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THE STABILITY AND STRATIFICATION OF A QUANTUM LIQUID MIXTURE

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A mixture of quantum liquids was investigated microscopically. The spectrum of collective excitations at finite temperature was determined. The form of this spectrum demonstrates whether there is a stability or stratification of the mixture. The influence of a relative motion of liquids on the spectrum was considered. It was demonstrated that beginning with some finite momentum, the spectrum of each component of the solution splits into two branches, one of which continues the spectrum into the single-particle region. The dynamic susceptibility, the dynamic form-factor, the coefficients of compressibility and the structure factor for the mixture of two Bose liquids were obtained. The integral relations that generalize some rules concerning the binary Bose solution was established.

1. *Introduction*

Much attention is given recently to the study of the mixture of quantum liquids. Usual object of these investigations is a He^3-He^4 solution which is usually considered in hydrodynamic or thermodynamic aspects [1, 2]. The microscopic foundation of the hydrodynamic equations was done by Galasiewicz [3].

The investigation of mixtures of Bose liquids is of special interest. This interest is stimulated by the fact that at a temperature of 2 mK the Fermi helium changes to the superfluid state [4, 5] resulting from the Cooper pairing of fermions. Therefore, at very low temperatures the He^3-He^4 solution is the mixture of two Bose liquids. In such a mixture, as was shown by Khalatnikov [6], the sound waves corresponding to the concentration oscillations also should appear beside the usual sound. The sound velocity of these oscillations in the case of superfluid He^3-He^4 solutions at zero temperature was also investigated [8, 9].

The other examples of quantum Bose solutions are the mixtures of He^4 and He^6, mesons in superdense matter, Cooper pairs of nucleons in the nucleus, the mixture of atoms of orthohelium and parahelium and the mixture of molecules of ortho-Hydrogen

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and para-Hydrogen. Neon occupies an intermidiate place between quantum and classic substances. Thus, the liquid mixture of isotopes of neon also needs the quantum-mechanic description.

Considering the polycomponent liquid, one must remember that it does not always exist as a uniform solution. Under certain circumstances the mixture can stratificate. The thermodynamics of the stratification of dilute Fermi–Bose solutions was described [2, 10]. It is possible to judge about the stratification of a mixture having arbitrary concentrations by the behaviour of the collective-excitation spectrum. Furthermore, the character of the collective spectrum provides an opportunity to examine the instability of the mixture under finite perturbations. The microscopic conditions of the stratification of superfluid liquids at zero temperature were obtained previously [11] in the Bogolubov and long-wave approximations.

In this paper we investigate in the pseudo-potential approximation the stability and the stratification of a mixture of normal quantum liquids at a finite temperature and also under the relative motion of components. We also analyse the special characteristics of the collective spectrum and the different dynamic characteristics of the mixture.

2. *Polycomponent liquid*

When describing the uniform polycomponent system in equilibrium we easily are able to find the individual spectrum for the particles of each component

$$\omega_i(k) = k^2/2m_i^0 + \Sigma_i(k, \omega_i) - \mu_i, \tag{1}$$

where the index enumerates the types of the particles, $\boldsymbol{k}$ is the momentum, m_i^0 is the mass of the particle, Σ_i is the self-energy and μ_i is the chemical potential.

We shall use here the causal Green functions, that is, the so-called propagators. If the spectrum (1) is real, these functions in the momentum-energy representations have the form

$$G_i(k, \omega) = \frac{1 \pm n_i(k)}{\omega - \omega_i + i0} \mp \frac{n_i(k)}{\omega - \omega_i - i0}, \tag{2}$$

in which

$$n_i(k) = [\exp(\beta\omega_i) \mp 1]^{-1}, \quad \beta\theta = 1 \tag{3}$$

and θ is the absolute temperature.

Seeking approximate expressions for the self-energy of a dense liquid, one cannot utilize the theory of perturbation based on the concept of free particles. But it is possible to renorm the interaction of particles by constructing the effective potential which permits the expansion of the self-energy in powers of that pseudopotential [12]. A number of diverse theoretical and empirical methods of constructing the effective potentials are widely used in the theories of liquids, crystals, metals, and in brief, in all cases where one deals with systems of strongly interacting particles [13–18].

We shall not concentrate here on the interaction pseudopotential $\phi_{ij}(r, \omega)$ in which r is the modulus of the Cartezian radius-vector $\boldsymbol{r}$. Let us suppose that such a pseudopotential is constructed and that it has the Fourier transform $\tilde{\phi}_{ij}(\mathrm{k}, \omega)$.

The set of variables $\{\boldsymbol{r}, t, i\}$, where t is the time, will be designated further by one figure. Using the Schwinger variational procedure we can derive the equation for the self-energy

$$\Sigma(12) = \delta(12) \int \phi(13) n(3) d(3) + i \int \phi(13) G(14) \Gamma(423) d(34), \tag{4}$$

where

$$n(1) = \pm i \lim_{2 \to 1} \lim_{\tau_{12} \to -0} G(12) \qquad (\tau_{12} \equiv t_1 - t_2),$$

$$\Gamma(123) = \delta(12)\delta(13) + \int \frac{\delta\Sigma(12)}{\delta G(45)} G(46) G(75) \Gamma(673) d(4567). \tag{5}$$

We investigate here the mixture with no chemical reactions so that the number of particles of each component is constant. Therefore, the propagator is diagonal with respect to the component indexes: $G_{ij} = \delta_{ij} G_i$. From Eqs. (4) and (5), it is clear in this case that the self-energy is also diagonal: $\Sigma_{ij} = \delta_{ij} \Sigma_i$. For example, the self-energy in the pseudo-Hartree approximation is

$$\Sigma_i^0(k, \omega) = \lim_{k \to 0} \lim_{\omega \to 0} \sum_j \tilde{\phi}_{ij}(k, \omega) n_j \tag{6}$$

and n_i is the average density of the particles. If the pseudo-potential is not a function of the propagator, then the pseudo-Fock approximation for the self-energy becomes

$$\Sigma_i^1(k, \omega) = \Sigma_i^0 + \frac{i}{(2\pi)^4} \int \tilde{\phi}_{ii}(\boldsymbol{k} - \boldsymbol{k}', \omega - \omega') G_i^0(k', \omega') dk' d\omega, \tag{7}$$

where Σ_i^0 is the same as that of Eq. (6), and G_i^0 is the propagator with this self-energy. In the next approximation

$$\Sigma_i^2(k, \omega) = \Sigma_i^1(k, \omega) - \frac{1}{(2\pi)^8} \int [\tilde{\phi}_{ii}(\boldsymbol{k} - \boldsymbol{k}'', \omega - \omega'') \pm \tilde{\phi}_{ii}(-\boldsymbol{k}', -\omega')]$$

$$\times \tilde{\phi}_{ii}(k', \omega') G_i^1(\boldsymbol{k} - \boldsymbol{k}', \omega - \omega') G_i^1(k'', \omega'') G_i^1(\boldsymbol{k}'' - \boldsymbol{k}', \omega'' - \omega') dk' dk'' d\omega' d\omega''. \tag{8}$$

It is known that the self-energy for He^3-He^4 mixtures can be presented in the form $\Sigma_i \cong \Sigma_i^0 + \gamma_i k^2/2$. One can obtain such a form by expanding Eq. (7) or (8) in powers of k. In such a situation for the spectrum (1) we have

$$\omega_i(k) = k^2/2m_i + \Sigma_i^0 - \mu_i, \qquad m_i = m_i^0/(1 + m_i^0 \gamma_i). \tag{9}$$

But the spectrum in the pseudopotential Hartree approximation has the same form only if the mass of a particle m_i^0 has been replaced by the mass of a quasiparticle m_i. We shall further state that the pseudo-Hartree approximation has an effective mass of m_i.

When the interaction is instantaneous, that is, when

$$\phi_{ij}(r, \omega) = \phi_{ij}(r) \lim_{\tau \to +0} e^{-i\omega\tau},$$

and spectrum (9) is real, then from Eq. (2) and the representation

$$G_i(k, \omega) = P(\omega-\omega_i)^{-1} - i\pi[1 \pm 2n_i(k)]\delta(\omega-\omega_i),$$

in which P is the symbol of the principal value, we can integrate over the frequency ω in the expressions of the self-energy and different macroscopic averages. For example, in this situation instead of (7) we obtain

$$\Sigma_i^1(k, \omega) = \Sigma_i^0 \pm \frac{1}{(2\pi)^3} \int n_i^0(k')\tilde{\phi}_{ii}(\mathbf{k}-\mathbf{k}')d\mathbf{k}'$$

and for the internal energy density we obtain

$$E = \frac{1}{2\varrho} \sum_i \left[\int \omega_i(k) n_i(k) \frac{d\mathbf{k}}{(2\pi)^3} + n_i \left(\frac{k_i^2}{2m_i^0} + \mu_i \right) \right], \tag{10}$$

where

$$\varrho = \sum_i n_i, \quad k_i^2 = 3m_i^0 \theta g_{5/2}(\alpha_i)/n_i \lambda_i^3,$$

$$g_n(\alpha) = \frac{\alpha}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}dx}{e^x \mp \alpha} = \sum_{l=1}^\infty (\pm 1)^{l-1} \frac{\alpha^l}{l^n},$$

$$\alpha_i = \exp[\beta(\mu_i - \Sigma_i^0)], \quad \lambda_i = \sqrt{2\pi/m_i^0\theta},$$

$\Gamma(n)$ is the gamma function.

3. *Dynamic susceptibility*

The collective spectrum of a system can be found by studying the poles of a dynamic susceptibility. Dealing with the Bose liquid we shall not isolate the Bose-Einstein condensate because its existence, as is well known, has a weak influence on the form of the collective spectrum. This will be verified again below. And more important, the isolation of the condensate with the application of Bogolubov approximation [19] or low-density one [20] leads to the fact that the sound velocity in the system becomes proportional to the square root of the number of condensate particles and thus must be zero at the temperature of the condensation. But the sound velocity should be proportional to the square root of the full density. A more detailed discussion of this problem has been published previously [21–23].

After shifting the self-energy with the infinitesimal source ($\Sigma \to \Sigma + \Delta$), we can define the response function as the variational derivative

$$\chi(123) = \delta G(12)/\delta\Delta(3). \tag{11}$$

After the variation in the equation of motion we can get the equation for function (11)

$$\chi(123) = G(13)G(32) + \int G(14)G(52) \frac{\delta\Sigma(45)}{\delta G(67)} \chi(673) d(4567). \tag{12}$$

Putting into Eq. (12) the self-energy in the pseudo-Hartree approximation, we find that

$$\chi(123) = G(13)G(32) \pm i \int G(14)G(42)\phi(45)\chi(553)d(45), \tag{13}$$

and in the pseudo-Fock approximation

$$\chi(123) = G(13)G(32) \pm i \int G(14)\phi(45)[G(42)\chi(553) \pm G(52)\chi(453)]d(45). \tag{14}$$

Going from Eq. (11) to the two-point function

$$\chi(12) = \pm i \lim_{3\to 1} \lim_{t_3 \to t_1+0} \chi(132) \tag{15}$$

and introducing the polarization function

$$\Pi(12) = \pm i \lim_{3\to 1} \lim_{t_3 \to t_1+0} G(12)G(23), \tag{16}$$

we obtain from (13) that

$$\chi(12) = \Pi(12) + \int \Pi(13)\phi(34)\chi(42)d(34). \tag{17}$$

Reverting to the momentum-energy representation instead of (16) and (17) we have

$$\Pi_{ij}(k, \omega) = \pm i \frac{\delta_{ij}}{(2\pi)^4} \int G_i(\mathbf{k}+\mathbf{k}', \omega+\omega')G_i(k', \omega')dk'd\omega',$$

$$\chi_{ij}(k, \omega) = \Pi_{ij}(k, \omega) + \Pi_{ii}(k, \omega) \sum_l \tilde{\phi}_{il}(k, \omega)\chi_{lj}(k, \omega). \tag{18}$$

If the considered system consists of two kinds of particles, then the dynamic susceptibilities given by Eq. (18) are

$$\chi_{11} = \Pi_{11}(1-\Pi_{22}\tilde{\phi}_{22})\,[(1-\Pi_{11}\tilde{\phi}_{11})\,(1-\Pi_{22}\tilde{\phi}_{22})-\Pi_{11}\Pi_{22}\tilde{\phi}_{12}^2]^{-1},$$

$$\chi_{12} = \Pi_{11}\Pi_{22}\tilde{\phi}_{22}[(1-\Pi_{11}\tilde{\phi}_{11})\,(1-\Pi_{22}\tilde{\phi}_{22})-\Pi_{11}\Pi_{22}\tilde{\phi}_{12}^2]^{-1}, \tag{19}$$

where for the sake of brevity the arguments k and ω are not written and it holds that $\phi_{12} = \phi_{21}$. Using the mutual replacement of the indices in Eq. (19) we may obtain the functions χ_{22} and $\chi_{21} = \chi_{12}$.

Let us think that the spectrum (1) is real which corresponds to the well-defined quaisparticles. Then the polarization function is described by the formula

$$\Pi_{ii} = \int \left\{ \frac{[1 \pm n_i(\mathbf{k}+\mathbf{k}')]n_i(k')}{\omega-\omega_i(\mathbf{k}+\mathbf{k}', k')+i0} - \frac{n_i(\mathbf{k}+\mathbf{k}')\,[1 \pm n_i(k')]}{\omega-\omega_i(\mathbf{k}+\mathbf{k}', k')-i0} \right\} \frac{dk'}{(2\pi)^3},$$

in which $\omega_i(k, k') = \omega_i(k) - \omega_i(k')$. We are interested here in the collective excitations with a small attenuation. Therefore, we shall examine first the real part of Π_{ii}, that comes from the precedent expression for Π_{ii} the following:

$$\mathrm{Re}\,\Pi_{ii}(k, \omega) = P \int \frac{n_i(\mathbf{k}+\mathbf{k}')-n_i(k')}{\omega_i(\mathbf{k}+\mathbf{k}', k')-\omega} \cdot \frac{dk'}{(2\pi)^3}$$

$$= P \int \frac{n_i(k')\,[\omega_i(\mathbf{k}+\mathbf{k}', k')-\omega_i(k', \mathbf{k}'-\mathbf{k})]}{[\omega_i(\mathbf{k}+\mathbf{k}', k')-\omega]\,[\omega_i(k', \mathbf{k}'-\mathbf{k})-\omega]} \cdot \frac{dk'}{(2\pi)^3}.$$

Substituting the spectrum $\omega_i(k)$ in the pseudo-Hartree approximation with effective mass into Π_{ii}, we finally derive that

$$\mathrm{Re}\,\Pi_{ii}(k,\omega) = \frac{k^2/m_i}{(2\pi)^3}\, P \int \frac{n_i^0(k')d\boldsymbol{k}'}{(\omega - \boldsymbol{k}\boldsymbol{k}'/m_i)^2 - (k^2/2m_i)^2}\,. \tag{20}$$

4. Collective spectrum

The distribution of particles $n_i^0(k)$ diminishes with an increase of k. This diminition happens quickly enough in the case of Bose statistics and also for the Fermi particles at high temperatures [24] or at a low concentration in a Bose liquid [25]. Consequently, for these situations the formula (20) should be approximately equal to the expression

$$\Pi_{ii}^0(k,\omega) = (n_i k^2/m_i)\,[\omega^2 - (k^2/2m_i)^2]^{-1} \tag{21}$$

obtained as the result of the substitution of $k' = 0$ into the $\boldsymbol{k}\boldsymbol{k}'$, which corresponds to the maximum of that distribution. Notice that the isolation of the Bose-condensate fraction using the transformation $n(k) \to n_c\delta(k) + n(k)$, where n_c is the density of the condensate fraction does not change the function (21). That is, the isolation of the condensate does not lead to the essential change of the collective spectrum.

Let us rewrite Eq. (20) as the product

$$\mathrm{Re}\,\Pi_{ii}(k,\omega) = \Pi_{ii}^0(k,\omega) f_i(k,\omega),$$

in which the factor

$$f_i(k,\omega) = \frac{\omega^2 - (k^2/2m_i)^2}{(2\pi)^3 n_i}\, P \int \frac{n_i^0(k')d\boldsymbol{k}'}{(\omega - \boldsymbol{k}\boldsymbol{k}'/m_i)^2 - (k^2/2m_i)^2} \tag{22}$$

contains the temperature dependence. Because of the limiting relations

$$\lim_{k\to 0} f_i(k,\omega) = 1 = \lim_{k\to\infty} f_i(k,\omega)$$

one must conclude that Eq. (21) is asymptotically equal to Eq. (20) in both alternative cases of low and high momenta.

Poles of the dynamic susceptibilities (19) dominate two branches of the collective spectrum for which we have the implicit form

$$\varepsilon_\pm^2 = \tfrac{1}{2}\left[\varepsilon_1^2 + \varepsilon_2^2 \pm \sqrt{(\varepsilon_1^2 - \varepsilon_2^2)^2 + 4\varepsilon_{12}^4}\right], \tag{23}$$

where

$$\varepsilon_i^2 = s_i^2 k^2 + (k^2/2m_i)^2, \qquad s_i^2 = (n_i/m_i)\tilde{\phi}_{ii}(k) f_i(k,\varepsilon_\pm),$$

$$\varepsilon_{12}^2 = s_{12}^2 k^2, \qquad s_{12}^2 = \sqrt{(n_1 n_2/m_1 m_2) f_1(k,\varepsilon_\pm) f_2(k,\varepsilon_\pm)}\;\tilde{\phi}_{12}(k).$$

To obtain the explicit function $\varepsilon_\pm(k)$ we have to use the iterative procedure to the right side of Eq. (23). However, the asymptotic behaviour of the spectrum can be found without such an iteration, that is in a general case only when $\tilde{\phi}_{ij}(k)$ is limited. In the long-wave limit two sound waves propagate

$$\varepsilon_\pm^2 \simeq s_\pm^2 k^2, \quad s_\pm^2 = \tfrac{1}{2}\left[s_1^2+s_2^2 \pm \sqrt{(s_1^2-s_2^2)+4s_{12}^4}\right]_{k=0},$$

and in the short-wave limit there are two single-particle motions

$$\varepsilon_\pm \simeq k^2/2m_i \quad (k \to \infty).$$

Now we shall discuss the stability of the collective excitations. The spectrum of stable excitations must satisfy the conditions

$$\operatorname{sign} \operatorname{Re} \varepsilon(k) = -\operatorname{sign} \operatorname{Im} \varepsilon(k), \qquad |\operatorname{Im} \varepsilon(k)/\operatorname{Re} \varepsilon(k)| < 1.$$

We find from Eq. (23) that two stable branches of excitations exist in the system when

$$\varepsilon_1^2(k)\varepsilon_2^2(k) > \varepsilon_{12}^4(k). \tag{24}$$

The instability threshold corresponds to the reduction of the inequality (24) to the equality, which gives us an equation for the threshold momentum

$$k_s^2 = 2\{\sqrt{[m_1^2 s_1^2(k_s)-m_2^2 s_2^2(k_s)]^2+4m_1^2 m_2^2 s_{12}^4(k_s)} - m_1^2 s_1^2(k_s) - m_2^2 s_2^2(k_s)\}. \tag{25}$$

When $\varepsilon_{12}^4 = \varepsilon_1^2\varepsilon_2^2$, only one branch survives: $\varepsilon_+^2 = \varepsilon_1^2+\varepsilon_2^2$, and the second one, describing the concentration oscillations, becomes zero: $\varepsilon_-^2 = 0$. The disappearance of concentration oscillations in a mixture means that the mixture stratificates.

To be definite let us consider that s_i^2 and s_{ij}^4 are real. Then three principal situations are possible. The first occurs when $s_1^2(k)s_2^2(k) \geqslant s_{12}^4(k)$ at all k. In that case, Eq. (25) does not have any positive solution. So, there are two branches of the spectrum in the whole of the momentum space. The mixture is absolutely stable. The other situation occurs when $s_1^2(0)s_2^2(0) > s_{12}^4(0)$ or, in the same situation when $\tilde{\phi}_{11}(0)\tilde{\phi}_{22}(0) > \tilde{\phi}_{12}^2(0)$, but beginning from some k_s, $s_1^2(k)s_2^2(k) < s_{12}^4(k)$. Then two branches exist in the interval $0 < k < k_s$ and only one in the region of $k \geqslant k_s$ up to k_p given by the equation $k_p^4 = 16 n_1 n_2 m_1 m_2 \tilde{\phi}_{12}^2(k_p)$, when the transmitted energy increases more than the interaction energy of particles. Thus the system is stable with respect to the perturbations with momenta of $k < k_s$, but it loses the stability and stratificates under the action of perturbations with the transmitted momenta lying in the interval $k_s < k < k_p$. Furthermore, when $s_1^2(0)s_2^2(0) \leqslant s_{12}^4(0)$, there is only one branch in the momentum space near $k \gtrsim 0$. The instability of a system with respect to infinitesimal perturbations means that the considered system does not represent a uniform solution. The system is absolutely unstable.

The relative motion of mixture components changes the collective spectrum and, consequently, the stratification conditions. This means the normal (not superfluid) motion due to external forces. This may be an electrical field acting on differently charged components of a mixture.

Let us limit this discussion to constant velocities $\boldsymbol{v}_i$ (in the laboratory system of coordinates) of compound components. Then in the equations of motion of immovable liquids we have to replace $-i\nabla \to -i\nabla + m_i \boldsymbol{v}_i$, which leads in all energy-representation functions to the shift in the frequency $\omega \to \omega - \boldsymbol{k}_i\boldsymbol{v}_i$ and a shif in the chemical potential $\mu_i \to \mu_i - m_i v_i^2/2$. Dealing with a binary solution we shall place the centre of the coordinate system into the first component and designate the velocity of the second component relative to the first by $\boldsymbol{v}$.

The poles of the dynamic susceptibility of the moving liquid give in the long-wave approximation the following equation for the sound velocity s:

$$s^4 - s^3 2v\cos\vartheta + s^2(v^2\cos^2\vartheta - s_1^2 - s_2^2) + ss_1^2 2v\cos\vartheta + s_1^2 s_2^2 - s_{12}^4 - s_1^2 v^2 \cos^2\vartheta = 0, \tag{26}$$

where ϑ is the angle between the vectors $\boldsymbol{k}$ and $\boldsymbol{v}$. Using the theorem of Descartes, Eq. (26) has two positive solutions if

$$s_1^2(s_2^2 - v^2\cos^2\vartheta) > s_{12}^4. \tag{27}$$

Hence, when condition (27) is correct, there are two sound modes in the system. Let us consider s_1, s_2, s_{12}^4 to be positive. This does not narrow the generality of the investigation, but only corresponds to the definite choice of a sign of square roots. Using the equation $u^2 = \sqrt{s_1^2 s_2^2 - s_{12}^4}$, we can rewrite condition (27) as $|\cos\vartheta| < u^2/vs_1$. Analysing these correlations, we see that if the immovable mixture was stratificated ($u^4 < 0$), then the moving one remains the same. If there were two sounds in the immovable two-component liquid ($u^2 > 0$), then at the low velocities of relative motion when $vs_1 < u^2$, the moving mixture will be stable. When increasing the velocity v, so that $vs_1 = u^2$, there are two sounds in the whole system except for the direction of $\vartheta = 0$. When $vs_1 > u^2$ the mixture will stratificate into a cone having an apex angle of $\vartheta_0 = \arccos(u^2/vs_1)$, then in the rest of the space the uniform mixture with two sounds will still exist. However the effect of the anisotropic stratification into a cone having an apex angle ϑ_0 might appear onlywhen the loss of stability occurs at $k \neq 0$. If the loss of stability is due to infinitesimal perturbations, i.e., when it happens at $k = 0$, then the consideration of stratification cones makes no sense. Then the mixture stratificates in the whole volume as soon as the velocity of the relative motion reaches the critical value $v_c = \sqrt{s_2^2 - s_{12}^4/s_1^2}$, where $k = 0$.

Let us emphasize that we have considered the normal motion of mixture components. When describing the superfluid motion of a binary solution, it is necessary to introduce not two but three velocities: the velocity of the system as a whole and the velocities of superfluid motions of two components.

We shall deal now with the question of the form of each collective branch in the immovable Bose liquid. It is known that the usual phonon-roton branch has the disintegration threshold of the elementary excitations [26]. However it is clear that the spectrum should have a continuation into the one-particle region. Such a continuation was observed by Cowley and Woods [27] using inelastic scattering of thermal neutrons from liquid He^4. The theoretical peculiarities of the spectrum continuation beyond the decay threshold at zero temperature were examined previously [28, 29]. The experiment [27] also shows that there are in the liquid He^4 two branches of collective excitations. One of them begins from $k = 0$ and it has a phonon-roton character and diminishes near $k \approx 3$ Å^{-1}. The other branch begins at $k \approx 1$ Å^{-1} and continues to the single-particle region. It is possible to explain the split of the spectrum in a one-component Bose liquid if one supposes the existence of bound roton pairs [30].

Let us show now that at a finite temperature each Bose component really has two stable spectra without the supposition of an appearance of bound excitations. To do this we expand in factor (22) the integrand but $n_i^0(k')$ in powers of k', limiting ourselves to the second order of the expansion. Then we obtain as an alternative of Eq. (22) the following:

$$f_i(k, \omega) = 1+\tfrac{1}{3}\left(\frac{k_i k}{2m_i}\right)^2 \frac{3\omega^2+(k^2/2m_i)^2}{[\omega^2-(k^2/2m_i)^2]^2}.$$

In the case of independent components, when $s_{12} = 0 = \phi_{12}$, we have from (23) $\varepsilon_+^2 = \varepsilon_1^2$, $\varepsilon_-^2 = \varepsilon_2^2$, which means, after taking into account the form of $f_i(k, \omega)$, that the collective spectrum of each component satisfies the equation

$$\varepsilon_i^2 = \frac{n_i}{m_i} k^2 \tilde{\Phi}_{ii}(k) \left\{1+\tfrac{1}{3}\left(\frac{k_i k}{2m_i}\right)^2 \frac{3\varepsilon_i^2+(k^2/2m_i)^2}{[\varepsilon_i^2-(k^2/2m_i)^2]^2}\right\}.$$

In solving the equation we made sure that each one-component Bose liquid can have two stable branches described by the formulas

$$\varepsilon_i^2 = c_i^2 k^2+(k^2/2m_i)^2 \tag{28}$$

with one of the functions

$$c_{i1}^2 = s_i^2(B_i^+ +B_i^- +\tfrac{1}{3}), \quad c_{i2}^2 = s_i^2\left[\tfrac{1}{3}-\tfrac{1}{2}(B_i^+ +B_i^-)-\frac{i\sqrt{3}}{2}(B_i^+ -B_i^-)\right],$$

where we used the notations

$$B_i^\pm = (K_i \pm L_i)^{1/3}, \quad K_i = \tfrac{1}{27}[1+\tfrac{9}{2} p_i^2(1+3q_i^2)],$$

$$L_i = \frac{p_i}{3\sqrt{3}}[q_i^2+p_i^2(2+\tfrac{9}{2} q_i^2+\tfrac{27}{4} q_i^4)-p_i^4]^{1/2},$$

$$p_i = k_i/m_i s_i, \quad q_i = k/m_i s_i, \quad s_i^2 = n_i \tilde{\phi}_{ii}(k)/m_i$$

To clear these complicated expressions let us simplify them by thinking that $p_i \ll 1$. Hence,

$$c_{i1}^2 \simeq s_i^2[1+p_i^2(1+q_i^2)], \quad c_{i2}^2 \simeq -s_i^2\left[\frac{p_i^2}{2}(1+q_i^2)+ip_i q_i\right]. \tag{29}$$

The first expression of Eqs. (29) contains the temperature dependence qualitatively agreeing with the low-temperature dependence of the sound velocity [31, 32], and at $k \to 0$ coinciding with the Etters sound velocity [33]. The second expression of Eqs. (29) corresponds to the stable branch of spectrum (28) only at high momenta when $k \gg k_i$. So, the second branch of spectrum (28) is of the one-particle kind

$$\varepsilon_i^{(2)} \simeq k^2/2m_i, \quad \mathrm{Im}\,\varepsilon_i^{(2)} \simeq -k_i k/m_i \quad (k \gg k_i). \tag{30}$$

As far as liquid He^4 is concerned, Eq. (29) may be applied to it only qualitatively because it follows from the measurements of the kinetic energy of the particles [34] that $p_i \sim 1$. However, one can see that beginning from $k_i \approx m_i s_i \approx 1$ Å^{-1} in liquid He^4 there can exist two spectrum branches what is in agreement with experimental data [27]. And below $k \lesssim k_i$ the second branch $\varepsilon_i^{(2)}$ strongly attenuates (if $s_i > 0$). When q_i, $p_i \simeq 1$ the real part of the spectrum (28) is of the same order as the imaginary one.

5. *Dynamic form-factor*

The information concerning the collective spectrum of a quantum liquid can be extracted from the shape of a dynamic form-factor which is the measurable quantity. One usually produces this by measuring the inelastic neutron scattering. The first quantum liquid, whose form-factor was investigated in detail both theoretically and experimentally, was liquid He^4. The latest papers on this subject are [27] and [35–37]. Recently the study of the dynamic properties of liquid He^3 was also started [38–47]. The analogous characteristics for mixtures of quantum liquids are not yet very well known. In paper [48], which also contains other references, the case of zero temperature was studied.

The dynamic form-factor is the Fourier transform of the correlation function "density-density". For a mixture of liquids we define the dynamic form-factor as the matrix having the elements

$$D_{ij}(k, \omega) = \frac{1}{N}\int R_{ij}(\mathbf{r}', 0, \mathbf{r}, t) \exp\left[i\mathbf{k}(\mathbf{r}-\mathbf{r}') - i\omega t\right] d\mathbf{r}\, d\mathbf{r}'\, dt, \tag{31}$$

where the indices enumerate the mixture components, N is the total number of particles in the system, the correlation function is

$$R_{ij}(\mathbf{r}, t, \mathbf{r}', t') = \langle \varrho_i(\mathbf{r}, t)\varrho_j(\mathbf{r}', t')\rangle, \qquad \varrho_i(\mathbf{r}, t) = \psi_i^\dagger(\mathbf{r}, t)\psi_i(\mathbf{r}, t) \tag{32}$$

and $\psi_i(\mathbf{r}, t)$ is the Heisenberg field operator.

Using the uniformity and the stability of the system we can transform function (32):

$$R_{ij}(\mathbf{r}, t, \mathbf{r}', t') = \int R_{ij}(k, \omega) \exp\left[i\mathbf{k}(\mathbf{r}-\mathbf{r}') - i\omega(t-t')\right] \frac{dk\, d\omega}{(2\pi)^4}.$$

Thus,

$$D_{ij}(k, \omega) = \varrho^{-1} R_{ij}(k, \omega).$$

Let us introduce the function

$$T(12) = -i\langle T\varrho(1)\varrho(2)\rangle,$$

where T is the chronological operator. This function is connected with the correlator $R(12)$ using the expression

$$T(12) = -i[\theta(\tau_{12})R(12) + \theta(\tau_{21})R(21)] \qquad (\tau_{12} \equiv t_1 - t_2),$$

in which $\theta(\tau)$ is the step function. Consequently,

$$R_{ij}(k,\omega) = -2\,\mathrm{Im}\,T_{ij}(k,\omega)/(1+e^{-\beta\omega}).$$

But with the Schwinger variational technique it is not difficult to make sure that

$$T(12) = \chi(12) - in(1)n(2).$$

Therefore,

$$\mathrm{Im}\,T_{ij}(k,\omega) = \mathrm{Im}\,\chi_{ij}(k,\omega) - (2\pi)^4 n_i n_j \delta(k)\delta(\omega).$$

Thus, we find that the connection between the correlator and the response function has the form

$$R_{ij}(k,\omega) = -\frac{2\,\mathrm{Im}\,\chi_{ij}(k,\omega)}{1+e^{-\beta\omega}} + (2\pi)^4 n_i n_j \delta(k)\delta(\omega). \tag{33}$$

Taking into account that on the complex ω-plane there is the representation

$$\chi_{ij}(k,\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\gamma_{ij}(k,\omega')}{\omega-\omega'}\,d\omega' \tag{34}$$

with the spectral function

$$\gamma_{ij}(k,\omega) = i[\chi_{ij}(k,\omega+i0) - \chi_{ij}(k,\omega-i0)], \tag{35}$$

we have

$$\mathrm{Im}\,\chi_{ij}(k,\omega) = -\tfrac{1}{2}\gamma_{ij}(k,\omega)\coth\left(\frac{\beta\omega}{2}\right). \tag{36}$$

Finally, the dynamic form-factor may be written as

$$D_{ij}(k,\omega) = (2\pi)^4 (n_i n_j/\varrho)\delta(k)\delta(\omega) + S_{ij}(k,\omega), \tag{37}$$

here

$$S_{ij}(k,\omega) = \gamma_{ij}(k,\omega)/\varrho(1-e^{-\beta\omega}) \tag{38}$$

is the Van Hove function.

Resulting from Eq. (19) and the approximation of (21) we look for complex ω

$$\chi_{11} = \frac{(n_1/m_1)(\omega^2-\varepsilon_2^2)k^2}{(\omega^2-\varepsilon_+^2)(\omega^2-\varepsilon_-^2)}, \quad \chi_{12} = \frac{\sqrt{n_1 n_2/m_1 m_2}\,\varepsilon_{12}^2 k^2}{(\omega^2-\varepsilon_+^2)(\omega^2-\varepsilon_-^2)}. \tag{39}$$

Eqs. (35) and (39) when $0 \neq \varepsilon_+ \neq \varepsilon_- \neq 0$ give us the spectral functions

$$\gamma_{11}(k,\omega) = (n_1/m_1)(\omega^2-\varepsilon_2^2)k^2\gamma(k,\omega),$$

$$\gamma_{12}(k,\omega) = \sqrt{n_1 n_2/m_1 m_2}\,\varepsilon_{12}^2 k^2 \gamma(k,\omega), \tag{40}$$

where the following notation is used:

$$\gamma(k,\omega) = 2\pi\delta[(\omega^2-\varepsilon_+^2)(\omega^2-\varepsilon_-^2)]$$

$$= \frac{\pi}{\varepsilon_+^2-\varepsilon_-^2}\left\{\frac{1}{\varepsilon_+}[\delta(\omega-\varepsilon_+)-\delta(\omega+\varepsilon_+)]-\frac{1}{\varepsilon_-}[\delta(\omega-\varepsilon_-)-\delta(\omega+\varepsilon_-)]\right\}.$$

6. *Sum rules*

In this paragraph we shall obtain the formulas generalized on a binary Bose solution those which are based on a mono-component liquid. We are interested in the integrals

$$S_{ij}(k) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} S_{ij}(k,\omega)d\omega,$$

$$K_{ij}(k) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} \omega S_{ij}(k,\omega)d\omega,$$

$$Q_{ij}(k) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} S_{ij}(k,\omega)\frac{d\omega}{\omega}. \tag{41}$$

Such integrals were examined previously [48], but in the case of zero temperature, and the explicit forms of Eqs. (41) were expressed through the macroscopic quantity of the energy density. We shall examine here the nonzero-temperature case and express these integrals through the microscopic characteristics, i.e., through the collective spectrum.

Using the properties of the isotropy of the system and of detailed balance we find that

$$S_{ij}(-k,\omega) = S_{ij}(k,\omega), \quad S_{ij}(k,-\omega) = e^{-\beta\omega}S_{ij}(k,\omega),$$

and separating the Van Hove function into symmetric and antisymmetric parts we have

$$S_{ij}(k,\omega) = S_{ij}^{(S)}(k,\omega)+S_{ij}^{(a)}(k,\omega),$$

$$S_{ij}^{(S)}(k,\omega) = \tfrac{1}{2}[S_{ij}(k,\omega)+S_{ij}(k,-\omega)] = \frac{1}{2\varrho}\gamma_{ij}(k,\omega)\coth\left(\frac{\beta\omega}{2}\right),$$

$$S_{ij}^{(a)}(k,\omega) = \tfrac{1}{2}[S_{ij}(k,\omega)-S_{ij}(k,-\omega)] = \frac{1}{2\varrho}\gamma_{ij}(k,\omega).$$

We can rewrite Eqs. (41) in the form

$$S_{ij}(k) = \frac{1}{2\pi\varrho}\int\limits_0^{\infty}\gamma_{ij}(k,\omega)\coth\left(\frac{\beta\omega}{2}\right)d\omega,$$

$$K_{ij}(k) = \frac{1}{2\pi\varrho}\int\limits_0^\infty \omega\gamma_{ij}(k,\omega)d\omega,$$

$$Q_{ij}(k) = \frac{1}{2\pi\varrho}\int\limits_0^\infty \gamma_{ij}(k,\omega)\frac{d\omega}{\omega}. \tag{42}$$

For the structure factor we get

$$S_{11}(k) = \frac{(n_1/m_1)k^2}{2\varrho(\varepsilon_+^2-\varepsilon_-^2)}\left[\frac{\varepsilon_+^2-\varepsilon_2^2}{\varepsilon_+}\coth\left(\frac{\beta\varepsilon_+}{2}\right) - \frac{\varepsilon_-^2-\varepsilon_2^2}{\varepsilon_-}\coth\left(\frac{\beta\varepsilon_-}{2}\right)\right],$$

$$S_{12}(k) = \frac{\sqrt{n_1n_2/m_1m_2}\,\varepsilon_{12}^2k^2}{2\varrho(\varepsilon_+^2-\varepsilon_-^2)}\left[\frac{1}{\varepsilon_+}\coth\left(\frac{\beta\varepsilon_+}{2}\right) - \frac{1}{\varepsilon_-}\coth\left(\frac{\beta\varepsilon_-}{2}\right)\right]. \tag{43}$$

When $\theta = 0$, then

$$S_{11}(k) = \frac{(n_1/m_1)(\varepsilon_+\varepsilon_- + \varepsilon_2^2)k^2}{2\varrho\varepsilon_+\varepsilon_-(\varepsilon_+ + \varepsilon_-)}, \quad S_{12}(k) = -\frac{\sqrt{n_1n_2/m_1m_2}\,\varepsilon_{12}^2k^2}{2\varrho\varepsilon_+\varepsilon_-(\varepsilon_+ + \varepsilon_-)}, \tag{44}$$

and in the long-wave limit $S_{ij}(k) \sim k$. But if $\theta \neq 0$, the longwave limit is

$$S_{11}(0) = \frac{(n_1/m_1)s_2^2(0)}{\beta\varrho s_+^2(0)s_-^2(0)}, \quad S_{12}(0) = \frac{\sqrt{n_1n_2/m_1m_2}\,s_{12}^2(0)}{\beta\varrho s_+^2(0)s_-^2(0)}. \tag{45}$$

Using Eqs. (43) we are able to verify the short-wave limit for any θ

$$\lim_{k=\infty}\sum_{ij} S_{ij}(k) = 1.$$

For the second and the third of the integrals (42) we have

$$K_{11}(k) = n_1k^2/2m_1\varrho, \quad K_{12}(k) = 0 \tag{46}$$

and the relation

$$Q_{ij}(k) = -\mathrm{Re}\,\chi_{ij}(k,0)/\varrho. \tag{47}$$

Since on the real ω-axis the response function has the spectral representation

$$\chi_{ij}(k,\omega) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\gamma_{ij}(k,\omega')\left[\frac{1+\nu(\omega')}{\omega-\omega'+i0} - \frac{\nu(\omega')}{\omega-\omega'-i0}\right]d\omega',$$

in which $\nu(\omega) = (e^{\beta\omega}-1)^{-1}$, the real part

$$\mathrm{Re}\,\chi_{ij}(k,\omega) = \frac{P}{2\pi}\int\limits_{-\infty}^{+\infty}\gamma_{ij}(k,\omega')\frac{d\omega'}{\omega-\omega'}$$

has the same form as Eq. (39).

Let us calculate the coefficients of compressibility. When shifting the self-energy of a system by the infinitesimal addition $\delta\Delta_i(\boldsymbol{r}, t)$ the particle density changes by the quantity

$$\delta n_i(\boldsymbol{r}, t) = n_i(\boldsymbol{r}, t) \int \kappa_i(\boldsymbol{r}', t', \boldsymbol{r}, t)\delta P_i(\boldsymbol{r}', t')d\boldsymbol{r}'dt',$$

which is expressed using the microscopic compressibility $\kappa_i(\boldsymbol{r}, t, \boldsymbol{r}', t')$ and the pressure change $\delta P_i(\boldsymbol{r}, t)$. In turn, the pressure change is stipulated by the shift of the self-energy

$$\delta P_i(\boldsymbol{r}, t) = -n_i(\boldsymbol{r}, t)\delta\Delta_i(\boldsymbol{r}, t).$$

Therefore, considering the definitions of the response function (11) and (15), it is evident that

$$\chi_{ii}(\boldsymbol{r}, t, \boldsymbol{r}', t') = -n_i(\boldsymbol{r}, t)\kappa_i(\boldsymbol{r}, t, \boldsymbol{r}', t')n_i(\boldsymbol{r}', t').$$

But for an equilibrium and uniform system the latest expression is equal to the following:

$$\chi_{ii}(\boldsymbol{r}, t, 0, 0) = -\kappa_i(\boldsymbol{r}, t, 0, 0)n_i^2.$$

Then, after the definition of the coefficients of compressibility

$$\kappa_i = \lim_{k\to 0} \lim_{\omega\to 0} \operatorname{Re} \int \kappa_i(\boldsymbol{r}, t, 0, 0)e^{-i(\boldsymbol{kr}-\omega t)}d\boldsymbol{r}dt,$$

we are able to establish the relations

$$\lim_{k\to 0} \operatorname{Re} \chi_{ii}(k, 0) = -n_i^2\kappa_i \tag{48}$$

between κ_i and the limiting values of the dynamic susceptibilities

$$\lim_{k\to 0} \operatorname{Re} \chi_{11}(k, 0) = -n_1 s_2^2(0)/m_1 s_+^2(0)s_-^2(0),$$

$$\lim_{k\to 0} \operatorname{Re} \chi_{12}(k, 0) = \sqrt{n_1 n_2/m_1 m_2}\, s_{12}^2(0)/s_+^2(0)s_-^2(0). \tag{49}$$

Comparing (48) and (49), it is clear that

$$\kappa_i = s_j^2(0)/n_i m_i s_+^2(0)s_-^2(0), \tag{50}$$

where $i \neq j$. When considering the limiting procedure $k \to 0$ for Eq. (47) and using the relation (48) we shall obtain the sum rules for the compressibilities.

One can see from (43) and (50) that if the instability, caused by the condensation of the soft mode, occurs at $k_s \neq 0$, that is when $\varepsilon_-(k_s) = 0$, then the structure factors diverge and the compressibility coefficients are finite. But if the system stratificates at $k_s = 0$, becoming absolutely instable, then the coefficients (50) and the structure factors in the long-wave limit both become infinite.

7. *Conclusion*

The behaviour of quantum crystals in many respects is very similar to those of quantum liquids. For example, in quantum crystals the diffusion processes play a much more important role than that in classic crystals [49], which leads to the necessity to take into account the jumps of particles from one lattice point to another [50]. Thus, it is supposed that properties investigated above of liquid Bose mixtures might be similar to the properties of solid Bose solutions, especially in the long-wave range, where the space structure of a system is not so essential.

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