NONLINEAR PROPAGATION OF ELECTROMAGNETIC WAVES IN STRONG MAGNETIC FIELDS

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The propagation of plane electromagnetic waves in strong constant magnetic fields is studied. It exhibits nonlinear effects (the generation of the second harmonics, field-dependent modification of the phase of the wave) due to the vacuum polarization.

The propagation of plane electromagnetic waves in space, in the presence of a strong homogeneous magnetic field, is studied here. According to quantum electrodynamics, this propagation is nonlinear as a result of virtual pair creation and annihilation. Due to those polarization effects the vacuum behaves as a medium in which nonlinear effects in the propagation of waves are enhanced by the presence of a strong constant field.

The propagation of a single photon (or a very weak wave) was studied previously ([1–3]) in connection with the analysis of the light spectrum from neutron stars, which are surrounded by very intense magnetic fields. More recently, [4], the scattering of light by light was studied in the limit of ultra intense waves. In the present study I am interested in the intermediate case, when the background magnetic field is very strong, even of the order of the critical field B_{cr} ($B_{cr} = m^2c^3/e\hbar = 4.41 \cdot 10^{13}$ G), but the wave is much weaker, though still strong enough to show appreciable nonlinear self-interaction. To describe this situation I use the effective Lagrangian derived from quantum electrodynamics by Heisenberg and Euler [5] and Weisskopf [6]. This Lagrangian, rederived later in a very elegant and concise way by Schwinger [7], has the following form,

$$L = \frac{S}{4\pi} - \frac{1}{8\pi^2} \int_0^\infty du u^{-3} e^{-m^2 u} \left[(eu)^2 P \frac{\operatorname{Re}\cosh(euX)}{\operatorname{Im}\cosh(euX)} - 1 + \tfrac{2}{3}(eu)^2 S \right], \qquad (1)$$

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where

$$S = -\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} = \tfrac{1}{2}(\boldsymbol{E}^2 - \boldsymbol{B}^2),$$

$$P = -\tfrac{1}{8}\, \varepsilon^{\mu\nu\lambda\varrho} F_{\mu\nu}F_{\lambda\varrho} = \boldsymbol{E}\cdot\boldsymbol{B},$$

$$X^2 = 2(-S+iP).$$

As in Ref. [2], I use the unrationalized electromagnetic units and put $c = 1 = \hbar$ ($\alpha = e^2$). The field $F_{\mu\nu}$ is the sum of the electromagnetic wave field $f_{\mu\nu}$ and the constant field $F^{(0)}_{\mu\nu}$,

$$F_{\mu\nu} = f_{\mu\nu} + F^{(0)}_{\mu\nu}.$$

The applicability of the Lagrangian (1) is restricted to electromagnetic fields which are not very rapidly varying in space and in time; the scale being set by the electron Compton wave length $\hbar/mc$ and corresponding time $\hbar/mc^2$. Thus, even X rays can be well described in this way. The following field equations, which are written in vector notation, follow from the nonlinear Lagrangian,

$$\partial_t \boldsymbol{B} + \operatorname{rot} \boldsymbol{E} = 0, \tag{2}$$

$$\partial_t \boldsymbol{D} - \operatorname{rot} \boldsymbol{H} = 0, \tag{3}$$

$$\operatorname{div} \boldsymbol{B} = 0, \tag{4}$$

$$\operatorname{div} \boldsymbol{D} = 0, \tag{5}$$

where

$$\boldsymbol{D} = \frac{\partial L}{\partial S}\boldsymbol{E} + \frac{\partial L}{\partial P}(\boldsymbol{B}+\boldsymbol{B}_0), \tag{6}$$

$$\boldsymbol{H} = \frac{\partial L}{\partial S}(\boldsymbol{B}+\boldsymbol{B}_0) - \frac{\partial L}{\partial P}\boldsymbol{E}. \tag{7}$$

Here $\boldsymbol{E}(z, t)$ and $\boldsymbol{B}(z, t)$ describe the plane electromagnetic wave and $\boldsymbol{B}_0$ is a constant background magnetic field. For the maximum effect I will assume, that $\boldsymbol{B}_0$ is orthogonal to the direction of propagation (i.e. to the z-axis).

Two linear polarizations of the wave are considered: the parallel mode ($\|$ mode) with the vector $\boldsymbol{B}(z, t)$ parallel to the direction of the constant field $\boldsymbol{B}_0$, and the perpendicular mode ($\perp$ mode) with the vector $\boldsymbol{B}(z, t)$ orthogonal to $\boldsymbol{B}_0$. They behave quite differently.

The parallel mode may propagate alone, not generating the wave with the other polarization. To prove this, I assume that at some instant $t = 0$ the $\perp$ mode is absent. It means that the component $E_{\|}(z, t)$ of $\boldsymbol{E}(z, t)$ along $\boldsymbol{B}_0$ and the component $B_{\perp}(z, t)$ of $\boldsymbol{B}(z, t)$ orthogonal to $\boldsymbol{B}_0$ vanish at $t = 0$. Then it follows from Eq. (2) that $\partial_t B_{\perp} = 0$ at $t = 0$. From Eq. (3) we get:

$$\left[\frac{\partial L}{\partial S} + 2\frac{\partial L}{\partial P^2}(B_{\|}+B_0)^2\right]_{t=0} \partial_t E_{\|}|_{t=0} = 0,$$

and hence

$$\partial_t E_{\|} = 0 \quad \text{at} \quad t = 0.$$

To obtain the above equality we used the following formula

$$\partial_t \left.\frac{\partial L}{\partial P}\right|_{t=0} = 2\,\frac{\partial L}{\partial P^2}(B_{\|}+B_0)\partial_t E_{\|}|_{t=0}$$

which is satisfied if the Lagrangian depends on P through P^2.

Thus, those components which were absent at $t = 0$ will not be generated at later times.

The situation is different for the perpendicular mode. The parallel mode is always generated by the perpendicular mode; the assumption that the $\perp$ mode is present and the $\|$ mode is absent contradicts the field equations.

Both cases are treated in this paper by a perturbation method based on the assumption that the ratio of the wave amplitude to the critical field B_{cr} is small. One may then use the expansion of the field equations into powers of the vectors $\boldsymbol{E}(z, t)$ and $\boldsymbol{B}(z, t)$.

For the $\|$ mode alone the field equations up to quadratic terms have the following form:

$$\partial_t B_{\|}(z, t) - \partial_z E_{\perp}(z, t) = 0, \tag{8}$$

$$\partial_t E_{\perp}(z, t) - n_{\|}^{-2}\partial_z B_{\|}(z, t) = \frac{B_0}{2\gamma_s}\left[\gamma_{ss}\partial_t(2E_{\perp}B_{\|}+E_{\perp}^2)+(-3\gamma_{ss}+\gamma_{sss}B_0^2)\partial_z(B_{\|}^2)\right], \tag{9}$$

where $n_{\|}$ is the index of refraction for the $\|$ mode in the case of a very weak wave, when the linearlized theory may be used [1, 2],

$$n_{\|} = [(\gamma_s - \gamma_{ss}B_0^2)/\gamma_s]^{-1/2} \simeq 1+\gamma_{ss}B_0^2/2\gamma_s = 1+\delta n_1. \tag{10}$$

By γ_s, γ_{ss}, γ_{sss} I denote the derivatives of the Lagrangian (1) evaluated in the case when only the constant field B_0 is present (i.e. when $P = 0$ and $S = -B_0^2/2$),

$$\gamma_s = \left.\frac{\partial L}{\partial S}\right|_{\substack{P=0\\ S=-B_0^2/2}}, \quad \gamma_{ss} = \left.\frac{\partial^2 L}{\partial S^2}\right|_{\substack{P=0\\ S=-B_0^2/2}}, \quad \gamma_{sss} = \left.\frac{\partial^3 L}{\partial S^3}\right|_{\substack{P=0\\ S=-B_0^2/2}}.$$

The following Ansatz solves Eq. (8),

$$E_{\perp}(z, t) = f(\tau, \eta)+g(\tau, \eta), \tag{11}$$

$$B_{\|}(z, t) = n_{\|}[-f(\tau, \eta)+g(\tau, \eta)], \tag{12}$$

$$\partial_\tau g(\tau, \eta) = \partial_\eta f(\tau, \eta), \tag{13}$$

which is consistent with the existence of e.m. potentials. I used here a convenient set of variables:

$$\tau = \omega(t-n_{\|}z), \quad \eta = \omega(t+n_{\|}z).$$

I assume that in the lowest order of perturbation $g = 0, f(\tau, \eta) = Af(\tau)$ and that in higher orders the function $f(\tau, \eta)$ has the form[1]: $f(\tau, \eta) = Af[\tau + \varphi(\tau, \eta)]$; the function $\varphi(\tau, \eta)$ will be sought in the form of a power series. This means that in the lowest order we have a plane monochromatic wave of arbitrary shape (function f is arbitrary) propagating along the positive z-axis. In higher orders we allow for modulations of the phase of the wave and for an additional term $g(\tau, \eta)$ which changes the ratio of the magnetic to the electric component of the wave. In the second order of this perturbation method I obtain from Eqs. (9) and (13):

$$4\omega\partial_\tau g = -\omega A^2 n_{\|}\gamma_1\partial_\tau f^2(\tau), \tag{14}$$

$$\partial_\tau g = A\partial_\tau f(\tau)\partial_\eta\varphi(\tau, \eta), \tag{15}$$

where

$$\gamma_1 = \frac{B_0}{2\gamma_s}[3\gamma_{ss}(1-n_{\|}^2)+\gamma_{sss}B_0^2 n_{\|}]. \tag{16}$$

The solution is:

$$\varphi(\tau, \eta) = -\tfrac{1}{2}n_{\|}\gamma_1 Af(\tau)\eta + \varphi(\tau), \tag{17}$$

$$g(\tau, \eta) = -\tfrac{1}{4}n_{\|}\gamma_1 A^2 f^2(\tau) + C(\eta), \tag{18}$$

with two arbitrary functions $\varphi(\tau)$ and $C(\eta)$. A special solution, obtained for $\varphi(\tau) = 0 = C(\eta)$, is

$$E_\perp(z, t) = Af[\tau - \tfrac{1}{2}n_{\|}\gamma_1 Af(\tau)\eta] - \tfrac{1}{4}n_{\|}\gamma_1 A^2 f^2(\tau), \tag{19}$$

$$B_{\|}(z, t) = -n_{\|}Af[\tau - \tfrac{1}{2}n_{\|}\gamma_1 Af(\tau)\eta] - \tfrac{1}{4}n_{\|}^2\gamma_1 A^2 f^2(\tau). \tag{20}$$

This solution describes the wave in the $\|$ mode generating the second harmonics of the same polarization. The phase of the original wave is modified, due to the appearance of the additional amplitude-dependent oscillating index of refraction, $n_{\|} = 1 + \delta n_1 + \delta n_2$,

$$\delta n_2 = \gamma_1 Af(\tau) = \chi E_\perp(\tau, t)/B_{cr}. \tag{21}$$

The dimensionless coeficient χ depends on the constant background field B_0 alone and is given by the formula which follows from Eqs. (1) and (10),

$$\chi = \gamma_1 B_{cr} \simeq 2\pi B_0^3 B_{cr}\gamma_{sss}$$

$$= \frac{\alpha}{2\pi}\frac{1}{w}\int_0^\infty du\, u^{-1}e^{-u/w}\left[\frac{3}{4u}\coth u + (\tfrac{3}{4} - u^2)\sinh^{-2}u - \tfrac{3}{2}\frac{u^2}{2\sinh^4 u}\right], \tag{22}$$

where $w = B_0/B_{cr}$.

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[1] I adopt this special form of the function f to avoid secular terms in the solution.

To obtain the approximate equality I have taken into account that $\gamma_s \simeq (4\pi)^{-1}$ and $3\gamma_{ss}^2 \ll \gamma_{sss}$. The coefficient χ was computed numerically and is plotted on Fig. 1. It attains its maximum value when the field B_0 is 3 times larger than the critical field. On Fig. 1 I show also δn_1 which measures the deviation of the index of refraction from unity due to the

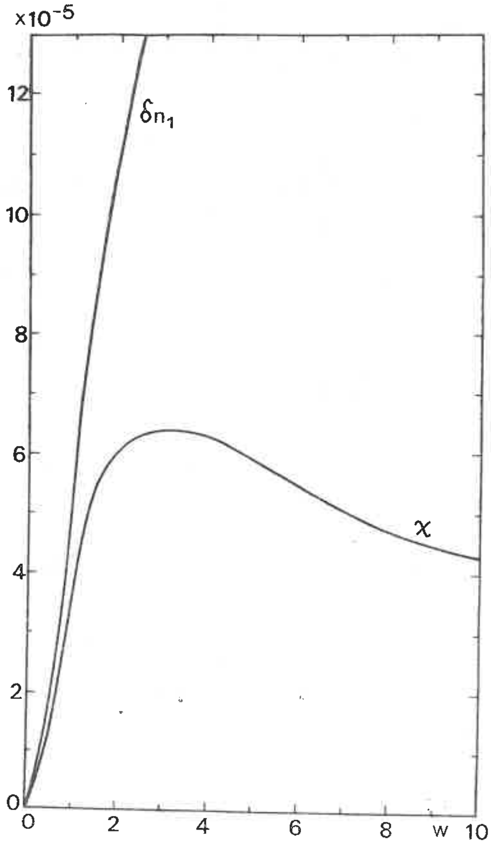

Fig. 1. The dimensionless coefficient χ which measures the change in the index of refraction caused by self-interaction is plotted together with the correction δn_1 to the index of refraction due to the background field B_0 alone. Both are functions of the ratio $w = B_0/B_{cr}$

background field alone. Coefficients χ and δn_1 are of the same order of magnitude[2] for B_0 up to B_{cr}. The ratio $\chi/\delta n_1$ attains its maximum value ($\sim$ 0.8) when $B_0 \simeq B_{cr}$.

To describe the propagation of the $\perp$ mode, field equations for both polarizations must be used. Up to terms quadratic in $\boldsymbol{E}$ and $\boldsymbol{B}$ they are of the form:

$$\partial_t B_\perp(z, t) + \partial_z E_\parallel(z, t) = 0, \tag{23}$$

[2] For very large B_0 ($w \to \infty$) the function δn_1 will eventually level off tending to the value $38.7 \cdot 10^{-5}$.

$$n_\perp^2 \partial_t E_\parallel(z,t) + \partial_z B_\perp(z,t) = \frac{B_0}{\gamma_s}\,[(\gamma_{ss} - \gamma_{pp} + \gamma_{spp} B_0^2)\partial_t(E_\parallel B_\parallel)$$

$$+\gamma_{ss}\partial_z(B_\parallel B_\perp) + \gamma_{pp}\partial_z(E_\parallel E_\perp)], \tag{24}$$

$$\partial_t B_\parallel(z,t) - \partial_z E_\perp(z,t) = 0, \tag{25}$$

$$\partial_t E_\perp(z,t) - n_\parallel^{-2}\partial_z B_\parallel(z,t) = \frac{B_0}{2\gamma_s}\,[\gamma_{ss}\partial_t(2E_\parallel B_\perp + E_\perp^2) - 2\gamma_{pp}\partial_t(E_\parallel B_\perp)$$

$$+(-3\gamma_{ss} + \gamma_{sss} B_0^2)\partial_z(B_\parallel^2) + (\gamma_{ss} - 2\gamma_{pp} + \gamma_{spp} B_0^2)\partial_z(E_\parallel^2) - \gamma_{ss}\partial_z(B_\perp^2)], \tag{26}$$

where $n_\perp$ is the index of refraction for the $\perp$ mode in the linearized theory,

$$n_\perp = [(\gamma_s + \gamma_{pp} B_0^2)/\gamma_s]^{1/2} \simeq 1 + \gamma_{pp} B_0^2/2\gamma_s, \tag{27}$$

and

$$\gamma_{pp} = \left.\frac{\partial^2 L}{\partial P^2}\right|_{\substack{P=0\\ S=-B_0^2/2}}, \qquad \gamma_{spp} = \left.\frac{\partial^3 L}{\partial S \partial P^2}\right|_{\substack{P=0\\ S=-B_0^2/2}}.$$

In the lowest order we take again a plane monochromatic wave of an arbitrary shape in pure $\perp$ mode, propagating along the positive z-axis. It means that in this order

$$E_\parallel(z,t) = Af(\tau'), \qquad E_\perp(z,t) = 0,$$
$$B_\perp(z,t) = n_\perp Af(\tau'), \qquad B_\parallel(z,t) = 0,$$

where $\tau' = \omega(t - n_\perp z)$.

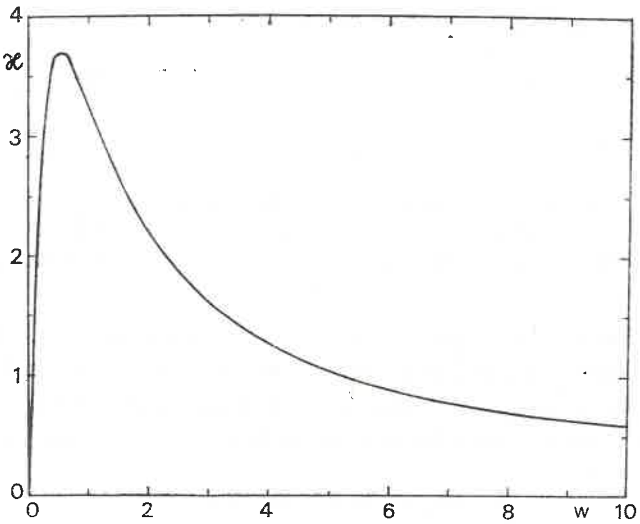

Fig. 2. The dimensionless coefficient χ, which measures the efficiency of the second harmonic generation of the $||$ mode by the $\perp$ mode is shown as a function of $w = B_0/B_{cr}$

We seen then that the r.h.s. of Eq. (24) has no second order terms, so that the wave in the $\perp$ mode does not undergo any change in this order of perturbation theory. The solution of Eqs. (25) and (26) for the $\|$ mode has the form:

$$E_{\perp}(z, t) = -\frac{n_{\perp}^{-1}\gamma_2}{1-n_{\perp}^2 n_{\|}^{-2}} A^2 f^2(\tau') = -\frac{n_{\perp}^{-1}\gamma_2}{1-n_{\perp}^2 n_{\|}^{-2}} E_{\|}^2(z, t), \tag{28}$$

$$B_{\|}(z, t) = -n_{\perp} E_{\perp}(z, t), \tag{29}$$

where

$$\gamma_2 = \frac{B_0}{2\gamma_s}[\gamma_{ss}(1-n_{\perp}^{-2})+\gamma_{spp}B_0^2]. \tag{30}$$

This means that the second harmonics with the $\|$ polarization is generated by the $\perp$ mode. The efficiency of this generation is a function of the background magnetic field. The ratio of the amplitude of the $\|$ mode to the amplitude of the $\perp$ mode is measured by a dimensionless coefficient κ,

$$E_{\perp}(z, t)/E_{\|}(z, t) = \kappa E_{\|}(z, t)/B_{\rm cr}, \tag{31}$$

$$\kappa = -\frac{n_{\perp}^{-1}\gamma_2 B_{\rm cr}}{1-n_{\perp}^2 n_{\|}^{-2}} = \frac{n_{\perp}^{-1} B_0 B_{\rm cr}[\gamma_{spp}+\gamma_{ss}\gamma_{pp}/(\gamma_s+\gamma_{pp}B_0^2)]}{2\gamma_s(\gamma_{pp}-\gamma_{ss}+\gamma_{ss}\gamma_{pp}B_0^2)} \simeq \frac{2\pi B_0 B_{\rm cr}\gamma_{spp}}{\gamma_{pp}-\gamma_{ss}}. \tag{32}$$

The functions γ_{spp}, γ_{pp} and γ_{ss} were calculated numerically from the Lagrangian (1). The coefficient κ is plotted on Fig. 2. It attains its maximum value ($\kappa_{\max} \simeq 3.7$) when $B_0 \simeq 0.5\, B_{\rm cr}$. The coefficient κ is quite large. It is greater than unity in the wide range of the values of B_0 from small fractions of $B_{\rm cr}$ to about $5B_{\rm cr}$. The reason for that is the smallness of the denominator in Eq. (32) which expresses the fact that the birefringence of vacuum caused by the presence of the background magnetic field is small; both modes propagate with almost the same velocities.

Although phenomena predicted by this calculation are not yet measurable, they are of interest since they are purely quantum effects of the vacuum polarization.

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