SELF-PRECESSION AND FREQUENCY SHIFT FOR AN ELECTROMAGNETIC WAVE IN A DISSIPATIVE PLASMA MEDIUM

BY S. K. CHANDRA

Department of Mathematics, Krishna Chandra College*

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Amplitude dependent frequency shifts and precessional frequencies have been obtained for an elliptically polarized transverse monochromatic wave propagating in a dissipative unmagnetised plasma medium. Two frequency shifts and two precessional frequencies are obtained, one of the frequency shifts and one of the precessional frequencies are due to dissipation in the medium. Numerical estimations for some values of the parameters have been performed.

1. *Introduction*

Nonlinear wave-length and frequency shifts for monochromatic waves travelling in a plasma have been obtained by several authors [1–4]. Recently Arons and Max [5] observed that when a transverse elliptically polarized wave propagates in a plasma medium, the nonlinear effects give rise to precession of the polarization ellipse such that the ellipticity remains constant. They also obtained the frequency shift and precessional frequency for an elliptically polarized wave in a cold, unmagnetised plasma. Here we have obtained the nonlinear frequency shifts and precessional frequencies for an elliptically polarized transverse monochromatic wave in a cold dissipative unmagnetised plasma, including relativistic corrections. Two frequency shifts and two precessional frequencies are obtained, one of the frequency shifts and one of the precessional frequencies are due to the dissipative medium. In the absence of dissipation, we recover the earlier results.

In the last section we have discussed some observations, by taking some numerical values of the parameters.

* Address: Department of Mathematics, Krishna Chandra College, P.O. Hetampur Rajbati, Dist. Birbhum, West Bengal, India 731124.

2. *Basic equations*

We consider a stationary, homogeneous, unmagnetised cold plasma which is charged but neutral before perturbation. The motion of the particle is relativistic and the effect of dissipation is included. We start with the set of cold plasma equations with Maxwell's equations, which are

$$\frac{\partial n}{\partial t} + \operatorname{div}(n\vec{v}) = 0, \tag{1}$$

$$\left[\frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla})\right]\frac{\vec{v}}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} + \frac{e\vec{E}}{m} + \frac{e}{mc}(\vec{v}\times\vec{H}) - \nu\vec{v} = 0, \tag{2}$$

$$\operatorname{div}\vec{E} = -4\pi e n, \tag{3}$$

$$\operatorname{div}\vec{H} = 0, \tag{4}$$

$$\operatorname{curl}\vec{E} = -\frac{1}{c}\frac{\partial\vec{H}}{\partial t}, \tag{5}$$

$$\operatorname{curl}\vec{H} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} - \frac{4\pi e(n+n_0)}{c}\vec{v}, \tag{6}$$

where n is the electron number density, $\vec{v}$ the electron velocity, $\vec{E}$ and $\vec{H}$ are the electric and magnetic field intensities, c the velocity of light, n_0 is the uniform background ion density and ν is the collisional frequency, which is constant and greater than zero. We seek a Krylov, Bogoliubov and Mitropolsky perturbation expansion in the form

$$\begin{aligned}\vec{E} &= \varepsilon\vec{E}_1 + \varepsilon^2\vec{E}_2 + \dots\\ \vec{H} &= \varepsilon\vec{H}_1 + \varepsilon^2\vec{H}_2 + \dots\\ \vec{v} &= \varepsilon\vec{v}_1 + \varepsilon^2\vec{v}_2 + \dots\\ n &= n_0 + \varepsilon n_1 + \varepsilon^2 n_2 + \dots\end{aligned} \tag{7}$$

Since the precession is generated by nonlinear interaction between the wave and the plasma, and the precession is absent in the linear approximation, we can write

$$\theta = \varepsilon^2\theta_2 + \varepsilon^3\theta_3 + \dots \tag{8}$$

where θ is the angle generated by the rotating axes in time t. For simplicity, henceforth, we shall write θ_2 as θ.

The connection terms are functions of the amplitudes and the phases of the elliptically polarized transverse monochromatic wave and ε is a perturbation parameter. We consider that the wave is propagating along the x direction. It is almost periodic, with period $2\pi/\omega$ and wave length $2\pi/k$, where ω and k are the frequency and wave number of the wave, respectively, in the linear approximation. In the spatial problems, let the linear approximation for $\vec{v}$ be taken as (Tanenbaum [6])

$$\vec{v}_1 = e^{\gamma x}(v' \cos \psi + v'' \sin \psi) \tag{9}$$

where $\psi = kx - \omega t$, v' and v'' are real constants. Supposing $k \gg \gamma$ and $\omega > \omega_p$, then the dispersion relations are found to be

$$\omega^2 \approx k^2c^2 + \omega_p^2 \quad \text{and} \quad \gamma \approx -\frac{\nu\omega_p^2}{2\omega^2 c}$$

where ω_p is the plasma frequency. To avoid some avoidable complications in our calculations, we can neglect γ from Eq. (9), because $|\gamma| \ll \nu$. Hence the first order approximations of the field equations becomes

$$\vec{v}_1 = [0, v' \cos \psi, v'' \sin \psi],$$

$$\vec{E}_1 = \left[0, -\frac{m}{e} v'(\nu \cos \psi + \omega \sin \psi), -\frac{m}{e} v''(\nu \sin \psi - \omega \cos \psi)\right],$$

$$\vec{H}_1 = \left[0, -\frac{mck}{e\omega} v''(\omega \cos \psi - \nu \sin \psi), -\frac{mck}{e\omega} v'(\omega \sin \psi + \nu \cos \psi)\right],$$

$$n_1 = 0. \tag{10}$$

The second order approximations are found to be

$$\vec{v}_2 = \left[-\frac{k(v'^2 - v''^2)\{(2\omega^2 - \omega_p^2)\nu \sin 2\psi - \omega(4\omega^2 - \omega_p^2 + 2\nu^2) \cos 2\psi\}}{(4\omega^2 - \omega_p^2)^2 + 4\omega^2\nu^2}, 0, 0\right],$$

$$\vec{E}_2 = \left[-\frac{\omega_p^2 mk(v'^2 - v''^2)\{(2\omega^2 - \omega_p^2)\nu \cos 2\psi + \omega(4\omega^2 - \omega_p^2 + 2\nu^2) \sin 2\psi\}}{2e\omega\{(4\omega^2 - \omega_p^2)^2 + 4\omega^2\nu^2\}}, 0, 0\right],$$

$$n_2 = \frac{n_0 k^2(v'^2 - v''^2)}{\omega}\left[\frac{\omega(4\omega^2 - \omega_p^2 + 2\nu^2) \cos 2\psi - \nu(2\omega^2 - \omega_p^2) \sin 2\psi}{(4\omega^2 - \omega_p^2)^2 + 4\omega^2\nu^2}\right]. \tag{11}$$

3. *The third order approximation*

If $\vec{E}$ denotes the sum of all parts of the electric vector upto the third order approximations, the equation upto the third order approximation reduces to

$$\left[\left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2 + \omega_p^2\right)\frac{\partial}{\partial t} + \nu\left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2\right)\right]\vec{E} = \frac{m\omega_p^2}{e}\frac{\partial}{\partial t}\left[-\frac{\partial}{\partial t}\left(\frac{\vec{v}_1 v_1^2}{2c^2}\right)\right.$$

$$\left. - (\vec{v}_2 \cdot \vec{\nabla})\vec{v}_1 - \frac{e}{mc}(\vec{v}_2 \times \vec{H}_1) + \frac{1}{n_0}\frac{\partial}{\partial t}(n_2\vec{v}_1)\right] + \frac{\nu m\omega_p^2}{en_0}\frac{\partial}{\partial t}(n_2\vec{v}_1). \tag{12}$$

Obviously

$$\left[\left(\frac{\partial^2}{\partial t^2}-c^2\nabla^2+\omega_p^2\right)\frac{\partial}{\partial t}+\nu\left(\frac{\partial^2}{\partial t^2}-c^2\nabla^2\right)\right]\vec{E}_1=\text{first harmonic part of the right-hand side of Eq. (12).} \tag{13}$$

Again, since the principal polarization axes of the elliptically polarized wave can rotate about the direction of propagation due to nonlinear sources as the wave advances in time or in space, we rotate the principal axes in the yz plane such that the wave can propagate along the direction of the x-axis. The ellipticity remains constant throughout the propagation. Let the axes rotate through an angle θ in time t. Then we can write (Lass [7])

$$\frac{\partial\vec{A}}{\partial t}=\frac{d\vec{A}}{d}+\frac{d\theta}{dt}(\boldsymbol{x}\times\vec{A}), \tag{14}$$

where $\vec{A}$ is the field variables in the rotating axes and $\frac{d}{dt}$ is the derivative with respective to the rotating axes. Neglecting higher terms, the second and the third derivatives with respect to time are found to be

$$\frac{\partial^2\vec{A}}{\partial^{\,2}}=\frac{d^2\vec{A}}{dt^2}+2\frac{d\theta}{dt}\left(\boldsymbol{x}\times\frac{d\vec{A}}{dt}\right),$$

$$\frac{\partial^3\vec{A}}{\partial t^3}=\frac{d^3\vec{A}}{dt^3}+3\frac{d\theta}{dt}\left(\boldsymbol{x}\times\frac{d^2\vec{A}}{dt^2}\right), \tag{15}$$

where $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ are the unit vectors along the three mutually perpendicular directions.

Since the effect of precession is observed to be present at the third order approximation, Eq. (13) reduces, in the rotating frame, to

$$\left[\left(\frac{d^2}{dt^2}+c^2k^2+\omega_p^2\right)\frac{d}{dt}+\nu\left(\frac{d^2}{dt^2}+c^2k^2\right)\right]\vec{E}_1+\frac{d\theta}{dt}\left[3\left(\boldsymbol{x}\times\frac{d^2\vec{E}_1}{dt^2}\right)\right.$$
$$\left.+c^2k^2(\boldsymbol{x}\times\vec{E}_1)+\omega_p^2(\boldsymbol{x}\times\vec{E}_1)+2\nu\left(\boldsymbol{x}\times\frac{d\vec{E}_1}{dt}\right)\right]$$
$$=\frac{m\omega_p^2}{e}\frac{d}{dt}\left[-\frac{d}{dt}\left(\frac{\vec{v}_1v_1^2}{2c^2}\right)-(\vec{v}_2\cdot\vec{\nabla})\vec{v}_1-\frac{e}{mc}(\vec{v}_2\times\vec{H}_1)+\frac{1}{n_0}\frac{d}{dt}(n_2v_1)\right]$$
$$+\nu\frac{m\omega_p^2}{en_0}\frac{d}{dt}(n_2\vec{v}_1). \tag{16}$$

4. Frequency shift and the precessional effect

We assume the total phase to be $kx-(\omega+\delta\omega)t$ where $\delta\omega$ is the frequency shift of the wave. Again, let $\frac{d\theta}{dt}$ be the precessional frequency of the wave. After some algebraic calculations we observe that Eq. (16) will be the linear combinations of the co-efficients of $y \sin \psi$, $y \cos \psi$, $z \sin \psi$, $z \cos \psi$ etc. Since the solution is to be periodic, it follows that the co-efficients of $\cos \psi$ and $\sin \psi$ must vanish independently. Therefore we only equate the co-efficients of $y \cos \psi$, $z \sin \psi$, $y \sin \psi$ and $z \cos \psi$ from both sides of Eq. (16), then four equations are obtained which are as follows

$$v'\delta\omega - v''\frac{d\theta}{dt} = \frac{\omega_p^2 k^2 \omega v'}{4(\omega^2-v^2)}\left[\frac{(v'^2-v''^2)(\omega^2+v^2)(4\omega^2-\omega_p^2)}{\omega^2\{(4\omega^2-\omega_p^2)^2+4\omega^2v^2\}} - \frac{3v'^2+v''^2}{4c^2k^2}\right] + \frac{v^2\omega_p^2 v'}{4\omega(\omega^2-v^2)}\left[2 - \frac{k^2(v'^2-v''^2)(2\omega^2-\omega_p^2)}{(4\omega^2-\omega_p^2)^2+4\omega^2v^2}\right], \tag{17}$$

$$v''\delta\omega - v'\frac{d\theta}{dt} = -\frac{\omega_p^2 k^2 \omega v''}{4(\omega^2-v^2)}\left[\frac{(v'^2-v''^2)(\omega^2+v^2)(4\omega^2-\omega_p^2)}{\omega^2\{(4\omega^2-\omega_p^2)^2+4\omega^2v^2\}} + \frac{3v''^2+v'^2}{4c^2k^2}\right] + \frac{v^2\omega_p^2 v''}{4\omega(\omega^2-v^2)}\left[2 + \frac{k^2(v'^2-v''^2)(2\omega^2-\omega_p^2)}{(4\omega^2-\omega_p^2)_2+4\omega^2v^2}\right], \tag{18}$$

$$v'\delta\omega - v''\frac{d\theta}{dt} = \frac{\omega_p^2 k^2(v'^2-v''^2)(2\omega^2-\omega_p^2)v'}{8\omega^2\{(4\omega^2-\omega_p^2)^2+4\omega^2v^2\}} - \frac{\omega_p^2 v'}{4\omega} \tag{19}$$

and

$$v''\delta\omega - v'\frac{d\theta}{dt} = -\frac{\omega_p^2 k^2(v'^2-v''^2)(2\omega^2-\omega_p^2)v''}{8\omega^2\{(4\omega^2-\omega_p^2)^2+4\omega^2v^2\}} + \frac{\omega_p^2 v''}{4\omega}. \tag{20}$$

In the nondissipative medium (when $v = 0$) Eqs. (19) and (20) do not exist and Eqs. (17) and (18) become identical to the earlier obtained results, if our procedure was adopted by Arons and Max [5].

The frequency shift and the precessional frequency from the equations (17) and (18) becomes

$$\delta\omega' = \frac{\omega_p^2\omega(v'^2+v''^2)}{4c^2(\omega^2-v^2)}\left[\frac{(\omega^2-\omega_p^2)(4\omega^2-\omega_p^2+2v^2)}{(4\omega^2-\omega_p^2)^2+4\omega^2v^2} - \tfrac{3}{4}\right] + \frac{v^2\omega_p^2}{2\omega(\omega^2-v^2)} \tag{21}$$

and

$$\frac{d\theta'}{dt} = \frac{\omega_p^2\omega v'v''}{2c^2(\omega^2-v^2)}\left[\frac{(\omega^2-\omega_p^2)(4\omega^2-\omega_p^2+2v^2)}{(4\omega^2-\omega_p^2)^2+4\omega^2v^2} - \tfrac{1}{4}\right]. \tag{22}$$

From Eqs. (19) and (20), the frequency shift and the precessional frequency becomes

$$\delta\omega'' = \frac{\omega_p^2}{8\omega}\left[2 + \frac{(v'^2+v''^2)k^2(2\omega^2-\omega_p^2)}{(4\omega^2-\omega_p^2)^2+4\omega^2v^2}\right] \tag{23}$$

and

$$\frac{d\theta''}{dt} = \frac{v'v''k^2\omega_p^2(2\omega^2-\omega_p^2)}{4\omega\{(4\omega^2-\omega_p^2)^2+4\omega^2\nu^2\}}. \tag{24}$$

Thus we get amplitude dependent two frequency shifts and two precessional frequencies, when a nonlinear elliptically polarized wave propagates in a dissipative medium. The frequency shift $\delta\omega''$ and the precessional shift $\dfrac{d\theta''}{dt}$ exists due to the dissipative media. If the medium is nondissipative, the co-efficient of $y \sin \psi$ and $z \cos \psi$ in Eqs. (16) becomes identically equal to zero. Hence in the nondissipative medium ($\nu = 0$) we shall get one frequency shift and one precessional shift which are

$$\delta\omega' = \frac{\omega_p^2(v'^2+v''^2)}{4\omega c^2}\left[\frac{\omega^2-\omega_p^2}{4\omega^2-\omega_p^2} - \tfrac{3}{4}\right] \tag{25}$$

and

$$\frac{d\theta'}{dt} = \frac{\omega_p^2 v'v''}{2\omega c^2}\left[\frac{\omega^2-\omega_p^2}{4\omega^2-\omega_p^2} - \tfrac{1}{4}\right]. \tag{26}$$

Eq. (25) satisfies the results of Sluijter and Montgomery [3] for the plane wave solution and Eq. (26) the results of Arons and Max [5].

The second terms in the third bracket of Eqs. (21) and (22) are due to the relativistic effects but no relativistic effects are to be observed in Eqs. (23) and (24) whose existence is due to the dissipation in the medium.

In the non-dissipative medium, the frequency shift looses some energy, because in equation (25) the relativistic part is greater than the nonrelativistic part.

5. *Wave member shift*

For the boundary value problem, a similar procedure is adopted. We assume here that the total phase is $(k+\delta k)x-\omega t$, where δk is the wave number shift of the wave. Let $d\theta/dx$ be the precessional frequency of the wave with respect to space. As before, we get two wave number shifts and two precessional frequencies which are

$$\delta k' = -\frac{\omega_p^2\omega^2(v'^2-v''^2)}{4c^4k(\omega^2-\nu^2)}\left[\frac{(\omega^2-\omega_p^2)(4\omega^2-\omega_p^2+2\nu^2)}{(4\omega^2-\omega_p^2)^2+4\omega^2\nu^2} - \tfrac{3}{4}\right] - \frac{\nu\omega_p^2}{2c^2k(\omega^2-\nu^2)} \tag{27}$$

$$\delta k'' = -\frac{\omega_p^2}{8kc^2}\left[\frac{(v'^2+v''^2)(2\omega^2-\omega_p^2)k^2\omega_p^2}{(4\omega^2-\omega_p^2)^2+4\omega^2\nu^2} - 2\right] \tag{28}$$

$$\frac{d\theta'}{dx} = \frac{v'v''\omega_p^2\omega^2}{2c^4k(\omega^2-\nu^2)}\left[\frac{(\omega^2-\omega_p^2)(4\omega^2-\omega_p^2+2\nu^2)}{(4\omega^2-\omega_p^2)^2+4\omega^2\nu^2} - \tfrac{1}{4}\right] \tag{29}$$

and

$$\frac{d\theta''}{dx} = \frac{v'v''k^2\omega_p^2(2\omega^2-\omega_p^2)}{4c^2k\{(4\omega^2-\omega_p^2)^2+4\omega^2v^2\}}. \tag{30}$$

The same conclusions can be drawn here as in the previous section.

6. *Numerical estimation*

We observed from our field equations of the first order approximations, that the Poynting flux averaged over time period is

$$\langle p\rangle = \frac{c}{4\pi}\cdot\frac{ckm^2}{2\omega e^2}(v^2+\omega^2)(v'^2+v''^2). \tag{31}$$

For a simple illustration of our result, we consider the magnitude of the Poynting flux to be $\approx 10^{23}$ ergs/cm^2 and the other numerical values as $\omega \approx 10^{15}$ cps, $\omega \approx 10^{11}$ cps, $k \approx 3.3\times10^4$. Using these values and assuming $\omega \gg v$, we obtained from Eq. (31) $v'+v'' \approx 2.6\times10^{19}$. Eqs (21) and (23) come out to be $\delta\omega' \approx -3.4\times10^4$ and $\delta\omega'' \approx 0.25\times10^7$. The negative sign in $\delta\omega'$ indicates that the propagating wave will have the red shift and the positive sign in $\delta\omega''$ indicates blue shift. The reason for the red shift in $\delta\omega'$ is that in Eq. (21) the relativistic effects predominate over the non-relativistic terms, causing thereby the negative sign in $\delta\omega'$ i.e. the red shift of the frequency, which means that the wave looses some energy.

Further, from Eqs (22) and (24), we obtained

$$\frac{d\theta'}{dt} \approx -\frac{\omega_p^2v'v''}{2\omega c^2}\left(\frac{3\omega_p^2}{16\omega^2}\right) \approx -1.4\times10^{-4}$$

and

$$\frac{d\theta_3''}{dt} \approx 5\times10^3$$

provided we assume $v' \approx v''$. Generally, the collisional frequency corresponding to the dissipation is very small in comparison with the frequency of the wave propagation. At present we can neglect the effect of dissipation from our illustration. By considering the different ratios of ω and ω_p, different values of the shifts are obtained which are shown in the following chart.

$\frac{\omega_p}{\omega} \approx$	$\delta\omega' \approx$	$\frac{d\theta'}{dt} \approx$	$\delta k' \approx$	$\frac{d\theta'}{dx} \approx$
10^{-1}	-3.6×10^{10}	-1.4×10^{8}	1.21	4.5×10^{-3}
10^{-3}	-3.6×10^{6}	-1.4	1.21×10^{-4}	4.5×10^{-11}
10^{-5}	-3.6×10^{2}	-1.4×10^{-8}	1.21×10^{-8}	4.5×10^{-19}
10^{-7}	-3.6×10^{-2}	-1.4×10^{-16}	1.21×10^{-12}	4.5×10^{-27}

The following conclusions can be drawn from the chart.

(*i*) as the difference between ω and ω_p becomes larger and larger, the frequency shift, wave number shift and the precessional frequencies become smaller and smaller,

(*ii*) the spatial precessional frequencies are much smaller than the temporal precessional frequencies,

(*iii*) the frequency shifts (both spatial and temporal) are greater than the precessional frequencies i.e.

$$|\delta\omega| > \left|\frac{d\theta}{dt}\right| \quad \text{and} \quad |\delta k| > \left|\frac{d\theta}{dx}\right| .$$

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