# ENERGY LEVELS OF DRESSED ATOMS AND RESONANCE PHENOMENA*

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A new universal algorithm for calculations of various characteristics of atomic $N$-level systems interacting with a single mode of the quantized radiation is introduced and applied to the study of the energy levels, resonance properties and the time evolution of the density matrix. The new algorithm is so simple, that the calculations can be carried out with the help of a programmable pocket calculator.

## *1. Introduction*

Semiclassical theories of electromagnetic processes, in which the electromagnetic field is treated not as a dynamical part of the system, but as an external agent described by a given function of space-time variables, have been very successful in describing a great variety of phenomena in quantum optics, atomic physics, solid state physics, etc. As compared to the treatment with the use of complete quantum electrodynamics, these theories offer substantial simplifications in the calculations. However, there is a price to be paid for that; the uniformity of the description is destroyed. The electromagnetic field is no longer treated on an equal footing with the atomic systems; its states are not described according to quantum theory by state vectors and its energy does not contribute to the Hamiltonian of the system. Owing to the time dependence of the electromagnetic field of a wave, the states of atoms interacting with such a field are not stationary, so that they cannot be sought as eigenstates of the Hamiltonian. Because of the importance of the energy eigenstates in the analysis of quantum systems, a special mathematical procedure

* A preliminary version of this work was presented in [4].

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has been introduced to deal with the interactions of atomic systems and monochromatic waves. In view of a formal similarity between monochromatic fields, which are periodic in time, and potential fields in a crystal, which are periodic in space, the analogs of Bloch wave functions can be defined. The corresponding eigenvalues are called quasienergies. We shall have more to say about it in Section 2.

The purpose of this paper is the introduction of a new computational method to evaluate the energy eigenvalues and eigenstates and to determine the time evolution of state vectors and density matrices. Our method starts from the quantum electrodynamic Hamiltonian describing an atomic (or molecular) system coupled to a single mode of the quantized radiation field. Next, this Hamiltonian is rewritten in the phase representation suggested by our earlier investigations of electromagnetic processes in intense photon beams [1–4]. The phase representation in the theory of electromagnetic field plays a role similar to that played by the coordinate representation in the Schrödinger wave mechanics. In the limit, when the number of photons becomes very large, we are able to convert the equations involving operators, represented usually by infinite matrices, into simple ordinary differential equations. Our theory, while preserving all the basic features of the quantum description, is equally simple as the semiclassical theory. As a matter of fact, many equations look the same in both theories, even though their physical interpretation is quite different.

In Section 2 we give a general description of the phase representation and we show its usefulness in the study of large photon numbers. In Section 3 we introduce a straightforward algorithm to calculate numerically the eigenvalues and the eigenfunctions for stationary states of an $N$-level system interacting with the electromagnetic radiation, described in the one mode approximation. In Section 4 we describe in some detail the energy levels of such a system for different orientations and polarizations of the field. The energy levels determine various resonance properties of the atomic system. In Section 5 we apply our algorithm to the study of the time evolution and Section 6 contains conclusions.

## *2. Phase representation*

In this paper we shall deal exclusively with a single mode of the electromagnetic radiation. Generalization to many modes is in principle possible, but the corresponding equations are more complicated and our simple algorithm for numerical calculations does not work.

It has been observed by Susskind and Glogower [5] that the phase of the electromagnetic field, unlike the position or the momentum of a particle, cannot be represented by a self-adjoint operator in the quantum theoretic description. A detailed analysis of this problem can be found in a review paper by Carruthers and Nieto [6]. In our earlier study of quantum electrodynamics of intense photon beams [3] we have discovered a practical method of bypassing these difficulties. We introduced the phase $\varphi$ merely as an argument of the wave function; calling $\psi(\varphi)$ the phase representation of the state vector. At the same time Levy–Leblond [7] independently introduced the same representation stressing its operator properties. Like in our previous publications, we shall concentrate here on practical

applications of the phase representation, referring the readers to the Levy–Leblond's paper for a very lucid analysis of this representation presented in a more general context of nonhermitian operators in quantum theory.

The standard description of the quantized radiation field employs the photon number states and the creation and annihilation operators. The photon number state vectors span the Fock space and the creation and annihilation operators act in this space as (conveniently normalized) raising and lowering operators. In order to translate this standard description into the language of the phase representation, it is sufficient to give the rules of correspondence for the basis state vectors and for the creation and annihilation operators. In view of the subsequent applications to intense photon beams, the labels of the harmonic wave functions, associated with the photon number states, will be shifted by $n$

$$|n+m\rangle \to e^{im\varphi}, \tag{1}$$

where the reference point $n$ is, in principle, arbitrary, but later it will be assumed to be large. The correspondence rule (1) enables us to represent any state vector $\psi$ in terms of the wave function $\psi(\varphi)$ in the phase representation

$$\psi \to \psi(\varphi) = \sum_{m=-n}^{\infty} c_m e^{im\varphi}, \tag{2}$$

where $c_m$ are the coefficients of $\psi$ in the Fock basis,

$$\langle n+m|\psi\rangle. \tag{3}$$

The annihilation and creation operators acting on the wave function $\psi(\varphi)$ take on a fairly complicated form

$$a \to e^{-i\varphi}\left(n+\frac{1}{i}\,\partial_\varphi\right)^{1/2}, \tag{4a}$$

$$a^\dagger \to \left(n+\frac{1}{i}\,\partial_\varphi\right)^{1/2} e^{i\varphi}. \tag{4b}$$

However, in the description of intense photon beams, we may greatly simplify these formulas by expanding the square roots appearing in expressions (4) into power series' in $1/n$ and retaining only the lowest order terms. This procedure is justified, whenever the variation of the photon number in the process under consideration is small as compared to the mean value of this number, taken as the reference point in the formulas (3) and (4). In the lowest approximation we obtain

$$a \sim e^{-i\varphi}\sqrt{n}\,, \tag{5a}$$

$$a^\dagger \sim e^{i\varphi}\sqrt{n}\,. \tag{5b}$$

Even though the corrections to formulas (5) are of the order $n^{-1/2}$, great care must be exercised when dealing with products of creation and annihilation operators; correction terms may easily survive in the limit, when $n \to \infty$, if they get multiplied by lowest order

terms. Thus for example, the exact expression for the photon number operator, obtained from (4), is

$$a^{\dagger}a = n + \frac{1}{i}\partial_{\varphi}, \tag{6}$$

which cannot be obtained by retaining only the lowest order terms in $a^{\dagger}$ and $a$.

The energy spectrum of the atomic system, interacting with a single mode of the quantized electromagnetic field, may be determined from the time-independent Schrödinger equation in the phase representation

$$\left[\omega\left(n + \frac{1}{i}\partial_{\varphi}\right) + H_{\mathrm{A}}(A(\varphi))\right]\psi(\varphi) = E\psi(\varphi), \tag{7}$$

where $H_{\mathrm{A}}$ is the atomic Hamiltonian in the presence of the electromagnetic field, in which the creation and annihilation operators appearing in the vector potential are expressed in the phase representation. In order to explain the connection with the method of quasienergies, let us compare Eq. (7) with the time-dependent Schrödinger equation for the same atomic system interacting with an external, monochromatic electromagnetic wave,

$$i\partial_t\psi(t) = H_{\mathrm{A}}(A(\omega t))\psi(t). \tag{8}$$

The spectrum of eigenvalues of the Hamiltonian is determined from Eq. (7) by imposing the periodicity condition,

$$\psi(2\pi) = \psi(0), \tag{9}$$

on the wave functions in the phase representation (cf. Eq. (2)). On the other hand, the values of the quasienergy are determined [8–10] by seeking the solution of Eq. (8) in the form

$$\psi(t) = e^{-iE_q t}\phi(t), \tag{10}$$

where $\phi$ is periodic in time with the period $T = 2\pi/\omega$. Changing the variable $t$ into $\varphi = \omega t$, we obtain from (8) and (10)

$$(i\omega\partial_{\varphi} + E_q)\phi(\varphi) = H_{\mathrm{A}}(A(\varphi))\phi(\varphi). \tag{11}$$

This equation becomes identical with the eigenvalue equation (7) when we set

$$E = n\omega + E_q. \tag{12}$$

However, the interpretation of the wave functions appearing in these two equations is totally different. The wave function $\psi(\varphi)$ in Eq. (7) describes stationary states of a closed (atom + field) system. This wave function can be used to calculate all quantum properties of both the atomic system and the electromagnetic field. The wave function $\phi(\varphi)$ appearing in Eq. (11) describes, through the relation (10), the time evolution of the atomic system driven by an external electromagnetic field. This wave function is not meant to describe any quantum properties of the electromagnetic field. Nonetheless, the values of the energy

and the quasienergy are the same. True energies of the system form a ladder; adding or removing one photon amounts to moving one step up or down the ladder. On the other hand, the quasienergies of the atomic are not defined uniquely. As seen from Eq. (10), we may always add a multiple of $\omega$ to $E_q$ and this has the same effect as changing the number of photons in the quantum description.

### 3. *Algorithm for numerical calculations of the energy levels of a dressed N-level atom*

In what follows we shall assume that in the limit, when $n \to \infty$, no derivatives with respect to $\varphi$ survive in the atomic Hamiltonian, so that it becomes a function of $\varphi$. In this case, the eigenvalue equation (7) possesses a simple, formal solution in terms of a phase-ordered exponential,

$$\psi(\varphi) = e^{i(\varepsilon - n)\varphi} P \exp\left(-i \int_0^{\varphi} d\varphi' h(\varphi')\right)\psi(0), \tag{13}$$

where the ordering operation $P$ applies to the dependence on the phase, and

$$\varepsilon = E/\omega, \qquad h = H_A/\omega. \tag{14}$$

The periodicity condition (9) imposed on the wave function (13) gives

$$P \exp\left(-i \int_0^{2\pi} d\varphi h(\varphi)\right)\psi = \exp(-2\pi i\varepsilon)\psi, \tag{15}$$

where $\psi = \psi(0)$ is the state vector in the Hilbert space of the atomic system alone. Thus, we have arrived at a new form of the energy eigenvalue equation which, in contradistinction to the standard form, contains a unitary operator instead of the Hamiltonian. Our new form of the eigenvalue equation will not be very helpful, unless we succeed in evaluating the phase-ordered exponential. This can be done easily numerically when the atomic system is described in the $N$-level approximation, so that its Hamiltonian is an $N \times N$ matrix.

Since the phase-ordered exponential is a unitary operator, it may be expressed in terms of a hermitian operator $H_{\text{eff}}$, playing the role of an effective Hamiltonian,

$$P \exp\left(-i \int_0^{2\pi} d\varphi h(\varphi)\right) = \exp(-2\pi i h_{\text{eff}}). \tag{16}$$

As seen from Eq. (15), the spectrum of the original full Hamiltonian, in the strong beam limit, coincides (modulo $n\omega$) with the spectrum of the effective atomic Hamiltonian $\omega h_{\text{eff}}$.

In order to evaluate the effective Hamiltonian numerically for an $N$-level system, we divide the interval $(0, 2\pi)$ into $m$ subintervals and we approximate the phase-ordered exponential by a finite product of unitary matrices

$$P \exp\left(-i \int_0^{2\pi} d\varphi h(\varphi)\right) \simeq \exp(-i\Delta\varphi h_m) \ldots \exp(-i\Delta\varphi h_1), \tag{17}$$

where

$$h_l = h(2\pi l/m). \tag{18}$$

Finally, the resulting unitary matrix must be diagonalized and its eigenvalues $\exp(-2\pi i\varepsilon)$ determine the spectrum of the Hamiltonian.

We shall now analyze in detail a simple case of an atomic system interacting with the electromagnetic field only via its magnetic moment. The electromagnetic field will be assumed to contain, in addition to an intense photon beam, also a level splitting, constant part $\boldsymbol{B}_0$. The Hamiltonian for this system has the form

$$H = \omega a^\dagger a - \gamma \boldsymbol{B}_0 \cdot \boldsymbol{S} - \gamma \boldsymbol{B}_{\text{rad}} \cdot \boldsymbol{S}, \tag{19}$$

where $\boldsymbol{S} = (S_x, S_y, S_z)$ is the spin operator and $\gamma$ is the gyromagnetic ratio.

The same model, with only minor notational changes, may be used to describe a two-level atom undergoing electric dipole transitions in the presence of a photon beam.

Let us choose the $z$-axis in the direction of the constant field $\boldsymbol{B}_0$. The most general polarization vector $\boldsymbol{\varepsilon}$ of a plane electromagnetic wave is a linear combination, with complex coefficients, of any two orthonormal vectors $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ which are perpendicular to the direction of the wave propagation. Without any loss of generality we may assume that one of these vectors, say $\boldsymbol{\varepsilon}_2$, is orthogonal to $\boldsymbol{B}_0$ and then choose the $y$-axis in its direction. The angle between the $z$-axis and the plane spanned by $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ will be denoted by $\vartheta$. Then, the polarization vector, written in terms of Cartesian basis vectors, has the form

$$\boldsymbol{\varepsilon} = e^{i\delta_1} \sin\delta(\boldsymbol{e}_x \sin\vartheta + \boldsymbol{e}_z \cos\vartheta) + e^{i\delta_2} \cos\delta \boldsymbol{e}_y. \tag{20}$$

Since the overall phase of the polarization vector has no significance, we may multiply (20) by $-ie^{-i\delta_2}$. The two parameters $\delta$ and $\psi = \delta_1 - \delta_2$ determine the shape and the orientation of the polarization ellipse. When the magnetic field $\boldsymbol{B}_{\text{rad}}$ is expressed in terms of $\boldsymbol{\varepsilon}$, the full Hamiltonian for our system takes on the form

$$\begin{aligned} H = {} & \omega a^\dagger a - \gamma B_0 S_z + \gamma(\omega/2V)^{1/2}[\cos\delta \sin\vartheta S_x(a + a^\dagger) \\ & - \sin\delta S_y(e^{i\psi}a + e^{-i\psi}a^\dagger) + \cos\delta \cos\vartheta S_z(a + a^\dagger)]. \end{aligned} \tag{21}$$

Hence, the atomic Hamiltonian in the phase representation is

$$h(\varphi) = \xi S_z + \eta(\cos\delta \sin\vartheta \cos\varphi S_x - \sin\delta \cos(\varphi - \psi) S_y + \cos\delta \cos\vartheta \cos\varphi S_z), \tag{22}$$

where $\xi = -\gamma B_0/\omega$ and $\eta = \gamma(2n/\omega V)^{1/2}$.

The phase-ordered exponential with this Hamiltonian has a simple geometric interpretation. Since the spin operators generate rotations, each term on the right hand side of the formula (17) represents an infinitesimal rotation. The composition of $m$ rotations is again a rotation and, therefore, the phase-ordered exponential may be written in the form

$$\exp(-i\boldsymbol{\alpha} \cdot \boldsymbol{S}), \tag{23}$$

where $\boldsymbol{\alpha}$ is the rotation vector, which determines the axis and the angle of the rotation. The composition law is the same in each representation of the group of rotations, so that we may determine the vector $\boldsymbol{\alpha}$ using the simplest representation, for which the generators of rotations are Pauli matrices divided by 2. In this case, known algebraic properties of

these matrices enable us to express all rotation matrices appearing in the formula (17) in a simple way

$$\exp(-i\Delta\varphi h_l) = \exp(-i\boldsymbol{\beta}\cdot\boldsymbol{\sigma}) = \cos\beta - \frac{i\boldsymbol{\beta}\cdot\boldsymbol{\sigma}}{\beta}\sin\beta. \tag{24}$$

The final multiplication of $m$ matrices was done numerically with the help of a recurrence relation based on the formula (24).

The eigenvalues of the projection of the spin matrix on any unit vector are equal to $-s, -s+1, \ldots, s$, where $s$ is the value of the spin. Hence, the eigenvalues of the effective Hamiltonian $h_{\text{eff}}$ are equal to

$$-s|\alpha|/2\pi, \quad (-s+1)|\alpha|/2\pi, \ldots, \quad s|\alpha|/2\pi. \tag{25}$$

The results of numerical calculations of these eigenvalues will be presented in the next section.

### *4. Energy levels and resonances of a two-level system*

In this section we shall present our results concerning the energy spectrum of the spin $\frac{1}{2}$ system, i.e. for a two-level system. As has been noted by Haroche [11] and much earlier by Majorana [12], the solution of the spin $s$ problem may be obtained from that of the spin $\frac{1}{2}$. In our formalism, the connection between different spin problems is described by the formula (25).

For intense radiation fields, the energy spectrum of the full Hamiltonian for the spin $\frac{1}{2}$ system has the form

$$E = (n \pm \alpha/4\pi)\omega. \tag{26}$$

In what follows, we shall give only the atomic energies $\pm\alpha\omega/4\pi$ to which always the photon beam energy $n\omega$ may be added.

There are too many (five) parameters in our atomic Hamiltonian to allow for a comprehensive, and at the same time comprehensible, analysis. Therefore, we shall restrict ourselves to the study of the dependence of the energy spectrum on a few parameters (sometimes just one parameter) at a time.

Let us begin with the standard case of the linear polarization and perpendicular static and wave magnetic vectors. In this case, the Hamiltonian (22) takes on the form

$$h(\varphi) = \tfrac{1}{2}(\xi\sigma_z + \eta\cos\varphi\sigma_x), \tag{27}$$

and the energy curves for a selected set of the values of $\eta$ are shown in Fig. 1. An exceptional feature of the energy curves shown in Fig. 1 is that the energy goes through zero and hence there is a degeneracy of the energy levels for some values of the static magnetic field. Such degeneracy is always present in the absence of the photon beam. The reason, why it is not removed by the interaction with the beam, is the existence of an additional sym-

metry of the Hamiltonian, which leads to a selection rule. In terms of the creation and annihilation operators, the operator commuting with the Hamiltonian has the form

$$P = \sigma_z(-1)^{a^\dagger a}. \tag{26}$$

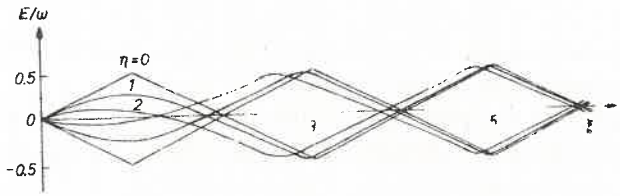


Fig. 1. Energy curves for different values of the intensity parameter $\eta$, when the magnetic vectors of the static field and the wave field are perpendicular

In the phase representation, this operator takes on the form

$$P = \sigma_z e^{\pi \hat{\varphi}}. \tag{30}$$

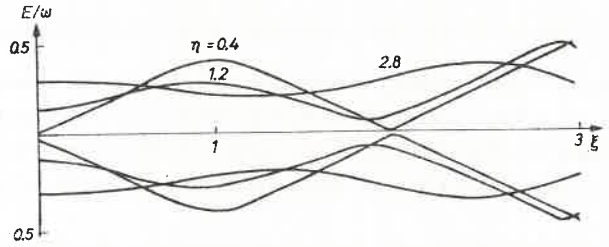


Fig. 2. Energy curves showing the phenomenon of anticrossing due to the elliptical polarization, $\delta = \pi/6$, $\vartheta = 0$, $\psi = 0$

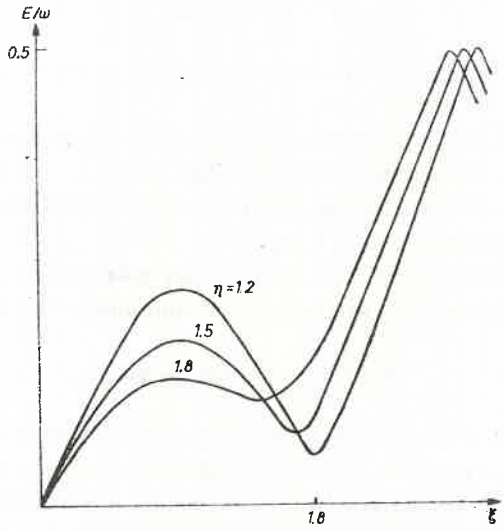


Fig. 3. Energy curves showing the phenomenon of anticrossing due to a departure from orthogonality of magnetic vectors, $\theta = 7\pi/18$

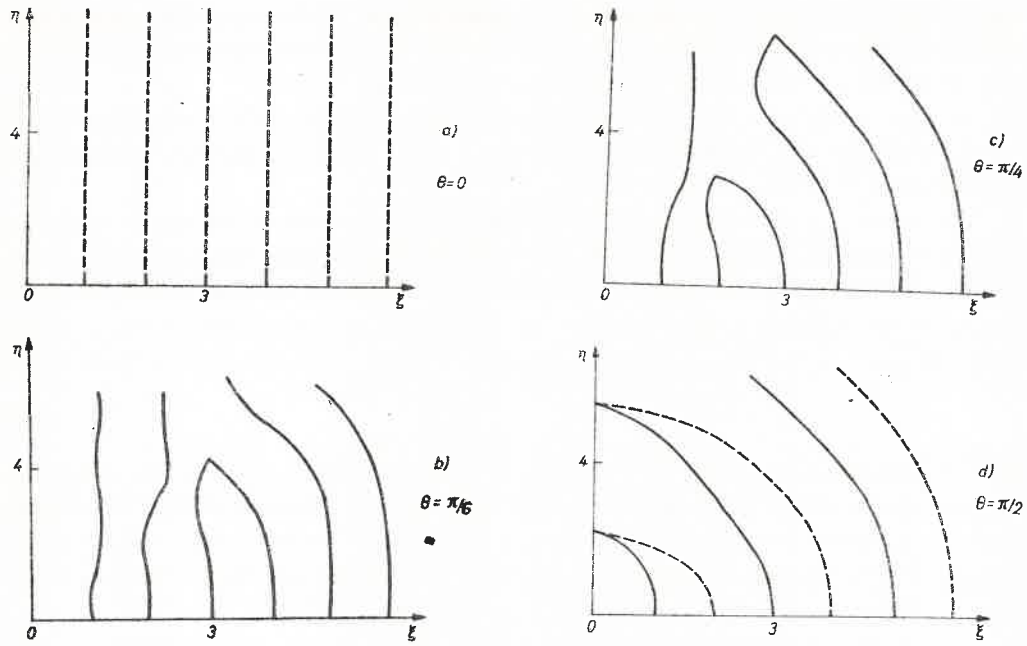


Fig. 4. Changes in the positions of resonances when the angle between the magnetic vectors varies from 0 to $\pi/2$. Dashed lines indicate crossings and solid lines indicate anticrossings

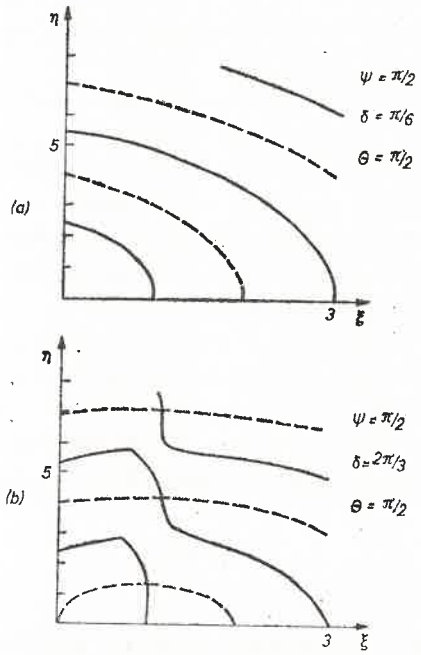


Fig. 5. Changes in the positions of resonances when the degree of ellipticity of the polarization varies

Like the parity operator, the operator $P$ has eigenvalues $+1$ or $-1$, and all energy eigenstates are either even or odd under the action of $P$. The even and odd levels do not mix and the phenomenon of the anticrossing does not occur.

In Figs. 2 and 3 we show typical energy curves, when the anticrossing does occur, either because the polarization is elliptical (Fig. 2), or because the magnetic vectors are not perpendicular (Fig. 3).

The resonance between the atomic system and the photon beam occurs when the transitions between the atomic levels are so frequent, that all levels become equally populated, and hence the average value of $S_z$ is equal to zero. On the other hand, differentiating the eigenvalue equation with respect to the static field, we obtain (Hellmann–Feynman theorem)

$$\langle S_z \rangle = -\frac{1}{\gamma}\frac{\partial E}{\partial B_0}. \tag{31}$$

Thus, the minima and maxima of the energy, plotted as a function of the magnetic field, determine the positions of the resonances. In Fig. 4 we show the positions of these resonances, when the polarization is linear and the angle between the magnetic field vectors varies. In Fig. 5 we give analogous curves, when the degree of ellipticity of the polarization varies.

### 5. *Use of the phase representation in the study of the time evolution*

The usefulness of the phase representation is not restricted to the problem of the energy spectrum. It may also be applied, with equal ease, to the study of the time evolution. In order to show the versatility of the phase representation method, we shall apply it to the study of the time evolution, not in the simpler case of the wave function, but in the more complex case of the density matrix.

The most general density matrix describing a spin $s$ system interacting with the radiation mode in the phase representation is a $(2s+1)$ by $(2s+1)$ matrix, whose elements are functions of two phase variables. However, we may greatly simplify this description, if we consider only those physical quantities, which depend on $\varphi$ and not on $\partial_\varphi$. In order to determine average values of such quantities, we only need the diagonal elements of density matrices in the $\varphi$ variable. In particular, to calculate the trace of the density matrix over the field variables, we integrate those diagonal elements over the phase. In what follows, we shall restrict ourselves to the study of only such diagonal matrix elements, denoting them by $\varrho(\varphi)$. These phase-diagonal matrices may be used to calculate all characteristics of the atomic system and also those characteristics of the radiation field which can be expressed as functions of $\varphi$, like, for example, the electromagnetic field $\cos\varphi$. The reduced atomic density matrix $\varrho_a$ may be written as the following integral over the phase [3]

$$\varrho_a = \int_0^{2\pi} \frac{d\varphi}{2\pi}\,\varrho(\varphi). \tag{32}$$

An important property of phase-diagonal density matrices is that they obey a closed evolution equation in the Dirac (interaction) picture. This is in marked contrast to situation

in the Schrödinger wave mechanics, where the $x$-diagonal elements of the density matrix (i.e. the squared modulus of the wave function) do not obey by themselves alone any evolution equation.

The general form of the evolution equation for the density matrix, the Liouville-von Neumann equation, in the Dirac picture is

$$i\partial_t \varrho(t) = [H_I(t), \varrho(t)]. \tag{33}$$

As we have assumed before, the interaction Hamiltonian is just a spin matrix dependent on the phase, $H_I(\varphi) = \omega h(\varphi)$. In this case the interaction Hamiltonian in the Dirac picture will be given by the same matrix, whose argument is shifted by $\omega t$. This follows directly from the difinition of the Dirac picture:

$$H_I(t) = e^{iH_0 t} H_I e^{-iH_0 t}, \tag{34}$$

$$h(\varphi, t) = e^{\omega t \partial_\varphi} h(\varphi) e^{-\omega t \partial_\varphi} = h(\varphi + \omega t). \tag{35}$$

The Liouville-von Neumann equation now takes on the form

$$i\partial_t \varrho(\varphi, t) = \omega[h(\varphi + \omega t), \varrho(\varphi, t)]. \tag{36}$$

We shall show in the simplest case of the spin $\frac{1}{2}$ system, how this equation can be solved with the help of techniques developed in this paper.

Let us write the density matrix as a linear combination of the Pauli matrices and the unit matrix ($\mathrm{Tr}\varrho = 1$)

$$\varrho(\varphi, t) = \tfrac{1}{2} + \boldsymbol{p}(\varphi, t) \cdot \boldsymbol{\sigma}. \tag{37}$$

We shall choose the Hamiltonian in the most general form, substituting $\frac{1}{2}\boldsymbol{\sigma}$ for general spin matrices $\boldsymbol{S}$. Using the commutation relations between Pauli matrices, we obtain from Eq. (36) a set of linear equations for the coefficients $p_x$, $p_y$ and $p_z$, which may be written in the following compact form

$$i\partial_t \boldsymbol{p}(\varphi, t) = \omega h_1(\varphi, t) \boldsymbol{p}(\varphi, t), \tag{38}$$

where

$$h_1(\varphi + \omega t) = \xi \Sigma_z + \eta(\cos\delta \sin\vartheta \cos(\varphi + \omega t)\Sigma_x$$
$$-\sin\delta \cos(\varphi + \omega t - \psi)\Sigma_y + \cos\delta \cos\vartheta \cos(\varphi + \omega t)\Sigma_z), \tag{39}$$

and $\Sigma_x$, $\Sigma_y$ and $\Sigma_z$ are spin 1 matrices expressed in the Cartesian basis. A formal solution of Eq. (38) is

$$\boldsymbol{p}(\varphi, t) = P \exp\left(-i \int_{\varphi}^{\varphi + \omega t} d\varphi' h_1(\varphi')\right) \boldsymbol{p}(\varphi, 0). \tag{40}$$

The phase ordered exponential appearing in this formula is just a slight extension of the expression considered in Section 4. This exponential may again be viewed as a composition of rotations and it can be evaluated using the simplest representation of rotations, that for spin $\frac{1}{2}$ system.

To this end, first, we write the phase-ordered exponential (40) in the form analogous to (23),

$$P\exp\left(-i\int_{\varphi_1}^{\varphi_2} d\varphi' h_1(\varphi')\right) = \exp\left(-i\boldsymbol{\alpha}(\varphi_1, \varphi_2)\cdot\boldsymbol{\Sigma}\right) \tag{41}$$

and we evaluate numerically the vector $\boldsymbol{\alpha}$ with the use of the spin $\frac{1}{2}$ representation, by dividing the interval $(\varphi_1, \varphi_2)$ into many subintervals. Next, with the help of algebraic properties of $\boldsymbol{\Sigma}$ matrices, we express the exponential (41) in the form

$$\exp\left(-i\boldsymbol{\alpha}\cdot\boldsymbol{\Sigma}\right) = \boldsymbol{n}\otimes\boldsymbol{n} + (1-\boldsymbol{n}\otimes\boldsymbol{n})\cos|\boldsymbol{\alpha}| - i\boldsymbol{n}\cdot\boldsymbol{\Sigma}\sin|\boldsymbol{\alpha}|, \tag{42}$$

where $\boldsymbol{n}\otimes\boldsymbol{n}$ is the dyadic product of the vector $\boldsymbol{n}$ by itself and $\boldsymbol{n}$ is the unit vector in the direction of $\boldsymbol{\alpha}$. The final result of this calculation may be written in the form

$$\boldsymbol{p}(t) = \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{p}(0))(1-\cos|\boldsymbol{\alpha}|) + \boldsymbol{p}(0)\cos|\boldsymbol{\alpha}| + \boldsymbol{n}\times\boldsymbol{p}(0)\sin|\boldsymbol{\alpha}|. \tag{43}$$

The same technique may be used to determine the time evolution of the wave function. The starting point is the analog of the formula (40) for the time evolution of the wave function in the Dirac picture

$$\phi(\varphi, t) = P\exp\left(-i\int_{\varphi}^{\varphi+\omega t} d\varphi' h(\varphi')\right)\phi(\varphi, 0). \tag{44}$$

The numerical evaluation proceeds along the same lines as for the density matrix. Transformation to the Schrödinger picture requires only a shift of the phase variable by $-\omega t$.

## 6. *Conclusions*

The main purpose of this paper was to describe a new computational method, which may be applied to various problems of $N$-level systems interacting with intense photon beams. Several applications of our method are described in this paper, others will be given in future publications.

As a byproduct we have obtained a clarification of the relationship between the quantum and the semiclassical descriptions of the radiation.

The central notion of our approach is the phase, which plays the role of the coordinate variable of standard quantum mechanics. It is worthwhile to point out that the phase averaging, directly related to our integrals over the phase, has been introduced before, as a somewhat ad hoc procedure, in the framework of the semiclassical description, by many authors. A recent discussion of this problem can be found in the paper by Moloney and Meath [13] and in references cited therein. The phase representation enables one to derive these phase averages from the quantum theoretic description of the electromagnetic radiation.

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