EXISTENCE OF CONTINUOUS PHASE TRANSITION IN ANISOTROPIC FERROMAGNETS WITH FIELD

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Renormalization-group arguments are used to study the phase transitions of anisotropic ferromagnets in an external magnetic field. The conditions for the existence of the XY-like and Ising-like continuous phase transitions are found. These conditions — some general relations between the interaction parameters and the external field — permit one to check the possibility of the occurrence of such phase transitions in any anisotropic ferromagnet with the field in an arbitrary direction.

1. *Introduction*

The phase transitions in anisotropic ferromagnets in the presence of an external uniform magnetic field have been studied for many years [1–13]. It has been shown by mean field (or Landau) and renormalization-group theories that there are magnetic second order phase transitions in some anisotropic ferromagnets, when the external field is directed in certain directions.

The mean-field theory has been used to study the phase transitions of uniaxial [1–5] and orthorhombic [6] ferromagnets with an external magnetic field of arbitrary direction; the three-axial (cubic) ferromagnet in the field $\boldsymbol{H}$ || [111], [100] and [110] directions [7, 10] and the four-axial (cubic) ferromagnet in a field of any direction in the plane {100} [9, 10]. The renormalization group method has been applied to the analysis of the phase transitions in uniaxial [2, 8, 12] and tetragonal [13] ferromagnets with the field perpendicular to the easy axis; the three-axial ferromagnet in the field on the plane formed by a main diagonal and an edge of the cube [11, 13]; the four-axial in $\boldsymbol{H}$ || [100] [12, 13].

As seen, there is, so far, no general theory which considers an arbitrary symmetry of a ferromagnet and direction of the external field. Therefore, it seems that it would be

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interesting to construct a general formalism for studying anisotropic ferromagnets with arbitrary symmetry in the presence of an external uniform magnetic field.

The purpose of this paper is to give a general treatment of the problem of phase transitions in the anisotropic ferromagnets by using the renormalization-group approach. We shall consider the general (to the fourth order in spin variables) Landau–Ginzburg–Wilson (LGW) Hamiltonian describing the anisotropic ferromagnets in an external field by using the Wilson–Fisher–Nelson formalism. We shall present some general relations between the coupling constants and the field components which must be fulfilled in order to cause the second order phase transition to occur. By specification of the coupling constants we shall be able to discuss the phase transitions in the ferromagnet with the arbitrary given symmetry. The results for some known symmetry will be presented in the next paper.

We shall consider ferromagnets which can be described by the three component vector model. Hence, the maximum number of critical variables could be, of course, three but if at least one of the variables is conjugate to the external field then we can have either one or two critical variables (one or two component order parameter) if any. Thus, the system under consideration can undergo the Ising-type (one critical variable) or the XY-type (two critical variables) phase transition. For convenience, we shall consider these two types of the critical behavior seperately and first find the conditions for the existence of the XY-like phase transition and then the Ising-like. The fixed points connected with Ising and XY-like critical behavior will be denoted by C_1 and C_2, respectively.

2. *Hamiltonian*

The most general LGW Hamiltonian for the anisotropic ferromagnets in an external uniform field including terms of up to the fourth order in spin variables may be written as

$$\mathscr{H}_0 = -\tfrac{1}{2}\sum_{i=1}^{3}\int_q (r_i^0+q^2)\sigma_q^i\sigma_{-q}^i + \sum_{i=1}^{3} h^i\sigma_0^i - \sum_{i,j=1}^{3} u_{ij}^0 \int_q\int_{q_1}\int_{q_2} \sigma_q^i\sigma_{q_1}^i\sigma_{q_2}^j\sigma_{-q-q_1-q_2}^j, \tag{1}$$

where $\boldsymbol{\sigma}_q$ is a three-component classical spin of wave vector $\boldsymbol{q}$, $\boldsymbol{h}$ denotes an external magnetic field and the integrals $\int_q$ are taken over $|q| < 1$ in $d = 4-\varepsilon$ dimensional space. The temperature variables are

$$r_i^0 = a_i(T-\Theta_i), \tag{2}$$

where Θ_i are respectively "critical temperatures" of σ^i in the free-field case.

By the choice of the parameters r_i^0 and u_{ij}^0 we can obtain the form (1) for several ferromagnets and so the Hamiltonian (1) describes:

1. isotropic ferromagnet if the following relations are fulfilled $r_i^0 = r^0$, $u_{ij}^0 = u^0 > 0$;
2. uniaxial ferromagnet if $r_1^0 \neq r_2^0 = r_3^0$, $u_{ij}^0 = u^0 > 0$;
3. orthorhombic ferromagnet if $r_1^0 \neq r_2^0 \neq r_3^0$, $u_{ij}^0 = u^0 > 0$;
4. cubic ferromagnet if $r_i^0 = r^0$, $u_{ij}^0 = u^0 > 0$ for $i \neq j$, $u_{ij}^0 = u^0+v^0 > 0$ for $i = j$;
5. tetragonal ferromagnet if $r_1^0 \neq r_2^0 = r_3^0$, $u_{ij}^0 = u^0 > 0$, $i \neq j$, $u_{ij}^0 = u^0+v^0 > 0$, $i = j$.

It is clear that in the presence of the external field the phase transition can occur if some spontaneous, independent of the field mechanism of ordering exists. In the case under consideration it means that some of the components of the spin variable must be critical (see [14]). In other words, one can say, that at least one "critical direction" (direction on which the projection of the spin vector is a critical component) must exist. As mentioned above in the anisotropic ferromagnets in the presence of the external field the maximum number of critical directions cannot be greater than two. In consideration of the renormalization of the spin components we will consider the cases with one and two critical directions (class C_1 and C_2 respectively) separately.

Before undertaking a renormalization of the Hamiltonian (1) we make two transformations. The first in order to eliminate the linear terms in the Hamiltonian (1). This amounts to defining new spin variables [14]

$$\sigma_q^i \to \sigma_q^i + M_i \delta_q, \tag{3}$$

where M_i are chosen so that the linear terms in σ_q^i vanish.

Upon inserting (3) into (1) and neglecting the spin independent terms we obtain

$$\tilde{\mathscr{H}} = -\tfrac{1}{2} \sum_{i,j=1}^{3} \int_q (\tilde{r}_{ij} + q^2 \delta_{ij}) \sigma_q^i \sigma_{-q}^j - \sum_{i,j=1}^{3} \tilde{w}_{ij} \int_q \int_{q_1} \sigma_q^i \sigma_{q_1}^j \sigma_{-q-q_1}^j$$
$$- \sum_{i,j=1}^{3} \int_q \int_{q_1} \int_{q_2} u_{ij}^0 \sigma_q^i \sigma_{q_1}^i \sigma_{q_2}^j \sigma_{-q-q_1-q_2}^j \tag{4}$$

where

$$\tilde{r}_{ij} = r_i^0 \delta_{ij} + 8 M_i M_j u_{ij}^0 + 4 \delta_{ij} \sum_k M_k^2 u_{ik}^0 \tag{5}$$

and

$$\tilde{w}_{ij} = 4 M_i u_{ij}^0, \tag{6}$$

The elimination of the linear terms in σ_q^i is subject to choosing M_i as the unique roots of

$$h_i = M_i (r_i^0 + 4 \sum_j M_j^2 u_{ij}^0) \tag{7}$$

which go linearly to zero with h_i, respectively.

It is clear that the vector $\boldsymbol{M} = \{M_i\}$ has the meaning of the magnetization vector in paramagnetic phase. Accordingly, the direction along the vector $\boldsymbol{M}$ cannot be the critical one in the presence of the external field, because the projection of the spin on this direction does not vanish at any finite temperature. Thus, if some critical directions exist, they must be perpendicular to the vector $\boldsymbol{M}$.

The second transformation is to make a rotation so that one of the new spin components, say S_q^1, is parallel to the $\boldsymbol{M}$ direction given by the angles φ and θ. Thus, on putting

$$\sigma_q^i = \sum_j t_{ij} S_q^j, \tag{8}$$

where t_{ij} are the elements of the matrix T

$$T = \begin{bmatrix} \cos\varphi & \sin\theta\cos\theta & \sin\varphi\sin\theta \\ -\cos\psi\sin\varphi & \cos\psi\cos\varphi\cos\theta-\sin\theta\sin\psi & \cos\psi\cos\varphi\sin\theta+\cos\theta\sin\psi \\ \sin\psi\sin\varphi & -\sin\psi\cos\varphi\cos\theta-\cos\psi\sin\theta & -\sin\psi\cos\varphi\sin\theta+\cos\theta\cos\psi \end{bmatrix} \tag{9}$$

(the additional angle ψ is introduced to represent the Hamiltonian in a diagonal form in S^2 and S^3 up to the quadratic terms) we obtain the Hamiltonian in the new spin variables

$$\begin{aligned}\tilde{\tilde{\mathscr{H}}} = & -\tfrac{1}{2}\int\limits_q \sum_{i,j}(r_{ij}+q^2\delta_{ij})S^i_q S^j_{-q} - \int\limits_q\int\limits_{q_1}\sum_{i,j} w_{ij}S^i_q S^j_{q_1} S^j_{-q-q_1} - 2w\int\limits_q\int\limits_{q_1} S^1_q S^2_{q_1} S^3_{-q-q_1} \\ & -\int\limits_q\int\limits_{q_1}\int\limits_{q_2}\sum_{i,j} u_{ij}S^i_q S^i_{q_1} S^j_{q_2} S^j_{-q-q_1-q_2} - 4\int\limits_q\int\limits_{q_1}\int\limits_{q_1}\sum_{i\neq j} v_{ij}S^i_q S^j_{q_1} S^j_{q_2} S^j_{-q-q_1-q_2} \\ & -\int\limits_q\int\limits_{q_1}\int\limits_{q_2}\sum_{i\neq j\neq k} m_{ijk}S^i_q S^i_{q_1} S^j_{q_2} S^k_{-q-q_1-q_2}. \end{aligned} \tag{10}$$

The new interaction parameters as functions of the original ones and the field (through $\boldsymbol{M}$) are given in the Appendix.

In the next two sections we shall consider the conditions for the existence of the phase transitions of the C_2 and C_1 classes by using the Hamiltonian (10).

3. *XY-like critical behavior*

In this section we shall analyze the conditions for the existence of the XY-like phase transitions in anisotropic ferromagnets with an external field. In other words we shall find some general relations between the effective interaction parameters (which are functions of the original interaction parameters and the field) which must be satisfied for the phase transition XY-like to occur.

In the previous section, without loss of generality, we assumed that the component S^1 is a projection of the spin variable on the $\boldsymbol{M}$ direction, this means that this component cannot be critical. The question is when (what relations must be satisfied) the components S^2 and S^3 are critical. It is clear (see [14]) that first of all the following inequalities must be fulfilled

$$r_{11} > r_{22}, \quad r_{11} > r_{33}. \tag{11}$$

We now follow the usual routine of the renormalization group in the system with two types of variables; critical ones and non-critical ones. We integrate out all spin variables with momenta such that $b|q| > 1$ ($b > 1$) and rescale $\boldsymbol{q}$ into $b\boldsymbol{q}$ and S^i_q into $c_i S^i_{bq}$, where c_i is chosen so that some coefficients in the Hamiltonian remain unchanged [14]. In our case the spin components S^2 and S^3 should be critical at a given fixed point, then we shall choose

$$c_2 = c_3 \approx b^{\frac{d+2}{2}} \tag{12}$$

so that the coefficients of $q^2 S_q^i S_{-q}^i$ ($i = 2, 3$) remain unchanged. The component S_q^1 does not have critical behavior, then we shall choose [14]

$$c_1 = b^{d/2}[1+0(\varepsilon)] \tag{13}$$

so as r_{11} remains unchanged.

Under the spin rescalings (12) and (13) the following parameters in (10) become strongly irrelevant

$$w_{i1};\, u_{i1};\, v_{1i};\, v_{i1};\, m_{ijk} \tag{14}$$

(they go to zero rapidly as the renormalization process progresses). This is easy to see because the appropriate rescaling factors are smaller than 1 for $d \geqslant 3$, for example

$$\begin{aligned} w'_{11} &= c_1^2 b^{-2d}\{w_{11}+0(\varepsilon)\} = b^{-d/2}\{w_{11}+0(\varepsilon)\},\\ u'_{11} &= c_1^4 b^{-3d}\{u_{11}+0(\varepsilon)\} = b^{-d}\{u_{11}+0(\varepsilon)\},\\ m'_{123} &= c_1^2 c_2 c_3 b^{-3d}\{m_{123}+0(\varepsilon)\} = b^{2-d}\{m_{123}+0(\varepsilon)\}. \end{aligned} \tag{15}$$

On the other hand the following parameters are relevant (the appropriate rescaling factors are larger than 1)

$$r_{ij};\, w_{ij} \quad (i, j = 2, 3). \tag{16}$$

The displaced temperature variables are r_{22} and r_{33} connected with the phase transitions for two distinct types of ordering. In order for a stable critical point to exist all the rest relevant parameters must vanish. Thus, we obtain the conditions

$$r_{ij} = 0 \quad \text{for} \quad i \neq j; \quad w_{ij} = 0 \quad \text{for} \quad i, j = 2, 3. \tag{17}$$

The other parameters not mentioned in (14) and (16) can be relevant or not at given fixed point. It follows from (14) (if relations (16) are fulfilled) that after a sufficient number of iterations, the Hamiltonian (10) becomes renormalized to the form

$$\begin{aligned} \mathscr{H}_{\text{eff}} = &-\tfrac{1}{2}\int_q r_{11} S_q^1 S_{-q}^1 - \tfrac{1}{2}\int_q \sum_{i=2}^{3} (r_{ii}+q^2) S_q^i S_{-q}^i - 2w \int_q \int_{q_1} S_q^1 S_{q_1}^2 S_{-q-q_1}^3 \\ &- \int_q \int_{q_1} \sum_i w_{1i} S_q^1 S_{q_1}^i S_{-q-q_1}^i - \int_q \int_{q_1} \int_{q_2} \sum_{i,j=2}^{3} u_{ij} S_q^i S_{q_1}^i S_{q_2}^j S_{-q-q_1-q_2}^j \\ &-4 \int_q \int_{q_1} \int_{q_2} \sum_{i\neq j\neq 1} v_{ij} S_q^i S_{q_1}^j S_{q_2}^j S_{-q-q_1-q_2}^j. \end{aligned} \tag{18}$$

According to the paper by Nelson and Fisher [14] we can now explicitly integrate over the non-critical variable S^1 and then we obtain

$$\begin{aligned} \mathscr{H}_{\text{red}} = &-\tfrac{1}{2}\int_q \sum_{i=2}^{3} (r_{ii}+q^2) S_q^i S_{-q}^i - \int_q \int_{q_1} \int_{q_2} \sum_{i,j=2}^{3} \bar{u}_{ij} S_q^i S_{q_1}^i S_{q_2}^j S_{-q-q_1-q_2}^j \\ &-4 \int_q \int_{q_1} \int_{q_2} \sum_{i\neq j\neq 1} \bar{v}_{ij} S_q^i S_{q_1}^j S_{q_2}^j S_{-q-q_1-q_2}^j, \end{aligned} \tag{19}$$

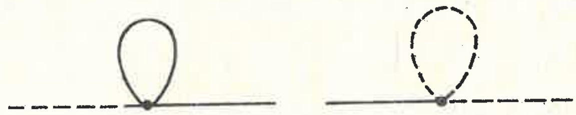
where

$$\bar{u}_{ii} = u_{ii} - \frac{w_{1i}^3}{2r_{11}}, \quad \bar{u}_{23} = \bar{u}_{32} = u_{23} - \frac{2w^2 + w_{12}w_{13}}{2r_{11}} \tag{20}$$

and

$$\bar{v}_{ij} = v_{ij} - \frac{ww_{1j}}{2r_{11}}. \tag{21}$$

It is easy to see that from v_{23} and v_{32} arise the following graphs



(22)

(solid line denotes the propagator $G_2(q) = (r_{22}+q^2)^{-1}$, broken line $G_3(q) = (r_{33}+q^2)^{-1}$ and we obtain the following recursion relation for $\bar{r}'_{23}$

$$\bar{r}'_{23} = b^2 3(\bar{v}_{32}+\bar{v}_{23}+ \dots). \tag{23}$$

Thus, after each stage of the iteration the terms $\bar{v}_{ij}S_q^i S_{q_1}^j S_{q_2}^j S_{-q-q_1-q_2}^j$ generate new quadratic terms in $\mathscr{H}_{\text{red}}(\bar{r}_{23}S_q^2 S_q^3)$. To avoid this the parameters v_{ij} must vanish

$$\bar{v}_{23} = \bar{v}_{32} = 0, \tag{24}$$

This leads to the reduced Hamiltonian

$$\mathscr{H}_{\text{red}} = -\tfrac{1}{2}\int_q \sum_{i=2}^{3} (r_{ii}+q^2)S_q^i S_{-q}^i - \int_q \int_{q_1} \int_{q_2} \sum_{i,j=2}^{3} \bar{u}_{ij} S_q^i S_{q_1}^i S_{q_2}^j S_{-q-q_1-q_2}^j. \tag{25}$$

This Hamiltonian has been studied by many authors (see e.g. [15]). It has been shown that the XY-like fixed point is the most stable one if

$$r_{22} = r_{33}. \tag{26}$$

This equality must, of course, be fulfilled for any value of the field, otherwise the Hamiltonian (25) describes a bicritical or tetracritical behavior [16].

Wilson and Fisher [17], while considering the Hamiltonian (25) with $r_{22} = r_{33}$ and $\bar{u}_{22} = \bar{u}_{33}$, found that the XY fixed point is stable for any initial condition with $\bar{u}_{23}$ in the range $0 < \bar{u}_{23} < 3\bar{u}_{22}$. A similar relation must be also satisfied in the general case ($\bar{u}_{22} \neq \bar{u}_{33}$)

$$0 < \bar{u}_{23} < \Lambda, \tag{27}$$

where Λ is some unknown function of $\bar{u}_{22}$ and $\bar{u}_{33}$ which we can only approximate from the above by

$$3\sqrt{\bar{u}_{22}\bar{u}_{33}}\,. \tag{28}$$

At $\bar{u}_{23} = 0$ and $u_{23} = \Lambda$ tricritical bahavior can occur.

Summarizing this section, we can conclude that the anisotropic ferromagnet in an external magnetic field described by the Hamiltonian (1) can undergo the XY-like continuous phase transition if the conditions (11), (17), (25), (27) and (28) are satisfied. In some cases the multicritical behavior can also occur.

4. Ising-like critical behavior

Let us consider now when only one of the spin components is critical variable. In other words consider what conditions must be fulfilled in order that the Hamiltonian (1) could describe Ising-like critical behavior.

Without loss of generality we can assume that the spin component S^3 is critical one whereas S^1 and S^2 are non-critical. For convenience, we shall introduce an additional rotation in the plane perpendicular to the critical direction:

$$\tilde{S}_q^1 = S_q^1 \cos\alpha + S_q^2 \sin\alpha, \quad \tilde{S}_q^2 = -S_q^1 \sin\alpha + S_q^2 \cos\alpha. \tag{29}$$

The form of the Hamiltonian (10) after transformation (29) remains unchanged and the interaction parameters are:

$$\begin{aligned} r_{11}^{(1)} &= r_{11}\cos^2\alpha + r_{22}\sin^2\alpha + 2r_{12}\sin\alpha\cos\alpha, \\ r_{22}^{(1)} &= r_{11}\sin^2\alpha + r_{22}\cos^2\alpha - 2r_{12}\sin\alpha\cos\alpha, \\ r_{12}^{(1)} &= (r_{22}-r_{11})\sin\alpha\cos\alpha + r_{12}(\cos^2\alpha - \sin^2\alpha), \\ r_{13}^{(1)} &= r_{13}\cos\alpha + r_{23}\sin\alpha, \\ r_{23}^{(1)} &= -r_{13}\sin\alpha + r_{23}\cos\alpha, \end{aligned} \tag{30}$$

and

$$w_{13}^{(1)} = w_{13}\cos\alpha + w_{23}\sin\alpha, \quad w_{23}^{(1)} = -w_{13}\sin\alpha + w_{23}\cos\alpha. \tag{31}$$

In the case under consideration only one critical variable exists (S^3), thus the condition analogous to (11) takes on the form

$$r_{33} < r_{11}^{(1)}, \quad r_{33} < r_{22}^{(1)} \tag{32}$$

and the appropriate rescaling factors are:

$$c_1 = c_2 = b^{d/2}[1+0(\varepsilon)], \quad c_3 \approx b^{\frac{d+2}{2}}. \tag{33}$$

Under these spin rescalings the following parameters:

$$w_{ij}^{(1)}, u_{ij}^{(1)}, \quad (i, j = 1, 2), \tag{34}$$

$$v_{ij}^{(1)}, m_{ijk}^{(1)}, \quad (i, j = 1, 2, 3) \tag{35}$$

become strongly irrelevant. It is clear that also the terms

$$\int_q \sum_{i=1}^{2} q^2 \tilde{S}_q^i \tilde{S}_{-q}^i \tag{36}$$

are irrelevant [14].

The parameters $r_{11}^{(1)}$, $r_{22}^{(1)}$ remain unchanged after the renormalization group transformation, whereas the following parameters are relevant:

$$r_{33},\ r_{13}^{(1)},\ r_{23}^{(1)},\ w_{33}. \tag{37}$$

Thus, we obtain the following conditions for the existence of the solutions of the C_1 class:

$$r_{13}^{(1)} = r_{23}^{(1)} = 0, \tag{38}$$

$$w_{33} = 0. \tag{39}$$

By using the transformation (29) we can diagonalize the quadratic terms of the Hamiltonian, which means that

$$r_{12}^{(1)} = 0 \tag{40}$$

and then we have

$$\mathscr{H}_{\mathrm{eff}} = -\tfrac{1}{2}\int_q \sum_{i=1}^{2} r_{ii}^{(1)}\tilde{S}_q^i\tilde{S}_{-q}^i - \tfrac{1}{2}\int_q (r_{33}+q^2)S_q^3S_{-q}^3$$
$$-\int_q\int_{q_1}\sum_{i=1}^{2} w_{i3}^{(1)}\tilde{S}_q^iS_{q_1}^3S_{-q-q_1}^3 - \int_q\int_{q_1}\int_{q_2} u_{33}S_q^3S_{q_1}^3S_{q_2}^3S_{-q-q_1-q_2}^3. \tag{41}$$

After integrating over the non-critical variables S^1 and S^2 we obtain the final form

$$\mathscr{H}_{\mathrm{red}} = -\tfrac{1}{2}\int_q (r_{33}+q^2)S_q^3S_{-q}^3 - \int_q\int_{q_1}\int_{q_2} \bar{u}_{33}S_q^3S_{q_1}^3S_{q_2}^3S_{-q-q_1-q_2}^3, \tag{42}$$

where

$$\bar{u}_{33} = u_{33} - \sum_{i=1}^{2} \frac{(w_{i3}^{(1)})^2}{2r_{ii}^{(1)}}. \tag{43}$$

This is the simplest nontrivial model [8] with an Ising-like stable fixed point for

$$\bar{u}_{33} > 0. \tag{44}$$

Thus, Hamiltonian (1) can describe the Ising-like critical behavior if the conditions (32), (34), (35), (38), (39), (40) and (44) are fulfilled.

5. *Conclusions*

We considered the three component vector model which can describe the critical behavior of several anisotropic ferromagnets in the presence of the external magnetic field. The appropriate LGW Hamiltonian has two types of fixed points: XY-like and Ising-like. We studied the conditions for the existence of these points separately and found some general relations between the interaction parameters and the external field which must be satisfied for the existence of the continuous phase transition to be possible.

The model considered in this paper has the XY-like critical behavior (phase transition is described by XY-like exponents) if:

I. $r_{22} = r_{33} < r_{11}$,
II. $r_{ij} = 0$, for $i \neq j$; (diagonalization of the Hamiltonian in the quadratic terms),
III. $w_{ij} = 0$, for $i, j = 2, 3$; (vanishing of the relevant third order in the critical variables terms),
IV. $v_{ij} - ww_{ij}/2r_{11} = 0$, for $i \neq j$ $(i, j = 2, 3)$; (vanishing of the terms odd with the respect of any of the critical variables in their products, e.g., the term $S_q^2 S_{q_1}^3 S_{q_2}^3 S_{-q-q_1-q_2}^3$ should vanish),
V. $0 < u_{23} - (2w^2 - w_{12}w_{13})/2r_{11} < \Lambda$; (determination of the region of the initial values of the interaction parameters). (45)

When relations (45) are not fulfilled the system can undergo the Ising-like phase transition if

I. $r_{33} < r_{11}^{(1)}, r_{22}^{(1)}$;
II. $r_{ij}^{(1)} = 0$, for $i \neq j$;
III. $w_{33} = 0$;
IV. $u_{33} - \sum_{i=1}^{2} [w_{i3}^{(1)}]^2 / 2r_{ii}^{(1)} > 0$.

In the other cases there is the first order phase transition, if any.

When taking into account the form of the effective interaction parameters (see Appendix) the conditions for the existence of the continuous phase transition (45) and (46) are rather complicated. However, they become considerably simplified for a given symmetry of the Hamiltonian. In fact, we can check them for a large class of anisotropic ferromagnets in an external magnetic field of arbitrary direction without difficulty. The calculations can, of course, be performed in any case but sometimes they are difficult.

In our next paper, using the conditions (45) and (46), we shall consider the phase transitions in some concrete anisotropic ferromagnets with the field having an arbitrary direction.

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APPENDIX

$$r_{11} = \tilde{r}_{11} \cos^2 \varphi + \tilde{r}_{22} \cos^2 \theta \sin^2 \varphi + \tilde{r}_{33} \sin^2 \theta \sin^2 \varphi + 2\tilde{r}_{12} \cos \theta \cos \varphi \sin \varphi$$
$$+ 2\tilde{r}_{13} \sin \theta \cos \varphi \sin \varphi + 2\tilde{r}_{23} \cos \theta \sin \theta \sin^2 \varphi; \qquad \text{(A1)}$$

$$r_{22} = \tilde{r}_{11} \cos^2 \psi \sin^2 \varphi + \tilde{r}_{22}(\sin^2 \psi \sin^2 \theta + \cos^2 \psi \cos^2 \theta \cos^2 \varphi$$
$$- 2\sin \psi \cos \psi \cos \theta \sin \theta \cos \varphi) + \tilde{r}_{33}(\sin^2 \psi \cos^2 \theta + \cos^2 \psi \cos^2 \varphi \sin^2 \theta$$
$$+ 2 \cos \psi \sin \psi \cos \theta \sin \theta \cos \varphi) + 2\tilde{r}_{12}(\cos \psi \sin \psi \sin \theta \sin \varphi - \cos^2 \psi \cos \theta \cos \varphi \sin \varphi)$$
$$- 2\tilde{r}_{13}(\cos^2 \psi \sin \theta \cos \varphi \sin \varphi + \cos \psi \sin \psi \cos \theta \sin \varphi) + 2\tilde{r}_{23}(\cos^2 \psi \cos \theta \sin \theta \cos^2 \varphi$$
$$+ \cos \psi \sin \psi \cos^2 \theta \cos \varphi - \cos \psi \sin \psi \sin^2 \theta \cos \varphi - \sin^2 \psi \cos \theta \sin \theta); \qquad \text{(A2)}$$

$$\begin{aligned}
r_{12} = &-\tilde{r}_{11}\cos\psi\cos\varphi\sin\varphi+\tilde{r}_{22})\cos\psi\cos^2\theta\cos\varphi\sin\varphi-\sin\psi\cos\theta\sin\theta\sin\varphi)\\
&+\tilde{r}_{33}(\cos\psi\sin^2\theta\cos\varphi\sin\varphi+\sin\psi\cos\theta\sin\theta\sin\varphi)+\tilde{r}_{12}(\cos\psi\cos\theta\cos^2\varphi\\
&-\cos\psi\cos\theta\sin^2\varphi-\sin\psi\sin\theta\sin\varphi)+\tilde{r}_{13}(\cos\psi\sin\theta\cos^2\varphi-\cos\psi\sin\theta\sin^2\varphi\\
&+\sin\psi\cos\theta\cos\varphi)+\tilde{r}_{23}(2\cos\psi\cos\theta\sin\theta\cos\varphi\sin\varphi+\sin\psi\cos^2\theta\sin\varphi\\
&-\sin\psi\sin^2\theta\sin\varphi); \qquad \text{(A3)}
\end{aligned}$$

$$\begin{aligned}
r_{23} = &-\tilde{r}_{11}\cos\psi\sin\psi\sin^2\varphi+\tilde{r}_{22}(\sin\psi\cos\psi\sin^2\theta-\sin\psi\cos\psi\cos^2\theta\cos^2\varphi\\
&+\sin^2\psi\cos\theta\sin\theta\cos\varphi-\cos^2\psi\cos\theta\sin\theta\cos\varphi)+\tilde{r}_{33}(\sin\psi\cos\psi\cos^2\theta\\
&-\sin\psi\cos\psi\sin^2\theta\cos^2\varphi+\cos^2\psi\cos\theta\sin\theta\cos\varphi-\sin^2\psi\cos\theta\sin\theta\cos\varphi)\\
&+\tilde{r}_{13}(2\cos\psi\sin\psi\sin\theta\cos\varphi\sin\varphi-\sin^2\psi\cos\theta\sin\varphi-\cos^2\psi\cos\theta\sin\psi)\\
&+\tilde{r}_{12}(2\cos\psi\sin\psi\cos\theta\cos\varphi\sin\varphi+\cos^2\psi\sin\theta\sin\varphi-\sin^2\psi\sin\theta\sin\varphi)\\
&-\tilde{r}_{23}(2\cos\psi\sin\psi\cos\theta\sin\theta\cos^2\varphi+2\cos\psi\sin\psi\cos\theta\sin\theta-(\cos^2\psi-\sin^2\psi)(\cos^2\theta\\
&-\sin^2\theta)\cos\varphi); \qquad \text{(A4)}
\end{aligned}$$

$$\begin{aligned}
w_{22} = &-\tilde{w}_{11}\cos^3\psi\sin^3\varphi+\tilde{w}_{21}\cos^2\psi\sin^2\varphi(\cos\psi\cos\varphi\cos\theta-\sin\theta\sin\psi)\\
&+\tilde{w}_{31}\cos^2\psi\sin^2\varphi(\cos\psi\cos\varphi\sin\theta+\sin\psi\cos\theta)+\tilde{w}_{22}(\cos\psi\cos\varphi\cos\theta-\sin\psi\sin\theta)^3\\
&-\tilde{w}_{12}\cos\psi\sin\varphi(\cos\psi\cos\varphi\cos\theta-\sin\psi\sin\theta)^2+\tilde{w}_{32}(\cos\psi\cos\varphi\sin\theta\\
&+\cos\theta\sin\psi)(\cos\psi\cos\varphi\cos\theta-\sin\psi\sin\theta)^2+\tilde{w}_{33}(\cos\psi\cos\varphi\sin\theta+\cos\theta\sin\psi)^3\\
&-\tilde{w}_{13}\cos\psi\sin\varphi(\cos\psi\cos\varphi\sin\theta+\sin\psi\cos\theta)^2+\tilde{w}_{23}(\cos\psi\cos\varphi\cos\theta-\sin\psi\sin\theta)\\
&\times(\cos\psi\cos\varphi\sin\theta+\sin\psi\cos\theta)^2; \qquad \text{(A5)}
\end{aligned}$$

$$\begin{aligned}
w_{23} = &-3\tilde{w}_{11}\cos\psi\sin^2\psi\sin^3\varphi+3\tilde{w}_{22}(\cos\psi\cos\varphi\cos\theta-\sin\psi\sin\theta)(\sin\psi\cos\varphi\cos\theta\\
&+\cos\psi\sin\theta)^2+3\tilde{w}_{33}(\cos\psi\cos\varphi\sin\theta+\sin\psi\cos\theta)(-\sin\psi\sin\theta\cos\varphi+\cos^2\psi\cos\theta)^2\\
&-\tilde{w}_{12}\{\cos\psi\sin\varphi(\sin\psi\cos\varphi\cos\theta+\sin\theta\cos\psi)^2+2\sin\psi\sin\varphi(\cos\psi\cos\varphi\cos\theta\\
&-\sin\psi\sin\theta)(\sin\psi\cos\varphi\cos\theta+\cos\psi\sin\theta)\}+\tilde{w}_{13}\{-\cos\psi\sin\varphi(-\sin\psi\cos\varphi\sin\theta\\
&+\cos\psi\cos\theta)^2+2\sin\psi\sin\varphi(\cos\psi\cos\varphi\sin\theta+\sin\psi\cos\theta)(-\sin\psi\cos\varphi\sin\theta\\
&+\cos\psi\cos\theta)\}+\tilde{w}_{23}\{(\cos\psi\cos\varphi\cos\theta-\sin\psi\sin\theta)(-\sin\psi\cos\varphi\sin\theta+\cos\psi\cos\theta)^2\\
&-2(\sin\psi\cos\varphi\cos\theta+\sin\theta\cos\psi)(\cos\psi\cos\varphi\sin\theta+\sin\psi\cos\theta)(-\sin\psi\cos\varphi\sin\theta\\
&+\cos\psi\cos\theta)\}+\tilde{w}_{21}\{\sin^2\psi\sin^2\varphi(\cos\psi\cos\varphi\cos\theta-\sin\psi\sin\theta)\\
&+2\cos\psi\sin\psi\sin^2\varphi(\sin\psi\cos\varphi\cos\theta+\cos\psi\sin\theta)\}\\
&+\tilde{w}_{31}\{\sin^2\psi\sin^2\varphi(\cos\psi\cos\varphi\sin\theta+\sin\psi\cos\theta)\\
&-2\cos\psi\sin\psi\sin^2\varphi(-\sin\psi\cos\varphi\sin\theta+\cos\psi\cos\theta)\}+\tilde{w}_{32}\{(\cos\psi\cos\varphi\sin\theta\\
&+\sin\psi\cos\theta)(\sin\psi\cos\varphi\cos\theta+\sin\theta\cos\psi)^2-2(-\sin\psi\cos\varphi\sin\theta\\
&+\cos\psi\cos\theta)(\cos\psi\cos\varphi\cos\theta-\sin\psi\sin\theta)(\sin\psi\cos\varphi\cos\theta+\sin\theta\cos\psi)\}; \qquad \text{(A6)}
\end{aligned}$$

$$w_{12} = 3\tilde{w}_{11}\cos^2\psi\cos\varphi\sin^2\varphi+3\tilde{w}_{22}\cos\theta\sin\varphi(\cos\psi\cos\varphi\cos\theta-\sin\psi\sin\theta)^2$$
$$+3\tilde{w}_{33}\sin\varphi\sin\theta(\cos\psi\cos\varphi\sin\theta+\sin\psi\cos\theta)^2+\tilde{w}_{12}\{\cos\varphi(\cos\psi\cos\varphi\cos\theta$$
$$-\sin\psi\sin\theta)^2-2\cos\psi\cos\theta\sin^2\varphi(\cos\psi\cos\varphi\cos\theta-\sin\psi\sin\theta)\}$$
$$+\tilde{w}_{13}\{\cos\varphi(\cos\psi\cos\varphi\sin\theta+\sin\psi\cos\theta)^2-2\cos\psi\sin\theta\sin^2\varphi(\cos\psi\cos\varphi\sin\theta$$
$$+\sin\psi\cos\theta)\}+\tilde{w}_{23}\{\cos\theta\sin\varphi(\cos\psi\cos\varphi\sin\theta+\sin\psi\cos\theta)^2$$
$$+2\sin\theta\sin\varphi(\cos\psi\cos\varphi\cos\theta-\sin\psi\sin\theta)(\cos\psi\cos\varphi\sin\theta+\sin\psi\cos\theta)\}$$
$$+\tilde{w}_{21}\{\sin^3\varphi\cos^2\psi\cos\theta-2\cos\psi\cos\varphi\sin\varphi(\cos\psi\cos\psi\cos\theta-\sin\psi\sin\theta)\}$$
$$\tilde{w}_{31}\{\sin^3\varphi\sin\theta\cos^2\psi-2\cos\psi\cos\varphi\sin\varphi(\cos\psi\cos\varphi\sin\theta+\cos\theta\sin\psi)\}$$
$$+\tilde{w}_{32}\{\sin\varphi\sin\theta(\cos\psi\cos\varphi\cos\theta-\sin\psi\sin\theta)^2+2\cos\theta\sin\varphi(\cos\psi\cos\varphi\cos\theta$$
$$-\sin\psi\sin\theta)(\cos\psi\cos\varphi\sin\theta+\sin\psi\cos\theta)\}, \tag{A7}$$

$$w = -3\tilde{w}_{11}\cos\varphi\sin^2\varphi\cos\psi\sin\psi+3\tilde{w}_{22}\cos\theta\sin\varphi(\cos\psi\cos\varphi\cos\theta$$
$$-\sin\psi\sin\theta)(\sin\psi\cos\varphi\cos\theta+\cos\psi\sin\theta)+3\tilde{w}_{33}\sin\varphi\sin\theta(\cos\psi\cos\varphi\sin\theta$$
$$+\sin\psi\cos\theta)(-\sin\psi\cos\varphi\sin\theta+\cos\psi\cos\theta)+\tilde{w}_{12}\{\cos\theta\sin^2\varphi\sin\psi(\cos\psi\cos\varphi\cos\theta$$
$$-\sin\psi\sin\theta)+\cos\theta\sin^2\varphi\cos\psi(\sin\psi\cos\varphi\cos\theta+\cos\psi\sin\theta)$$
$$-\cos\varphi(\cos\psi\cos\varphi\cos\theta-\sin\psi\sin\theta)(\sin\psi\cos\varphi\cos\theta+\cos\psi\sin\theta)\}$$
$$+\tilde{w}_{13}\{\cos\varphi(\cos\psi\cos\varphi\sin\theta+\sin\psi\cos\theta)(-\sin\psi\cos\varphi\sin\theta+\cos\psi\cos\theta)$$
$$-\cos\psi\sin^2\varphi\sin\theta(-\sin\psi\cos\varphi\sin\theta+\sin\psi\cos\theta)+\sin\psi\sin^2\varphi\sin\theta(\cos\psi\cos\varphi\sin\theta$$
$$+\sin\psi\cos\theta)\}+\tilde{w}_{23}\{\cos\theta\sin\varphi(\cos\psi\cos\varphi\sin\theta+\sin\psi\cos\theta)$$
$$\times(-\sin\psi\cos\varphi\sin\theta+\cos\psi\cos\theta)+\sin\varphi\sin\theta(\cos\psi\cos\varphi\cos\theta$$
$$-\sin\psi\sin\theta)(-\sin\psi\cos\varphi\sin\theta+\cos\psi\cos\theta)-\sin\varphi\sin\theta(\sin\psi\cos\varphi\cos\theta$$
$$+\sin\theta\cos\psi)(\cos\psi\cos\varphi\sin\theta+\sin\psi\cos\theta)\}+\tilde{w}_{21}\{-\cos\theta\sin^2\varphi\cos\psi\sin\psi$$
$$+\cos\varphi\sin\varphi\sin\psi(\cos\psi\cos\varphi\cos\theta-\sin\psi\sin\theta)$$
$$+\cos\varphi\sin\varphi\cos\psi(\sin\psi\cos\varphi\cos\theta+\cos\psi\sin\theta)\}$$
$$+\tilde{w}_{31}\{-\cos\psi\sin\psi\sin^3\varphi\sin\theta+\sin\psi\cos\varphi\sin\varphi(\cos\psi\cos\varphi\sin\theta+\sin\psi\cos\theta)$$
$$-\cos\psi\cos\varphi\sin\varphi(-\sin\psi\cos\varphi\sin\theta+\cos\psi\cos\theta)\}$$
$$+\tilde{w}_{32}\{-\sin\varphi\sin\theta(\cos\psi\cos\varphi\cos\theta-\sin\psi\sin\theta)(\sin\psi\cos\varphi\cos\theta+\sin\theta\cos\psi)$$
$$-\sin\varphi\cos\theta(\cos\psi\cos\varphi\sin\theta+\sin\psi\cos\theta)(\sin\psi\cos\varphi\cos\theta+\cos\psi\sin\theta)$$
$$+\sin\varphi\cos\theta(\cos\psi\cos\varphi\cos\theta-\sin\psi\sin\theta)(-\sin\psi\cos\varphi\sin\theta+\cos\psi\cos\theta)\}; \tag{A8}$$

$$u_{22} = u^0_{11}\cos^4\psi\sin^4\varphi+u^0_{22}(\cos\psi\cos\varphi\cos\theta-\sin\psi\sin\theta)^4$$
$$+u^0_{33}(\cos\psi\cos\varphi\sin\theta+\sin\psi\cos\theta)^4+2u^0_{12}\cos^2\psi\sin^2\varphi(\cos\psi\cos\varphi\cos\theta$$
$$-\sin\psi\sin\theta)^2+2u^0_{13}\cos^2\psi\sin^2\varphi(\cos\psi\cos\varphi\sin\theta+\sin\psi\cos\theta)^2$$
$$+2u^0_{23}(\cos\psi\cos\varphi\cos\theta-\sin\psi\sin\theta)^2(\cos\psi\cos\varphi\sin\theta+\sin\psi\cos\theta)^2; \tag{A9}$$

$$u_{23} = 3u_{11}^0 \cos^2\psi \sin^2\psi \sin^4\varphi + 3u_{22}^0(\sin\psi\cos\varphi\cos\theta + \sin\theta\cos\psi)^2$$
$$\times(\cos\psi\cos\varphi\cos\theta - \sin\psi\sin\theta)^2 + 3u_{33}^0(-\sin\psi\cos\varphi\sin\theta + \cos\theta\cos\psi)^2$$
$$\times(\cos\psi\cos\varphi\sin\theta - \sin\psi\cos\theta)^2 + u_{12}^0\{\cos^2\psi\sin^2\varphi(\sin\psi\cos\varphi\cos\theta + \cos\psi\sin\theta)^2$$
$$+\sin^2\psi\sin^2\varphi(\cos\psi\cos\varphi\cos\theta - \sin\psi\sin\theta)^2 + 4\cos\psi\sin\psi\sin^2\varphi$$
$$\times(\cos\psi\cos\varphi\cos\theta - \sin\psi\sin\theta)(\sin\psi\cos\varphi\cos\theta + \cos\psi\sin\theta)\}$$
$$+u_{13}^0\{\cos^2\psi\sin^2\varphi(-\sin\psi\cos\varphi\sin\theta + \cos\psi\cos\theta)^2$$
$$+\sin^2\psi\sin^2\varphi(\cos\psi\cos\varphi\sin\varphi + \sin\psi\cos\theta)^2 - 4\cos\psi\sin\psi\sin^2\varphi$$
$$\times(\cos\psi\cos\varphi\sin\theta + \sin\psi\cos\theta)(-\sin\psi\cos\varphi\sin\theta + \cos\psi\cos\theta)\}$$
$$+u_{23}^0\{(\cos\psi\cos\varphi\cos\theta - \sin\psi\sin\theta)^2(\sin\psi\cos\varphi\sin\theta - \cos\psi\cos\theta)^2$$
$$+(\sin\psi\cos\varphi\cos\theta + \cos\psi\sin\theta)^2(\cos\psi\cos\varphi\sin\theta + \sin\psi\cos\theta)^2$$
$$-4(\cos\psi\cos\theta\cos\varphi - \sin\psi\sin\theta)(\sin\psi\cos\varphi\cos\theta + \sin\theta\cos\psi)$$
$$\times(\cos\psi\cos\varphi\sin\theta + \sin\psi\cos\theta)(-\sin\psi\cos\varphi\sin\theta + \cos\theta\cos\psi)\}; \tag{A10}$$

$$v_{23} = -u_{11}^0 \cos\psi\sin^3\psi\sin^4\varphi - u_{22}^0(\sin\psi\cos\varphi\cos\theta + \sin\theta\cos\psi)^3$$
$$\times(\cos\psi\cos\varphi\cos\theta - \sin\psi\sin\theta) + u_{33}^0(-\sin\psi\cos\varphi\sin\theta + \cos\psi\cos\theta)^3$$
$$\times(\cos\psi\cos\varphi\sin\theta + \sin\psi\cos\theta) - u_{12}^0\{\sin^2\psi\sin^2\varphi(\cos\psi\cos\varphi\cos\theta - \sin\psi\sin\theta)$$
$$\times(\sin\psi\cos\varphi\cos\theta + \cos\psi\sin\theta) + \cos\psi\sin\psi\sin^2\varphi(\sin\psi\cos\varphi\cos\theta + \cos\psi\sin\theta)^2\}$$
$$+u_{13}^0\{\sin^2\psi\sin^2\varphi(\cos\psi\cos\varphi\sin\theta + \sin\psi\cos\theta)(-\sin\psi\cos\varphi\sin\theta + \cos\psi\cos\theta)$$
$$-\cos\psi\sin\psi\sin^2\varphi(-\sin\psi\cos\varphi\sin\theta + \cos\psi\cos\theta)^2\} + u_{23}^0\{(\sin\psi\cos\varphi\cos\theta$$
$$+\sin\theta\cos\psi)^2(\cos\psi\cos\varphi\sin\theta + \sin\psi\cos\theta)(-\sin\psi\cos\varphi\sin\theta + \cos\psi\cos\theta)$$
$$-(-\sin\psi\cos\varphi\sin\theta + \cos\psi\cos\theta)^2(\cos\psi\cos\varphi\cos\theta - \sin\psi\sin\theta)$$
$$\times(\sin\psi\cos\varphi\cos\theta + \sin\theta\cos\psi)\}. \tag{A11}$$

After introducing the following transformations

$$\mathrm{I}:\ \cos\psi \to -\sin\psi, \quad \sin\psi \to \cos\psi, \tag{A12}$$
$$\mathrm{I}^*:\ \cos\psi \to \sin\psi, \quad \sin\psi \to -\cos\psi, \tag{A13}$$

we can write the remaining relevant parameters in the form

$$r_{33} = I(r_{22}) = I^*(r_{22}); \qquad w_{32} = I(w_{23}) = -I^*(w_{23});$$
$$r_{13} = I(r_{12}) = -I^*(r_{12}); \qquad w_{13} = I(w_{12}) = I(w_{12});$$
$$w_{33} = I(w_{22}) = -I^*(w_{22}); \qquad u_{33} = I(u_{22}) = I^*(u_{22});$$
$$v_{32} = -I(v_{23}) = -I^*(v_{23}). \tag{A14}$$

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