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A REAL-SPACE RENORMALIZATION GROUP METHOD FOR CONTINUOUS-VARIABLE ISING MODEL. I*

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A new effective continuous-variable Hamiltonian for the Ising model is introduced. A real-space renormalization group method, for systems described by this effective Hamiltonian, is presented and some approximations are considered.

1. *Introduction*

The application of renormalization transformation concepts [1] to critical phenomena of spin lattice systems has been achieved by means of the so-called real-space renormalization group (RSRG) methods, first introduced by Niemeijer and van Leeuwen [2, 3]. The most important advantages of these methods consist in their validity in any real lattice dimension d, and their ability to evaluate not only critical exponents, but also other quantities of interest such as critical temperature, thermodynamic functions, and scaling functions. Among the various RSRG approaches, the decimation transformation is conceptually the simplest one [4–9]. However, the use of this transformation in the study of systems exhibiting critical behaviour encounters a difficulty. It lies in that a non-trivial fixed point can be reached only if $d-2+\eta = 0$, where η is the critical exponent characterizing the behaviour of the two-point correlation function at the critical point. In general, this condition is not satisfied and the decimation transformation cannot give the correct value of η, although it is useful for calculating the critical exponent ν characterizing the critical behaviour of the correlation length [7, 9]. To overcome the above mentioned difficulty, the decimation transformation has been modified by employing a linear weight factor, which involves an adjustable parameter p [6, 10]. Then, the restriction $d-2+\eta = 0$ was removed by a suitable choice of p (see also [11]).

The application of RSRG approaches has largely been confined to Ising systems described by discrete spin variables. In this paper, we present a renormalization group approach to Ising systems with continuous variables defined on a lattice. Our method

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consists of two steps. The first one is the pure decimation transformation in the integral form, and the second one is the variable rescaling, determined by the transformation of the two-point correlation function. Such a rescaling of the variables enables us to avoid the difficulty arising from the calculation of the critical index η. We also present a procedure for computing η, and introduce an approximate calculation technique based on the cumulant expansion involving a non-Gaussian distribution.

2. *The renormalization group transformation*

The partition function of the discrete-spin Ising Hamiltonian on a lattice can be written in integral form using the so called classical effective Hamiltonian, expressed in terms of continuous lattice variables [12, 13].

In this paper, we consider in nearest neighbour approximation the discrete spin-S Ising systems in zero magnetic field. These systems can be described by the following continuous-variable effective Hamiltonian on a lattice with N sites

$$H\{x_i\} = a_1 \sum_{i=1}^{N} x_i^2 - a_2 \sum_{i=1}^{N} x_i^4 + \ldots + \sum_t K_t \sum_{s \in t} x_s, \tag{2.1}$$

with each x_i varying from $-\infty$ to $+\infty$. Here, K_t denotes the coupling constant of a particular interaction type, s is a subset of lattice sites, the s summation runs over all interaction terms of a particular type, and x_s is defined as $x_s = \prod_{\substack{n=1 \\ (i_n \in s)}}^{m} x_{i_n}^{l_n}$ with l_n being natural numbers such that the sum $\sum_{n=1}^{m} l_n$ is an even natural number. For example, confining ourselves to the nearest neighbour (n.n.) and next nearest neighbour (n.n.n.) effective pair interactions, we have

$$\sum_t K_t \sum_{s \in t} x_s = K_{2,1,1} \sum_{\langle i,j \rangle} x_i x_j + K_{2,1,2} \sum_{\langle i,j \rangle} (x_i^3 x_j + x_i x_j^3) + K_{2,1,3} \sum_{\langle i,j \rangle} x_i^2 x_j^2 + \ldots$$
$$+ K_{2,2,1} \sum_{(i,j)} x_i x_j + K_{2,2,2} \sum_{(i,j)} (x_i^3 x_j + x_i x_j^3) + K_{2,2,3} \sum_{(i,j)} x_i^2 x_j^2 + \ldots, \tag{2.2}$$

where the parameters $K_{2,1,p}$, $K_{2,2,p}$ stand for the n.n. and n.n.n. interactions, respectively, and the symbols $\langle i,j \rangle$ and (i,j) denote that the summations are over the n.n. and n.n.n. pairs of lattice sites. The Hamiltonian (2.1) can be derived exactly (see Appendix). It should be noted, that the classical effective Hamiltonians derived in [12, 13] differ from that introduced by us.

In defining the renormalization group transformation for our continuous-variable Ising model, we take advantage of the decimation transformation. It consists in an integration over lattice variables x_i belonging to a set $\mathscr{S}_0$ such that the remaining variables $y_i \equiv x_i$, which are elements of the complementary set $\mathscr{S}_1$, form a new lattice, isomorphic to the original lattice. Then, the partition function remains the same, and the original Hamiltonian $H\{x_i\}$ is transformed into the new Hamiltonian $H'\{y_i\}$. It can be easily verified, that $H'\{y_i\}$ has the same symmetry as $H\{x_i\}$. The decimation transformation of the two-point correlation function Γ leads to the relation [5, 6]

$$\Gamma(r; \boldsymbol{K}) = \Gamma(r/b; \boldsymbol{K}'), \tag{2.3}$$

where r is the distance between two lattice sites, b denotes the length rescaling factor $\boldsymbol{K}$ and $\boldsymbol{K}'$ symbolize parameters of $H\{x_i\}$ and $H'\{y_i\}$, respectively. From relation (2.3), it follows that, if the decimation transformation were performed exactly, a non-trivial fixed point could exist, if and only if, the condition $d-2+\eta = 0$ were satisfied [5, 6]. It is obvious that this condition is not, in general, fulfilled. We remove the above difficulty by rescaling the variables y_i as follows (cf. [1])

$$y_i \to b^{-(d-2+\eta)/2} y_i'. \tag{2.4}$$

Then, our renormalization group transformation consists of two steps. The decimation transformation is the first one and the variable rescaling (2.4) the second one. These two steps can be put together by writing

$$e^{E+H'\{y_i'\}} = \Big(\int_{-\infty}^{+\infty} \prod_i dx_i \prod_{i \in \mathscr{S}_1} [\delta(x_i - y_i)] e^{H\{x_i\}} \Big)_{y_i \to b^{1-\frac{d}{2}-\frac{\eta}{2}} y_i'}, \tag{2.5}$$

where E is a constant independent of y_i. Since the Hamiltonians $H\{x_i\}$ and $H'\{y_i'\}$ are parametrized by the sets of coefficients $\boldsymbol{K} = \{a_1, a_2, \ldots, K_t\}$ and $\boldsymbol{K}' = \{a_1', a_2', \ldots, K_t'\}$, one can view Eq. (2.5) as a mapping in parameter space

$$\boldsymbol{K}' = \mathscr{R}\boldsymbol{K}. \tag{2.6}$$

According to (2.5), the transformation relation for the two-point correlation function becomes

$$\Gamma(r; \boldsymbol{K}) = b^{-d+2-\eta} q(r/b; \boldsymbol{K}'). \tag{2.7}$$

In order to find the fixed point $\boldsymbol{K}^*$, we need to determine first the critical exponent η. This can be done as follows. We specify the distance r in (2.7) putting, e.g., $r = b\delta'$ with δ' being the nearest neighbour distance in the new lattice. Then, at the fixed point, we have

$$\Gamma(b\delta'; \boldsymbol{K}^*) = b^{-d+2-\eta} \Gamma(\delta'; \boldsymbol{K}^*). \tag{2.8}$$

It is easy to check that this relation can be written in the form

$$\left(\frac{\partial}{\partial K^*_{2,2,1}} - b^{2-\eta} \frac{\partial}{\partial K^*_{2,1,1}} \right) \ln Z^* = 0, \tag{2.9}$$

where Z^* is the partition function at the critical point. On calculating Z^* in some approximation, one can use (2.9) to express η as a function of the parameters of the critical Hamiltonian. Thus, we can write formally the exponent η as

$$\eta = \eta(\boldsymbol{K}^*). \tag{2.10}$$

Now, inserting (2.10) into the recursion relations (2.6) one can find the fixed point $\boldsymbol{K}^*$, and, subsequently, using once again (2.10) the exponent η can be evaluated. Other critical indices can be calculated by standard methods [3]. It is to be noted, that in an exact calculation, η should be independent of $\boldsymbol{K}^*$ [6]. Thus, a legitimate approximation should yield $\eta(\boldsymbol{K}^*)$ to be approximately constant.

It should be pointed out, that in the investigation of the critical behaviour of the continuous-variable Ising systems described by the Hamiltonian (2.1) also other RSRG approaches can be applied. For example, we may use the one-hypercube approximation [14].

3. *Approximations*

In general, the renormalization group transformation (2.5) cannot be performed exactly, and certain approximations are necessary for obtaining the recursion relations. Here, we utilize the cumulant approximation [3]. Splitting the Hamiltonian $H\{x_i\}$ into a zeroth part $H_0\{x_i\}$ and a perturbational part $V\{x_i\}$

$$H\{x_i\} = H_0\{x_i\}+V\{x_i\}, \tag{3.1}$$

we can write the transformation relation (2.5) in this approximation as

$$E+H'\{y_i'\} = \langle V\{x_i\}\rangle_0+\tfrac{1}{2}\langle(V\{x_i\}-\langle V\{x_i\}\rangle_0)^2\rangle_0+\ldots, \tag{3.2}$$

where the average $\langle\ldots\rangle_0$ is defined by

$$\langle A\rangle_0 = J_1/J_0, \tag{3.3}$$

with

$$J_n = \Big(\int\limits_{-\infty}^{+\infty}\prod_i dx_i \prod_{i\in\mathscr{S}_1}[\delta(x_i-y_i)]A^n \exp H_0\{x_i\}\Big)_{y_i\to b^{1-\frac{d}{2}-\frac{\eta}{2}}y_i'} \tag{3.4}$$

It should be noticed, that the cumulant expansion (3.2) is well justified when $V\{x_i\}$ may be treated as small in comparison to $H_0\{x_i\}$. This restriction of the applicability of the cumulant approximation corresponds to the case of small coupling constants in $V\{x_i\}$. We take the zeroth part of $H\{x_i\}$ in the form

$$H_0\{x_i\} = a_1\sum_i x_i^2-a_2\sum_i x_i^4, \tag{3.5}$$

with a_2 assumed to be positive. We note, that $H_0\{x_i\}$ cannot be adopted in the Gaussian form when $a_1 > 0$. To evaluate the cumulants in (3.2), we must know the averages

$$\langle x^n\rangle_0 = I_n/I_0, \quad n = 2, 4, 6, \ldots, \tag{3.6}$$

with

$$I_n = a_2^{-\frac{n+1}{4}}\int\limits_{-\infty}^{+\infty} dx x^n e^{2gx^2-x^4}, \tag{3.7}$$

where $g = a_1/2\sqrt{a_2}$. The moments I_n can be represented as (cf. [15])

$$I_n = a_2^{-\frac{n+1}{4}}\left[\tfrac{1}{2}\Gamma\left(\frac{n+1}{4}\right){}_1F_1\left(\frac{n+1}{4}, \tfrac{1}{2}; g^2\right)+\Gamma\left(\frac{n+3}{4}\right)g\,{}_1F_1\left(\frac{n+3}{4}, \tfrac{3}{2}; g^2\right)\right], \tag{3.8}$$

where ${}_1F_1(\alpha, \beta; z)$ denotes the confluent hypergeometric function

$$ {}_1F_1(\alpha, \beta; z) = 1 + \frac{\alpha}{\beta}\frac{z}{1!} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)}\frac{z^2}{2!} + \dots . \tag{3.9}$$

According to (3.8) and (3.9), each moment I_n can be expressed as a convergent power series in g. The convergence of these series is rapid for small n and slow for large n. Dividing each I_n by I_0, one can write each average $\langle x^n \rangle_0$ as a convergent power series in g

$$\langle x^n \rangle_0 = \sum_k b_k^{(n)} g^k. \tag{3.10}$$

For the calculational purposes, it is convenient to carry out the summation of (3.10). This can be realized by the Padé–Borel–Leroy method of summation [16]. The cumulant expansion based on the separation (3.1) determined by (3.5) and the above procedure of calculating the averages $\langle x^n \rangle_0$ can also be used in computing Z^*. The details of the calculations and the results will be presented elsewhere.

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APPENDIX

The continuous-variable Ising Hamiltonian used in this paper can be derived as follows. We start from the spin-$\frac{1}{2}$ Ising Hamiltonian

$$H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j, \tag{A1}$$

where the sum runs over nearest neighbour pairs of lattice sites, and the spin variables σ_i take on the values -1 and $+1$. We express the spin variables in terms of Bose operators a_i, $a_i^\dagger$

$$\sigma_i = -1 + 2a_i^\dagger a_i, \tag{A2}$$

$$[a_i, a_j^\dagger] = \delta_{i,j}. \tag{A3}$$

Then, the partition function is

$$Z = \sum_{\{n_i\}} \langle \{n_i\} | \exp H(\{a_i^\dagger\}, \{a_i\}) P | \{n_i\} \rangle, \tag{A4}$$

$$H(\{a_i^\dagger\}, \{a_i\}) = K \sum_{\langle i,j \rangle} (-1+2a_i^\dagger a_i)(-1+2a_j^\dagger a_j), \quad K = J/k_B T, \tag{A5}$$

where $|\{n_i\}\rangle$ denotes the normalized boson states

$$|\{n_i\}\rangle = \prod_i |n_i\rangle = \prod_i [(n_i!)^{-1/2} (a_i^\dagger)^{n_i} |0_i\rangle], \quad n_i = 0, 1, 2, \dots, \infty, \tag{A6}$$

and

$$P = \prod_i (P_i^0 + P_i^1), \tag{A7}$$

with P_i^0 and P_i^1 being the projection operators defined by

$$P_i^0|n_j\rangle = \delta_{i,j}\delta_{n\ ,0_i}|0_i\rangle, \qquad (A8)$$

$$P_i^1|n_j\rangle = \delta_{i,j}\delta_{n_j,1_i}|1_i\rangle. \qquad (A9)$$

The operator P eliminates the reduntant contributions to (A4) with $n_i = 2, 3, \ldots, \infty$. Now, using the coherent state formalism, see e.g. [17], we can rewrite (A4) in the form

$$Z = \int d\mu(\{\alpha_i\}) \langle\{\alpha_i\}| \exp H(\{a_i^\dagger\}, \{a_i\})P|\{\alpha_i\}\rangle \equiv \int d\mu(\{\alpha_i\}) \exp \tilde{H}(\{\alpha_i^*\}, \{\alpha_i\}), \qquad (A10)$$

where $d\mu(\{\alpha_i\}) = \prod_i \frac{d^2\alpha_i}{\pi}$, and the coherent states $|\{\alpha\}\rangle$ are given by

$$|\{\alpha_i\}\rangle = \prod_i |\alpha_i\rangle = \prod_i [\exp(-\tfrac{1}{2}|\alpha_i|^2)\exp(\alpha_i a_i^\dagger)|0_i\rangle]. \qquad (A11)$$

Equation (A10) defines the Hamiltonian $\tilde{H}(\{\alpha_i^*\}, \{\alpha_i\})$, which is the coherent state representation of $H(\{a_i^\dagger\}, \{a_i\})$:

$$\tilde{H}(\{\alpha_i^*\}, \{\alpha_i\}) = \ln \langle\{\alpha_i\}| \exp H(\{a_i^\dagger\}, \{a_i\})P|\{\alpha_i\}\rangle. \qquad (A12)$$

This effective Hamiltonian is a function of the continuous complex variables α_i. Inserting (A5) into (A12), using the formulae

$$\langle\{\alpha_i\}| (-1+2a_j^\dagger a_j)^n P|\{\alpha_i\}\rangle = \prod_i [(1+|\alpha_i|^2)\exp(-|\alpha_i|^2)], \quad n = 0, 2, 4, \ldots, \qquad (A13)$$

$$\langle\{\alpha_i\}| (-1+2a_j^\dagger a_j)^n P|\{\alpha_i\}\rangle = \frac{-1+|\alpha_j|^2}{1+|\alpha|_j^2} \prod_i [(1+|\alpha_i|^2)\exp(-|\alpha_i|^2)], \quad n = 1, 3, 5, \ldots, \qquad (A14)$$

and performing a cumulant expansion of the average in (A12), we arrive at

$$\tilde{H}(\{\alpha_i^*\}, \{\alpha_i\}) = \tilde{E}_0 - \sum_i |\alpha_i|^2 + \sum_i \ln(1+|\alpha_i|^2) + c_1 \sum_{\langle i,j\rangle} f_i f_j$$
$$+ c_2 \sum_{(i,j)} f_i f_j + c_1^{(2)} \sum_{\langle i,j\rangle} f_i^2 f_j^2 + \ldots, \qquad (A15)$$

with

$$f_i = \frac{-1+|\alpha_i|^2}{1+|\alpha_i|^2}, \qquad (A16)$$

$$\tilde{E}_0 = \tfrac{1}{2} Nz \cosh K, \qquad (A17)$$

$$c_1 = \sum_{m=1}^{N_1} a_m^{(1)} \frac{1}{m!} u^m, \qquad (A18)$$

$$c_2 = \sum_{m=2}^{N_2} a_m^{(2)} \frac{1}{m!} u^m, \tag{A19}$$

$$c_1^{(2)} = \sum_{m=2}^{N_{1,2}} a_m^{(1,2)} \frac{1}{m!} u^m, \tag{A20}$$

$$u = \operatorname{tgh} K, \tag{A21}$$

where z is the number of nearest neighbours, the parameters c_1 and $c_1^{(2)}$ are the n.n. constant couplings, and c_2 is the n.n.n. constant coupling. The coefficients $a_m^{(1)}$, $a_m^{(2)}$ and $a_m^{(1,2)}$ are numbers dependent on the lattice symmetry. Inserting (A15) into the second equality of (A10) and substituting $\alpha_i \to \exp(\frac{1}{2}x_i + i\varphi_i)$ $(0 \leqslant \varphi_i \leqslant 2\pi)$ we have

$$Z = e^{\tilde{E}_0} \int_{-\infty}^{+\infty} \prod_i dx_i e^{\tilde{H}\{x_i\}}, \tag{A22}$$

where

$$\tilde{H}\{x_i\} = \sum_i x_i - \sum_i e^{x_i} + \sum_i \ln(1+e^{x_i}) + c_1 \sum_{\langle i,j\rangle} g_i g_j$$
$$+ c_2 \sum_{(i,j)} g_i g_j + c_1^{(2)} \sum_{\langle i,j\rangle} g_i^2 g_j^2 + \ldots, \tag{A23}$$

with

$$g_i = \operatorname{tgh} \tfrac{1}{2} x_i. \tag{A24}$$

The partition function (A22) can be rewritten as follows

$$Z = e^{\tilde{E}_0} \int_{-\infty}^{+\infty} \prod_i dx_i \tfrac{1}{2} (e^{\tilde{H}\{x_i\}} + e^{\tilde{H}\{-x_i\}}) \equiv e^{E_0} \int_{-\infty}^{+\infty} \prod_i dx_i e^{H\{x_i\}}. \tag{A25}$$

Finally, from (A25), (A23) and (A24) we obtain

$$E_0 = \tilde{E}_0 - N(1 - \ln 2), \tag{A26}$$

$$H\{x_i\} = a_1 \sum_i x_i^2 - a_2 \sum_i x_i^4 + K_{2,1,1} \sum_{\langle i,j\rangle} x_i x_j + K_{2,2,1} \sum_{(i,j)} x_i x_j + K_{2,1,3} \sum_{\langle i,j\rangle} x_i^2 x_j^2 + \ldots . \tag{A27}$$

In (A27), we have expanded g_i in a power series in x_i. The first three coefficients in (A27) are

$$a_1 = -\tfrac{1}{4}, \tag{A28}$$

$$a_2 = \tfrac{13}{96}, \tag{A29}$$

$$K_{2,1,1} = \frac{c_1 + 1}{4}. \tag{A30}$$

Similarly, we may derive the effective continuous-variable Ising Hamiltonian for any spin. For example, in the case of $S = 1$, the coefficients a_1, a_2 and $K_{2,1,1}$ become

$$a_1 = -(3\sqrt{2}-4)+O(K^2), \tag{A31}$$

$$a_2 = (6-\tfrac{49}{12}\sqrt{2})+O(K^4), \tag{A32}$$

$$K_{2,1,1} = c_1'+6-4\sqrt{2}). \tag{A33}$$

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