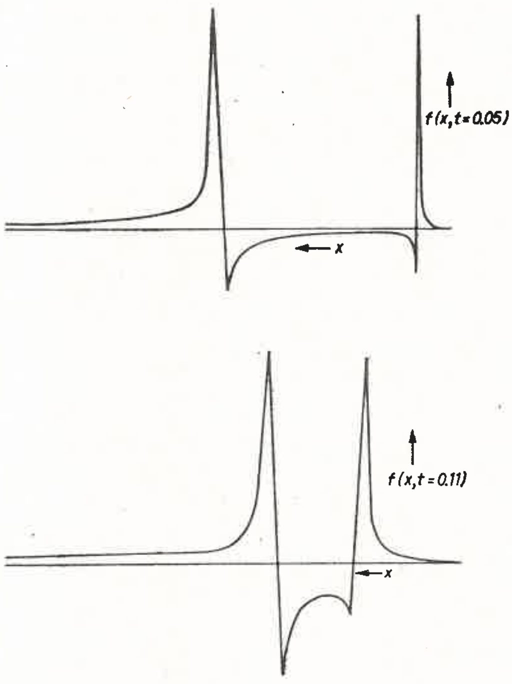


Fig. 2

Also the equations of Thirring model law can be written as

$$q_{1\xi} = g|q_2|^2 q_1; \qquad q_{2\eta} = g|q_1|^2 q_2. \tag{7}$$

Equations (6) and (7) can be treated in the same manner. In fact we have observed that by choosing $\lambda(\xi)$ and $\mu(\eta)$ to be higher order polynomials we can generate multi soliton solutions.

## REFERENCES

[1] M. J. Ablowitz et al., *Stud. Appl. Math.* **53**, 261 (1974).
[2] R. Hirota, in *Backlund Transformation*, Lecture Notes in Mathematics, Springer Verlag, Vol. 515.