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GROWTH AND DECAY OF WEAK MHD WAVES IN GASES AT VERY HIGH TEMPERATURE

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The propagation of weak MHD waves has been studied in gases at very high temperature. The law of propagation has been determined and the growth and decay of waves has been discussed. A Bernoulli type differential equation governing the local and global behaviour of the wave amplitude has been obtained. Some particular cases of interest have also been discussed.

1. Introduction

In many new technological developments the effects of thermal radiation play a very important role in the determination of the flow field. A good deal of work has been done on the problems of radiation-gas dynamics with radiative heat flux effects [1–4], but very little work has been done by accounting for radiation stresses. In this paper our main academic interest is to study the growth and decay of weak MHD waves by accounting for the effects of radiation pressure and radiation energy and to examine the local and global behaviour of the wave amplitude during propagation.

We assume an optically thick gray gas with such a high temperature and low pressure that the radiation pressure number R_p is not negligible, but the profiles structured by radiant heat-transfer terms are assumed to be imbedded in the discontinuities. The flow field is restricted to be an ideal radiating gas in which viscous and heat conducting terms are negligible. When the temperature of the gas is of the order of 10^5 K or more, the radiation pressure and radiation energy density must be taken into account in determining the flow-behaviour.

Now we assume that there exists a moving singular surface $\Sigma(t)$ of a weak discontinuity across which the flow and field parameters are continuous, but their first and higher order derivatives are discontinuous. Such a discontinuity is defined as a weak wave or a discontinuity. In the subsequent analysis we shall study the law of wave propagation and the global behaviour of its amplitude during the course of propagation.

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2. *Law of propagation*

The fundamental differential equations governing the symmetric MHD flow under consideration are

$$\frac{\partial \varrho}{\partial t}+u\frac{\partial \varrho}{\partial x}+\varrho\frac{\partial u}{\partial x}+\frac{\alpha \varrho u}{x}=0, \tag{2.1}$$

$$\varrho\frac{\partial u}{\partial t}+\varrho u\frac{\partial u}{\partial x}+\frac{\partial h}{\partial x}+(1+4R_{\mathrm{p}})\frac{\partial p}{\partial x}-4\frac{p}{\varrho}R_{\mathrm{p}}\frac{\partial \varrho}{\partial x}+(1-n)\frac{2h}{x}=0, \tag{2.2}$$

$$\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+2h\frac{\partial u}{\partial x}+\frac{2\alpha n h u}{x}=0, \tag{2.3}$$

$$\{1+12(\nu-1)R_{\mathrm{p}}\}\frac{\partial p}{\partial t}-\{\nu+16(\nu-1)R_{\mathrm{p}}\}\frac{p}{\varrho}\frac{\partial \varrho}{\partial t}+\{1+12(\nu-1)R_{\mathrm{p}}\}u\frac{\partial p}{\partial x}$$

$$-\{\nu+16(\nu-1)R_{\mathrm{p}}\}\frac{up}{\varrho}\frac{\partial \varrho}{\partial x}=0, \tag{2.4}$$

where x denotes the distance from the centre of symmetry 0. p, ϱ, u, h and ν respectively denote the pressure, the density, the velocity of the gas, the magnetic pressure and the heat exponent of the gas. R_p is the radiation pressure number which is defined as

$$R_{\mathrm{p}}=\frac{\text{radiation pressure}}{\text{gas pressure}}=\frac{p_{\mathrm{R}}}{p}$$

$\alpha=0, 1, 2$ for planar, cylindrical and spherical symmetry and $n=0, 1$ for azimuthal and axial magnetic field respectively. The system of equations (2.1) to (2.4) is quasi-linear and admits discontinuities in the flow field. We assume the existence of a weak discontinuity as a singular surface $\Sigma(t)$ with boundary conditions

$$[p]=0,\quad [\varrho]=0,\quad [u]=0,\quad [h]=0,$$

$$\left[\frac{\partial p}{\partial x}\right]\neq 0,\quad \left[\frac{\partial \varrho}{\partial x}\right]\neq 0,\quad \left[\frac{\partial u}{\partial x}\right]\neq 0,\quad \left[\frac{\partial h}{\partial x}\right]\neq 0. \tag{2.5}$$

$[Z]$ denotes the discontinuity in the quantity enclosed. The geometrical and kinematical conditions of first order for a singular surface of order one can be expressed in the form [5]

$$\left[\frac{\partial Z}{\partial x}\right]=B,\quad \left[\frac{\partial Z}{\partial t}\right]=-BG, \tag{2.6}$$

where Z stands for any of the flow variables, B is a scalar function defined over $\Sigma(t)$. G is the velocity of the surface $\Sigma(t)$ into a uniform medium at rest.

Taking jumps in equations (2.1) to (2.4) with the help of (2.5) and (2.6) we get

$$(u-G)\zeta+\varrho\lambda = 0, \tag{2.7}$$

$$\varrho(u-G)\lambda+\eta+(1+4R_p)\xi-4\frac{p}{\varrho}R_p\zeta = 0, \tag{2.8}$$

$$(u-G)\eta+2h\lambda = 0, \tag{2.9}$$

$$\{1+12(\nu-1)R_p\}(u-G)\xi+\frac{p}{\varrho}\{\nu+16(\nu-1)R_p\}(G-u)\zeta = 0, \tag{2.10}$$

where

$$\xi = \left[\frac{\partial p}{\partial x}\right], \quad \lambda = \left[\frac{\partial u}{\partial x}\right], \quad \zeta = \left[\frac{\partial \varrho}{\partial x}\right], \quad \eta = \left[\frac{\partial h}{\partial x}\right].$$

From equations (2.7), (2.8) and (2.9) we get

$$\lambda = \frac{-(u-G)}{\varrho}\zeta = \frac{-(u-G)}{2h}\eta = \frac{(1+4R_p)(u-G)}{\{2h-\varrho(u-G)^2-4pR_p\}}\xi. \tag{2.11}$$

Substituting from (2.11) in (2.10) we get,

$$\lambda[\{1+12(\nu-1)R_p\}\varrho\{b^2-(u-G)^2\}+\nu p+20(\nu-1)pR_p+16(\nu-1)pR_p^2] = 0. \tag{2.12}$$

The assumption that $\Sigma(t)$ is a regular singular surface, implies that $\lambda \neq 0$. Hence we obtain

$$(u-G)^2 = b^2+C_R^2, \tag{2.13}$$

where

$$C_R^2 = \frac{p}{\varrho}\left[\frac{\nu+20(\nu-1)R_p+16(\nu-1)R_p^2}{\{1+12(\nu-1)R_p\}}\right]$$

is the speed of sound in radiating gases and b is the Alfvén speed. We assume that the medium in front of the propagating surface $\Sigma(t)$ is uniform and at rest. For this case the speed of propagation is given by

$$G^2 = C_e^2, \tag{2.14}$$

where

$$C_e^2 = b^2+C_R^2$$

which is the effective speed of sound of a radiating gas. Consequently the relations (2.11) reduce to the forms

$$\lambda = \frac{G\zeta}{\varrho} = \frac{G\eta}{2h} = \frac{-(1+4R_p)G\xi}{\{2h-\varrho G^2-4pR_p\}}. \tag{2.15}$$

3. *Growth equation*

Differentiating equations (2.1) to (2.4) with regard to x and taking jumps across $\Sigma(t)$ and making use of the geometrical and kinematical compatibility conditions of second order due to Thomas [5] we get,

$$-G\bar{\zeta}+\varrho\bar{\lambda} = -\left\{\frac{\delta\zeta}{\delta t}+2\zeta\lambda+\frac{\alpha\varrho\lambda}{X}\right\}, \tag{3.1}$$

$$\varrho\frac{\delta\lambda}{\partial t} = \varrho G\bar{\lambda}-(1+4R_p)\bar{\xi}+4\frac{p}{\varrho}R_p\bar{\zeta}+\frac{32}{\varrho}\zeta\xi R_p-\bar{\eta}-12\frac{R_p}{p}\xi^2-20\frac{p}{\varrho^2}\zeta^2R_p-(1-n)\frac{2}{X}\eta, \tag{3.2}$$

$$-G\bar{\eta}+\frac{\delta\eta}{\delta t}+3\eta\lambda+2h\bar{\lambda}+\frac{2\alpha nh\lambda}{X} = 0, \tag{3.3}$$

$$\begin{aligned}\{1+12(\nu-1)R_p\}\frac{\delta\xi}{\delta t} &= \{1+12(\nu-1)R_p\}G\bar{\xi}+36(\nu-1)\frac{R_p}{p}G\xi^2-48\frac{R_p}{\varrho}G\xi\zeta\\ &-\{\nu+16(\nu-1)R_p\}p\left(\bar{\lambda}+\frac{\alpha\lambda}{X}\right)+64(\nu-1)p\frac{R_p}{\varrho}\lambda\zeta\\ &-\{(\nu+1)+76(\nu-1)R_p\}\lambda\xi,\end{aligned} \tag{3.4}$$

where

$$\bar{\lambda} = \left[\frac{\partial^2 u}{\partial x^2}\right],\quad \bar{\zeta} = \left[\frac{\partial^2 \varrho}{\partial x^2}\right],\quad \bar{\xi} = \left[\frac{\partial^2 p}{\partial x^2}\right],\quad \bar{\eta} = \left[\frac{\partial^2 h}{\partial x^2}\right],$$

and X gives the position of the wave whose configuration can be represented by $X = \Sigma(t)$.

The equations (3.1) to (3.4) can be rewritten after some manipulations in the forms

$$\frac{\partial\zeta}{\partial t} = G\bar{\zeta}-\varrho\bar{\lambda}+D, \tag{3.5}$$

$$\varrho\frac{\delta\lambda}{\delta t} = \varrho G\bar{\lambda}-(1+4R_p)\bar{\xi}+4\frac{p}{\varrho}R_p\bar{\zeta}-\bar{\eta}+C, \tag{3.6}$$

$$\frac{\delta\eta}{\delta t} = G\bar{\eta}-2h\bar{\lambda}+F, \tag{3.7}$$

$$A\frac{\delta\xi}{\delta t} = AG\bar{\xi}-B\bar{\lambda}+E, \tag{3.8}$$

where

$$A = \{1+12(\nu-1)R_p\}, \quad B = \{\nu+16(\nu-1)R_p\},$$

$$C = \frac{-12R_p}{p}\xi^2 + \frac{32R_p}{\varrho}\xi\zeta - 20\frac{p}{\varrho^2}R_p\zeta^2 - \frac{(1-n)2\eta}{X},$$

$$D = -\left(2\lambda\zeta + \frac{\alpha\varrho\lambda}{X}\right), \quad F = -\left(3\eta\lambda + \frac{2\alpha nh\lambda}{X}\right),$$

$$E = 36(\nu-1)\frac{R_p}{p}G\xi^2 - 48(\nu-1)\frac{R_p}{\varrho}G\xi\zeta + 64(\nu-1)p\frac{R_p}{\varrho}\lambda\zeta$$

$$-\{\nu+64(\nu-1)R_p\}\lambda\xi - A\lambda\xi - \frac{B\alpha\lambda}{X}.$$

Eliminating the time-derivatives in equations (3.5) to (3.8) by using equation (2.15) we get

$$G\bar{\zeta} - \varrho\bar{\lambda} = \frac{(1+4R_p)}{2\varrho AG^2}(\varrho E - BD) + \frac{F}{2G^2} + \frac{C}{2G} - \frac{D}{2}\left(1 + \frac{b^2}{G^2}\right), \tag{3.9}$$

$$\bar{\zeta} - \frac{A}{B}\varrho\bar{\xi} = \frac{\varrho E}{BG} - \frac{D}{G}, \tag{3.10}$$

$$b^2\bar{\zeta} - \bar{\eta} = -b^2\frac{D}{G} + \frac{F}{G}. \tag{3.11}$$

Using equations (3.9), (3.10), (3.11) and (2.15) in equations (3.1) to (3.4) we get,

$$\frac{\delta\lambda}{\delta t} = -\varrho\frac{\mu_2}{G}\lambda^2 - \frac{\lambda G\mu_1}{X}, \tag{3.12}$$

$$\frac{\delta\zeta}{\delta t} = -\mu_2\zeta^2 - \frac{G\mu_1\zeta}{X}, \tag{3.13}$$

$$\frac{\delta\xi}{\delta t} = -\frac{\varrho\mu_2 A}{b^2}\xi^2 - \frac{G\mu_1\xi}{X}, \tag{3.14}$$

$$\frac{\delta\eta}{\delta t} = -\frac{\mu_2}{b^2}\eta^2 - \frac{G\mu_1\eta}{X}, \tag{3.15}$$

where

$$\mu_1 = \left\{\frac{\alpha}{2} + (1-n)M_f^2 - \frac{\alpha}{2}(1-n)M_f^2\right\},$$

$$\mu_2 = \frac{(1+4R_p)}{2\varrho AG}\left[\frac{-36(\nu-1)R_pB^2}{p\varrho A^2} + \frac{48(\nu-1)R_pB}{\varrho A} - \frac{64(\nu-1)R_pp}{\varrho}\right.$$

$$\left. + \{(\nu+1)+76(\nu-1)R_p\}\frac{B}{\varrho A} - \frac{2B}{\varrho}\right]$$

$$+\left\{\frac{G}{\varrho} + \frac{6R_p}{Gp\varrho^2}\frac{B^2}{A^2} - \frac{16R_pB}{AG\varrho^2} + \frac{10pR_p}{G\varrho^2} + \frac{1}{2\varrho}M_f^2C_e\right\},$$

$$M_f = \frac{b}{C_e}.$$

For the case $M_f = \dfrac{b}{C_e} = 0$, the equations (3.12), (3.13) and (3.14) are in agreement with the corresponding ones in [7]. The equations (3.12), (3.13), (3.14) and (3.15) are the fundamental differential equations for the variations of the quantities λ, ζ, ξ and η along the normal trajectories of the wave front $\Sigma(t)$.

Now we define $\lambda = \left[\dfrac{\partial u}{\partial x}\right]$ as the amplitude of the wave which undergoes continuous distortion during propagation. The amplitude λ of the wave is a function of time and we expect it to either grow or decay in time. If $X_0 = \Sigma(t_0)$ represents the position of the wave at some fixed time t_0 before it breaks down, the position of the wave at time t is given by

$$X = X_0+C_e(t-t_0), \tag{3.16}$$

where C_e is the speed of propagation into a constant state at rest. In view of (3.16) we can write

$$\frac{\delta\lambda}{\delta t} = C_e\frac{d\lambda}{d\sigma}, \tag{3.17}$$

where

$$\sigma = X-X_0.$$

Using (3.16) and (3.17) in (3.12) we get,

$$\frac{d\lambda}{d\sigma} + \frac{\mu_1}{X_0+\sigma}\lambda + \frac{\mu_2\varrho}{C_e^2}\lambda^2 = 0. \tag{3.18}$$

The equation (3.18) is the required growth equation which governs the growth and decay of the wave. The equation (3.18) is reducible to linear form and has a solution of the form:

$$\lambda(\sigma) = \frac{\left(1+\dfrac{\sigma}{X_0}\right)^{-\mu_1}}{\dfrac{1}{\lambda_0} + \dfrac{\varrho\mu_2X_0}{(1-\mu_1)C_e^2}\left\{\left(1+\dfrac{\sigma}{X_0}\right)^{1-\mu_1}-1\right\}}, \tag{3.19}$$

where λ_0 is the initial wave amplitude at time t_0.

4. *Local and global behaviour of wave amplitude*

The solution (3.19) for the wave amplitude $\lambda(\sigma)$ can be put in the form

$$\lambda(\sigma) = 1-\left(\frac{\mu_1}{X_0}+\frac{\varrho\mu_2\lambda_0}{C_e^2}\right)\sigma+O(\sigma^2). \tag{4.1}$$

For the local behaviour of λ the last term of the right hand side of (4.1) is insignificant and hence we conclude that:

(*i*) If λ_0 is positive in the case of an expansion wave, the wave amplitude λ will locally decay.

(*ii*) If λ_0 is negative in the case of a compressive wave there exists a critical value $\lambda_c = \dfrac{\mu_1 C_e^2}{\varrho\mu_2 X_0}$ of $|\lambda_0|$ such that the wave amplitude λ will locally grow for $|\lambda_0| > \lambda_c$ and decay for $\lambda_0 < \lambda_c$. For the global behaviour of $\lambda(\sigma)$ we observe from (3.19) that it will continuously decay and ultimately tends to zero in the case of $\lambda_0 > 0$. When $\lambda_0 < 0$ the amplitude λ tends to infinity after a critical time t_c given by

$$t_c = t_0-\frac{X_0}{C_e}+\frac{X_0}{C_e}\left\{1+\frac{(1-\mu_1)C_e^2}{\varrho\mu_2X_0|\lambda_0|}\right\}^{1/(1-\mu_1)}.$$

The parameter $\mu_1 \leqslant 1$ for all practical problems in MHD flows. When $\lambda \to \infty$ as $t \to t_c$, a compressive wave discontinuity will break down and a shock type discontinuity will appear spontaneously. The underlying fact is that as a consequence of exceedingly large gradients the flow parameters themselves become discontinuous and the flow can not be maintained without the presence of a shock wave. When $\lambda_0 < 0$ and $|\lambda_0| < \lambda_c$ a compressive weak wave will locally decay for a short time and later on it will grow and terminate into a shock wave after a finite time t_c.

5. *Particular cases of interest*

(*i*) When $R_p = 0$, $A = 1$, $B = v_p$, $M_f = 0$, $\mu_1 = 0$, $\mu_2 = \left(\dfrac{v+1}{2\varrho}\right)C_e$, the growth equation (3.19) reduces to that of Thomas [6] and hence all his conclusions will follow.

(*ii*) When $M_f = 0$ and $\alpha = 0, 2$ the solution (3.19) is in full agreement with that of Srinivasan and Ram [7] and hence their conclusions follow.

(*iii*) For cylindrical symmetry with an axial magnetic field ($\alpha = 1$, $n = 1$), the solution (3.19) assumes the form:

$$\lambda(\sigma) = \frac{\lambda_0\left(1+\dfrac{\sigma}{X_0}\right)^{-1/2}}{1+2\dfrac{\mu_2\varrho\lambda_0X_0}{C_e^2}\left(\sqrt{1+\dfrac{\sigma}{X_0}}-1\right)},$$

with the critical time

$$t_c = t_0 - \frac{X_0}{C_e} + \frac{X_0}{C_e}\left\{1 + \frac{C_e^2}{2\varrho\mu_2 X_0|\lambda_0|}\right\}^2.$$

(*iv*) For a cylindrical symmetry with azimuthal magnetic field ($n = 0$, $\alpha = 1$), the solution (3.19) assumes the form:

$$\lambda(\sigma) = \frac{\lambda_0\left(1 + \frac{\sigma}{X_0}\right)^{-1/2(1+M^2{}_f)}}{1 + \frac{2\varrho\mu_2\lambda_0 X_0}{(1-M_f^2)C_e^2}\left\{\left(1+\frac{\sigma}{X_0}\right)^{1/2(1+M_f{}^2)}\right\}}$$

with critical time

$$t_c = t_0 - \frac{X_0}{C_e} + \frac{X_0}{C_e}\left\{1 + \frac{(1-M_f^2)C_e^2}{2\varrho\mu_2 X_0|\lambda_0|}\right\}^{2/(1-M_f{}^2)}.$$

(*v*) For a planar wave case with axial magnetic field ($\alpha = 0$, $n = 1$) the solution (3.19) assumes the form:

$$\lambda(\sigma) = \lambda_0\{1 + \varrho\mu_2\lambda_0\sigma/C_e^2\}^{-1}$$

with critical time

$$t_c = t_0 + \frac{C_e}{\varrho\mu_2|\lambda_0|}.$$

Thus we see that all particular cases of interest are derivable from our result (3.19).

REFERENCES

[1] S. I. Pai, C. K. Taso, Proc. 3rd Intl. Heat Transfer Conf., Chicago 1966.

[2] S. M. Scala, D. H. Sampson, *Heat Transfer in Hypersonic Flow with Radiation and Chemical Reaction in Supersonic Flow, Chemical Process and Radiative Transfer*, Pergamon Press, Oxford 1966, p. 319–354.

[3] S. I. Pai, *Radiation Gasdynamics*, Springer Verlag, New York 1966.

[4] S. Şrinivasan, Rishi Ram, *Z. Angew. Math. Mech.* **57**, 191 (1977).

[5] T. Y. Thomas, *J. Math. Mech.* **6**, 311 (1957).

[6] T. Y. Thomas, *J. Math. Mech.* **6**, 455 (1957).

[7] S. Srinivasan, Rishi Ram, *Z. Angew. Math. Phys.* **26**, 307 (1975).