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THE SPATIAL DISTRIBUTION OF THE INTERACTION CONTRIBUTION TO THE MAGNETIC-FIELD ENERGY ASSOCIATED WITH TWO MOVING CHARGES

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The interaction contribution to the magnetic-field energy associated with two moving charges q_1 and q_2 with velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ separated by a spatial vector $\boldsymbol{s}$ is evaluated to order (v_1v_2/c^2) by direct integration over all space. It is shown that the contribution to the magnetic interaction energy from a spherical shell of radius $r > s$ centred on one particle is equal to $(2q_1q_2/3c^2)(\boldsymbol{v}_1 \cdot \boldsymbol{v}_2)r^{-2}dr$, while for a shell with $r < s$ the contribution is $(q_1q_2/3c^2)[3(\boldsymbol{v}_1 \cdot \boldsymbol{s})(\boldsymbol{v}_2 \cdot \boldsymbol{s})/s^2-(\boldsymbol{v}_1 \cdot \boldsymbol{v}_2)]s^{-3}rdr$. After integration the negative of the usual expression $(-q_1q_2/2sc^2)[(\boldsymbol{v}_1 \cdot \boldsymbol{v}_2)+(\boldsymbol{v}_1 \cdot \boldsymbol{s})(\boldsymbol{v}_2 \cdot \boldsymbol{s})/s^2]$ for the interaction contribution to the Hamiltonian for two moving charged particles is obtained. The change in the electric-field energy due to effects of retardation on electric fields does not contain any terms proportional to (v_1v_2/c^2), and so the convention sometimes adopted of calling $[-(q_1q_2/sc^2)(\boldsymbol{v}_1 \cdot \boldsymbol{v}_2)]$ the magnetic interaction and attributing the remainder of the interaction, viz. $(q_1q_2/2sc^2)[(\boldsymbol{v}_1 \cdot \boldsymbol{v}_2)-(\boldsymbol{v}_1 \cdot \boldsymbol{r})(\boldsymbol{v}_2 \cdot \boldsymbol{r})/s^2]$, to retardation appears to be misleading.

1. Introduction

The generally accepted expression for the terms of second order in the particle velocities in the Hamiltonian for the interaction between two charged particles was first derived by Darwin [1]. For particles with charges q_1 and q_2 and velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ at $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, this interaction H_D can be written in the form

$$H_D = (-q_1q_2/2sc^2)\,[(\boldsymbol{v}_1 \cdot \boldsymbol{v}_2)+(\boldsymbol{v}_1 \cdot \boldsymbol{s})\,(\boldsymbol{v}_2 \cdot \boldsymbol{s})/s^2], \tag{1}$$

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where $s = (r_2 - r_1)$. Darwin obtained this result by considerations involving magnetic vector potentials and retarded electric potentials in a Lorentz gauge. In this gauge H_D can be decomposed into two terms,

$$H_D = H_1 + H_2, \tag{2}$$

where

$$H_1 = (-q_1 q_2 / sc^2)(v_1 \cdot v_2) \tag{3}$$

arises from the unretarded interaction involving the magnetic vector potential, and

$$H_2 = (-q_1 q_2 / 2sc^2)[(v_1 \cdot s)(v_2 \cdot s)/s^2 - (v_1 \cdot v_2)] \tag{4}$$

is the contribution to the interaction due to effects of retardation on the scalar potential. Quantum mechanical generalizations of this result have been made by various authors, either by simply writing down the operator equivalent to the velocity [2] or by use of perturbation theory in quantum electrodynamics, with the electromagnetic interaction with the particles treated as a perturbation [3, 4][1].

The term H_1 of (3) is sometimes referred to as the magnetic energy [6] and the term H_2 of (4) is referred to as the retardation energy [4, 6]. However Jackson [7], using vector and scalar potentials in a Coulomb gauge, has shown that the whole of the interaction H_D can be obtained without consideration of effects of retardation. Thus, when calculations are made by use of potentials, whether part of the interaction is attributed to retardation or not appears to depend on the choice of gauge.

Since electric and magnetic fields are normally regarded as more physical quantities than potentials, some insight into the problem of the interaction between moving charged particles may be obtained by integrating the interaction contribution to the magnetic-field energy over all space. Performing such an integration enables us to show that the total Darwin interaction H_D is equal to the negative of the magnetic-field interaction energy. We also show that there are no terms proportional to $(v_1 v_2 / c^2)$ due to retardation in the electric-field interaction energy.

During the course of performing the integrations we obtain some information about the spatial distribution of the interaction contribution to the magnetic-field energy, on which we report. A similar calculation giving information about the spatial distribution of the electric-field energy for the Coulomb interaction between two charged particles

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[1] It has been shown that for heavy atoms the expectation value of the quantum mechanical operator corresponding to H_2 is of the opposite sign to, and has a magnitude of the order of 10% of, the expectation value of the operator corresponding to H_1 [4]. Inclusion of H_2 in the Hamiltonian is an important factor in helping to provide agreement between theory and experiment for K-shell binding energies of heavy atoms. For the atoms W, Hg and Pb, the term H_2 in the Hamiltonian gives a contribution to K-shell binding energies varying from 1.18 to 1.59 Ry [4]. With inclusion of this contribution, agreement between theoretical atomic K-shell binding energies and experimental values stated to be obtained from results on solids by adding the work functions for solids is on average within about 0.1 Ry for the three atoms mentioned. However, if other necessary corrections [5] to the solid-state K-shell binding energies were made besides the correction for work functions, it is possible that the agreement would be less accurate.

has been made recently by one of the authors [8]. For this case it was shown that there is no net contribution to the interaction energy from any spherical volume centred on one of the charges and having a radius less than the distance between the charges.

In Section 2 we outline our procedure for integrating the interaction contribution I to the magnetic-field energy, and show why there is no contribution of order $(v_1 v_2/c^2)$ to the electric-field interaction energy. The reason that H_D is equal to $-I$ rather than $+I$ is discussed briefly. In Section 3 we show some graphs and contour plots demonstrating the spatial distribution of the magnetic-field interaction energy for two special cases, $\boldsymbol{v}_1 || \boldsymbol{v}_2 || \boldsymbol{s}$ and $\boldsymbol{v}_1 || \boldsymbol{v}_2 \perp \boldsymbol{s}$.

We use equations appropriate for the unrationalised Gaussian system of units.

2. *Field-interaction energies of order* $(v_1 v_2/c^2)$

Let us consider two charges q_i at points $\boldsymbol{r}_i$ moving with velocities $\boldsymbol{v}_i$ $(i = 1, 2)$. In order to avoid confusion with the notation H for a Hamiltonian, we use the symbol $\boldsymbol{F}$ to denote magnetic fields. To order (v_i/c), the magnetic field $\boldsymbol{F}_i$ at $\boldsymbol{r}$ due to the charge q_i is given by [9]

$$\boldsymbol{F}_i = [q_i \boldsymbol{v}_i \times (\boldsymbol{r} - \boldsymbol{r}_i)]/(c|\boldsymbol{r} - \boldsymbol{r}_i|^3). \tag{5}$$

We choose a system of coordinates in which

$$\boldsymbol{r}_1 = (0, 0, 0), \quad \boldsymbol{r}_2 = (0, 0, s); \tag{6}$$

thus

$$\boldsymbol{s} \equiv (\boldsymbol{r}_2 - \boldsymbol{r}_1) = (0, 0, s). \tag{7}$$

We write

$$\boldsymbol{v}_i = (l_i, m_i, n_i) v_i, \tag{8}$$

where l_i, m_i and n_i are the directional cosines of $\boldsymbol{v}_i$. Then, introducing spherical polar coordinates with $\boldsymbol{s}$ as polar axis, θ as colatitude and ϕ as azimuth measured from the x axis, we find from (5) to (8) that, to order (v_i/c)

$$\begin{aligned} \boldsymbol{F}_1 = {} & (q_1 v_1/cr^2)\,[m_1 \cos\theta - n_1 \sin\theta \sin\phi,\ n_1 \sin\theta \cos\phi - l_1 \cos\theta, \\ & l_1 \sin\theta \sin\phi - m_1 \sin\theta \cos\phi], \end{aligned} \tag{9}$$

$$\begin{aligned} \boldsymbol{F}_2 = {} & q_2 v_2 c^{-1} (r^2 - 2rs\cos\theta + s^2)^{-(3/2)} \\ & \times [m_2(r\cos\theta - s) - n_2 r \sin\theta \sin\phi,\ n_2 r \sin\theta \cos\phi - l_2(r\cos\theta - s), \\ & r(l_2 \sin\theta \sin\phi - m_2 \sin\theta \cos\phi)]. \end{aligned} \tag{10}$$

The interaction contribution I to the magnetic-field energy can be obtained by integrating $(\boldsymbol{F}_1 \cdot \boldsymbol{F}_2/4\pi)$ successively over ϕ, θ and r. The only non-zero contributions remaining after integration over ϕ are those involving the products l_1l_2, m_1m_2 and n_1n_2. Introducing the notation

$$A = (q_1q_2v_1v_2/c^2)(l_1l_2+m_1m_2+2n_1n_2) = (q_1q_2/c^2)[(\boldsymbol{v}_1\cdot\boldsymbol{v}_2)+(\boldsymbol{v}_1\cdot\boldsymbol{s})(\boldsymbol{v}_2\cdot\boldsymbol{s})/s^2] \quad (11)$$

and

$$B = (q_1q_2v_1v_2/c^2)(l_1l_2+m_1m_2) = (q_1q_2/c^2)[(\boldsymbol{v}_1\cdot\boldsymbol{v}_2)-(\boldsymbol{v}_1\cdot\boldsymbol{s})(\boldsymbol{v}_2\cdot\boldsymbol{s})/s^2], \quad (12)$$

we find what

$$I = (1/4\pi)\int(\boldsymbol{F}_1\cdot\boldsymbol{F}_2)dV$$

$$= \int_0^\infty dr\int_{-1}^1 d\lambda[Ar-2Bs\lambda+(2B-A)r\lambda^2]/[4(r^2+s^2-2rs\lambda)^{(3/2)}], \quad (13)$$

where $\int dV$ denotes an integral over all space and we have written

$$\cos\theta = \lambda. \quad (14)$$

We perform the integral over λ by rewriting the numerator in (13) as a sum of terms proportional to $(r^2+s^2-2rs\lambda)^n$ $(n = 0, 1, 2)$ with coefficients independent of λ. We find

$$I = \int_0^\infty dr\{[(2B-A)/48r^2s^3][|r+s|^3-|r-s|^3]+[(A/8r^2s)-(2B-A)/8s^3][|r+s|-|r-s|]$$

$$+(1/16r^2s^3)(r^2-s^2)[2B(r^2+s^2)-A(r^2-s^2)][|r-s|^{-1}-|r+s|^{-1}]\}$$

$$= I_1+I_2, \quad (15)$$

where I_1 and I_2 are contributions from inside and outside a sphere of radius s. Using (11), (12) and (15) we find that

$$I_1 = (q_1q_2/c^2)\int_0^s(1/3)[3(\boldsymbol{v}_1\cdot\boldsymbol{s})(\boldsymbol{v}_2\cdot\boldsymbol{s})/s^2-(\boldsymbol{v}_1\cdot\boldsymbol{v}_2)]s^{-3}rdr$$

$$= (1/6)(q_1q_2/sc^2)[3(\boldsymbol{v}_1\cdot\boldsymbol{s})(\boldsymbol{v}_2\cdot\boldsymbol{s})/s^2-(\boldsymbol{v}_1\cdot\boldsymbol{v}_2)], \quad (16)$$

and

$$I_2 = (q_1q_2/c^2)\int_s^\infty(2/3)(\boldsymbol{v}_1\cdot\boldsymbol{v}_2)r^{-2}dr = (2/3)(q_1q_2/sc^2)(\boldsymbol{v}_1\cdot\boldsymbol{v}_2). \quad (17)$$

From (15) to (17) we see that $I = -H_D$, where H_D is given by (1).

To second order in particle velocities, the Hamiltonian H for an electromagnetic field plus charged particles can be written as

$$H = H_f+H_p+H_{int}, \quad (18)$$

where [10]

$$H_f = (1/8\pi)\int (E_v^2 - E_s^2 + B^2)dV \tag{19}$$

is the Hamiltonian of the electromagnetic field, H_p denotes the particle Hamiltonian and H_{int} denotes the Hamiltonian for the interaction between the particles and fields. In (19) E_v and E_s denote the magnitudes of the parts of the electric field associated in a Lorentz gauge with the time derivative of the magnetic vector potential and with the gradient of the scalar potential respectively, and B is the magnitude of the magnetic induction[2]. To second order in particle velocities, the quantity I of (15) to (17) represents the interaction contribution to the magnetic-field part of H_f of (19).

After the fields due to the particles have been eliminated, the terms in the interaction between particles which are of order $(v_1 v_2/c^2)$ have been shown [1–4] to be equal to H_D of (1), while our work shows that the interaction contribution to the magnetic-field energy is equal to $-H_D$. Thus H_{int} alone must contribute $2H_D$ to the particle-particle interaction, and then the field interaction energy $-H_D$ brings the net interaction between particles down to H_D.

In order to see that there is no contribution of order $(v_1 v_2/c^2)$ to the electric-field interaction energy we merely note that, to order (v_i^2/c^2), the electric field $\boldsymbol{E}_i$ at r due to a charge q_i at $\boldsymbol{r}_i$ is given by

$$\boldsymbol{E}_i = [q_i(\boldsymbol{r}-\boldsymbol{r}_i)/|\boldsymbol{r}-\boldsymbol{r}_i|^3]\,[1-\tfrac{1}{2}(v_i/c)^2(3\cos^2\theta_i - 1)], \tag{20}$$

where θ_i denotes the angle between $\boldsymbol{v}_i$ and and $(\boldsymbol{r}-\boldsymbol{r}_i)$. This result may be obtained from the general expression for the field due to a moving charge [11] by expanding in powers of the velocity. Since the first correction to the static field is of order $(v_i/c)^2$, there can be no contribution to the electric-field energy associated with two particles proportional to $(v_1 v_2/c^2)$. We have not verified by direct integration that there are no terms in the interaction contribution to the field energy proportional to v_1^2 or v_2^2 separately, but without performing the integrations we can make use of the empirical analysis by Ampère of interactions between electric circuits reported by Whittaker [12] to infer that there will be no such terms.

Thus the total interaction contribution to the magnetic- plus electric-field energies is equal to I given by (15) to (17). This quantity $I = -H_D$. The total Darwin interaction is brought to H_D by two terms equal to H_D associated with the interaction between the particles and the field potentials at the particles. Since we have shown that the contribution $-H_D$ to the interaction due to the pure-field part of the Hamiltonian has nothing to do with retardation, and for the contribution $2H_D$ from the particle-field interactions it depends on the choice of gauge for the potentials whether retardation appears to be involved, we consider that to call H_1 of (2) the magnetic energy and H_2 of (3) the retardation energy is misleading.

[2] Despite the negative sign in front of E_s^2 in (19), the expectation value of the total Hamiltonian for charged particles at rest including their electric fields is proportional to the same integral of E_s^2 with a positive sign, since e.g. for two particles the interaction of each particle with the potential due to the other is equal in magnitude to the integral over all space of $(1/8\pi)E_s^2$.

3. *Spatial distribution of the magnetic interaction energy*

Since particle-field interactions are normally written in terms of charges or currents multiplied by potentials, the question of the location of the energy is avoided. However, we do have definite information about the location of the interaction contribution to the pure-field term in the Hamiltonian. If the absolute magnitude of this interaction energy changes, then the absolute magnitude of the value of the interaction of each particle with the field will change by the same amount. Thus it is plausible to assume that the total velocity-dependent interaction energy between the particles should be considered to have the same type of distribution in space as the interaction contribution to the magnetic-field energy. In this section we discuss the spatial distribution of this magnetic interaction energy. Knowledge of this distribution is of interest from the point of view of forming a picture of the processes leading to particle interactions, although not necessary for calculations of forces between the particles.

First we consider the average distribution over spherical shells centred on one particle. For a shell between radii r and $(r+dr)$ we see from (17) that for $r > s$ the magnetic interaction energy, which we write as $\Delta(r)dr$, is proportional to $(1/r^2)dr$, while from (16) we

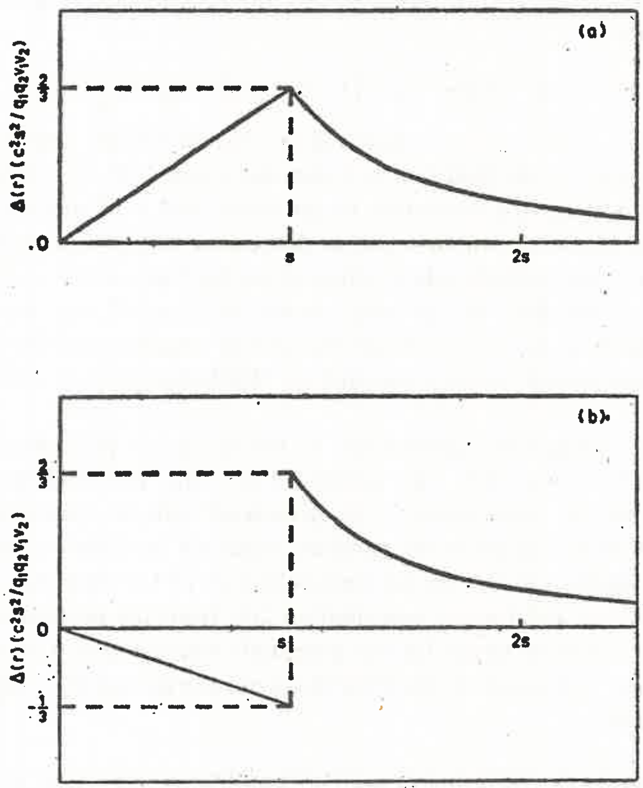


Fig. 1. Plots of the function $\Delta(r)$, where $\Delta(r)dr$ is the contribution to the magnetic-field interaction energy I from a shell lying between radii r and $(r+dr)$: (a) $v_1 || v_2 || s$; (b) $v_1 || v_2 \perp s$

see that for $r < s$, $\Delta(r)dr$ is proportional to rdr. For a fixed angle between $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ the contributions from shells with $r > s$ are independent of the direction of $\boldsymbol{s}$, while the contributions from shells with $r < s$ change in both magnitude and sign as the direction of $\boldsymbol{s}$ changes. If $\boldsymbol{v}_1$ is parallel to $\boldsymbol{v}_2$, $\Delta(r)$ is continuous at $r = s$ if $\boldsymbol{s}$ is parallel to the velocities, but for other directions of $\boldsymbol{s}$ there is a discontinuity. In Fig. 1 we plot $\Delta(r)$ for $\boldsymbol{v}_1 || \boldsymbol{v}_2$ for two cases (*i*) $\boldsymbol{s} || \boldsymbol{v}_i$ and (*ii*) $\boldsymbol{s} \perp \boldsymbol{v}_i$. The existence of terms proportional to r for shells with $r < s$ contrasts with results for the electric-field energy associated with the Coulomb interaction, which show that there is no contribution to this energy from shells with $r < s$ [8].

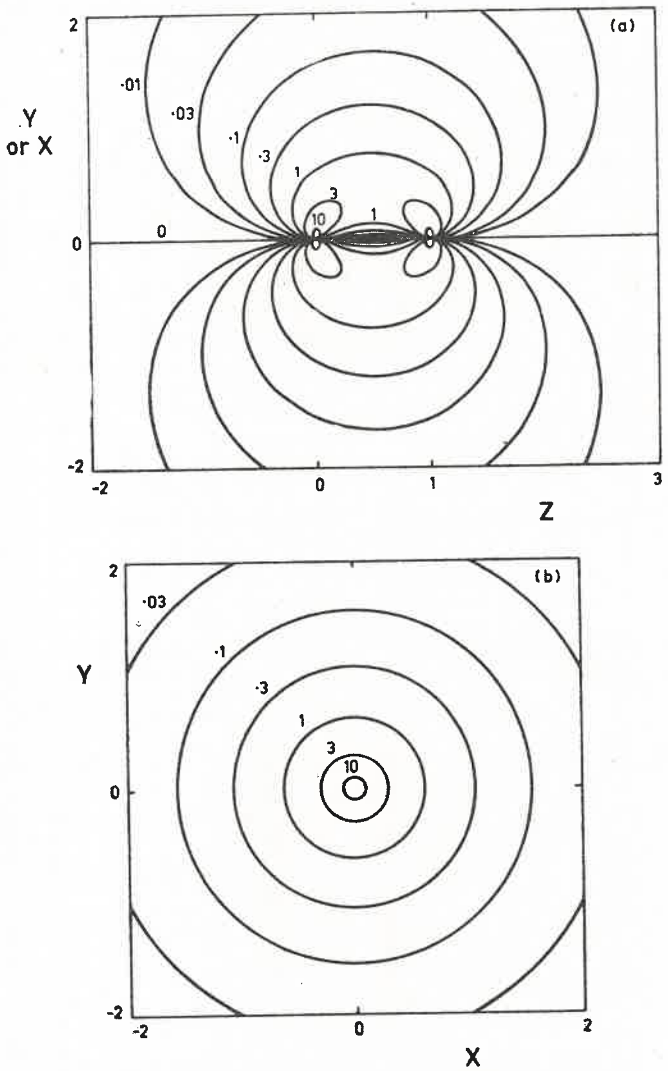


Fig. 2. Contour plots of the function $f(X, Y, Z)$ defined by (21) and (22) for special planes for $\boldsymbol{v}_i = (0, 0, v_i)$ $(i = 1, 2)$, $\boldsymbol{s} = (0, 0, s)$, i.e. $\boldsymbol{v}_1 || \boldsymbol{v}_2 || \boldsymbol{s}$. (a) $X = 0$ (or $Y = 0$); (b) $Z = 0$

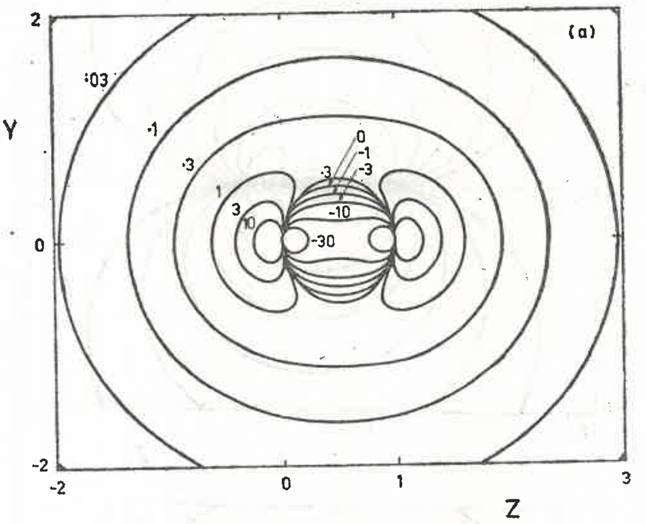
To obtain a more detailed picture of the distribution of magnetic-field interaction energy, we use (9) and (10) to construct some contour plots of a function $f(X, Y, Z)$ defined by

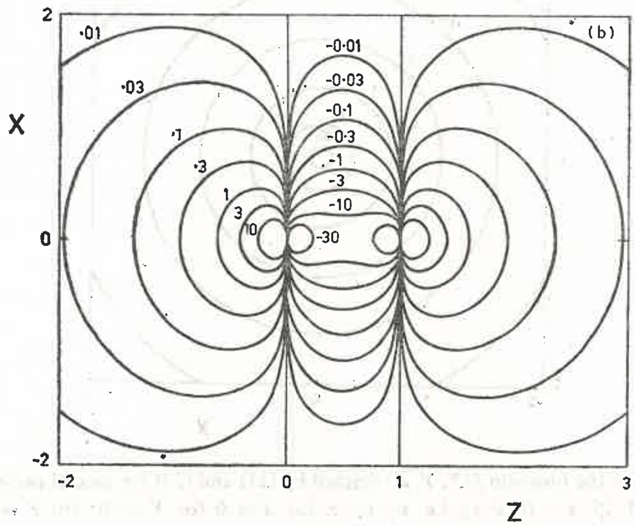
$$f(X, Y, Z) \equiv (c^2 s^4 / q_1 q_2 v_1 v_2)(\boldsymbol{F}_1 \cdot \boldsymbol{F}_2), \tag{21}$$

where

$$X = x/s,\ Y = y/s,\ Z = z/s. \tag{22}$$

We consider two special cases (*i*) $\boldsymbol{v}_1 = (0, 0, v_1)$, $\boldsymbol{v}_2 = (0, 0, v_2)$ (i.e. $\boldsymbol{v}_1 || \boldsymbol{v}_2 || \boldsymbol{s}$, remembering (7)), and (*ii*) $\boldsymbol{v}_1 = (v_1, 0, 0)$, $\boldsymbol{v}_2 = (v_2, 0, 0)$ (i.e. $\boldsymbol{v}_1 || \boldsymbol{v}_2 \perp \boldsymbol{s}$). These plots are shown in





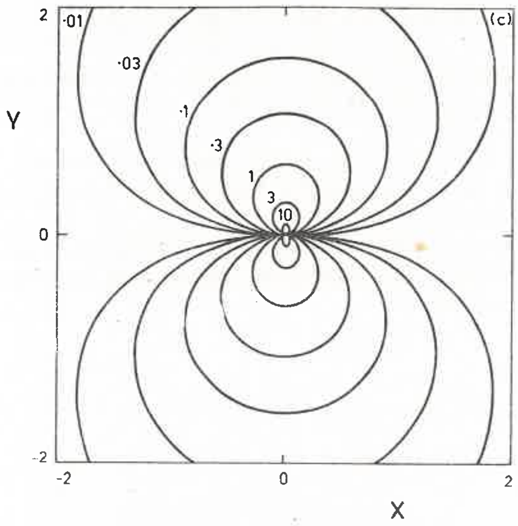


Fig. 3. As Fig. 2, but with $v_i = (v_i, 0, 0)$ $(i = 1, 2)$, $s = (0, 0, s)$, i.e. $v_1 \| v_2 \perp s$. (a) $X = 0$; (b) $Y = 0$; (c) $Z = 0$

Figs. 2 and 3. For comparison we note that the interaction contribution to the static electric-field energy for particles with the same separation has contour forms of the type of Fig. 3(a) for $X = 0$ or $Y = 0$ but has forms of the type of Fig. 2(b) for the plane $Z = 0$.

4. *Conclusions*

It appears to be misleading to attribute part of the Darwin interaction between moving charged particles to retardation. When calculations are made by use of potentials, whether part of the interaction is attributed to retardation depends on the choice of gauge for the potentials. The total Darwin interaction is equal to the negative of the integral over all space of the interaction contribution to the magnetic-field energy. A high proportion of this interaction energy is located in a region outside a sphere centred on one charge with a radius equal to the separation between the charges. The contribution to the magnetic-field interaction energy from this region has the same sign as the product $q_1 q_2(\boldsymbol{v}_1 \cdot \boldsymbol{v}_2)$, and has a magnitude at least twice as large as that of the contribution from within the same sphere. The latter can have the same sign as $q_1 q_2(\boldsymbol{v}_1 \cdot \boldsymbol{v}_2)$ or the opposite one, depending on the relative orientation of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{s} = (\boldsymbol{r}_2 - \boldsymbol{r}_1)$.

We should like to thank P. Lalousis for writing a computer program to provide the contour plots shown in Figs. 2 and 3.

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