PERTURBED SOLITON SOLUTIONS IN SELF-INDUCED TRANSPARENCY WITH FINITE RELAXATION TIME

By A. Roy Chowdhury, T. Roy and R. S. Banerjee

Department of Physics, Jadavpur University, Calcutta*

(*Received December 28, 1978; final version received August 8, 1979*)

The behaviour of soliton-like solutions regarding equations of self-induced transparency, when the relaxation time of the system is finite, was analysed using the perturbation theory of inverse scattering formalism. The time evolution of the phase, amplitude and velocity of the soliton were determined and the implications discussed.

During the last few years the exact solution of nonlinear evolution equations have been exhaustively studied extensively using the inverse scattering transform formalism [1]. Of all the equations treated, those of self-induced transparency (SIT) are of greatest importance which describe the propagation of ultra short optical pulses through a material medium [2]. But it is interesting to study the approximations involved in reducing the equations of SIT to a form amenable to analysis by the inverse scattering transform (IST). Firstly, we have to consider the one space variable and, secondly, the relaxation time of the medium has to be infinite. It is this second point on which we focus in this communication because till now no methodology is available for the extension of the IST formalism to more than one space dimension. The closest practical approach to a physical situation is the case of SIT with a finite relaxation time. Unfortunately, the SIT equation cannot be exactly solved if the relaxation time is kept finite. So one must take recourse to the perturbation technique [3] within the framework of IST. The present work is organised as follows: In Section 1, the modified equations of SIT are described and the equation for the electric field $E(x, t)$ is deduced. In Section 2, we apply the perturbation technique of Kaup [3] for analysing the time variation of the physical parameters of soliton. In Section 3, we discuss the physical implications of the effects of finite relaxation time.

1. Governing equations

In recent times there have been umpteen formulations of self-induced transparency from different points of view. But the basic idea is always to give an accurate description of the 2π-pulse, starting from several visions of Maxwell–Block equations. Among the

* Address: Department of Physics, Jadavpur University, Calcutta — 700032, India.

various approaches worth mentioning are those of Diels and Hahn's [4], Lamb [2], and that of Eilbeck et al. [5]. In the last one a description of the sech profile of the 2π-pulse is obtained at a density of the medium equal to 10^{18} atoms cm^{-3} within the accuracy of pico-second range. Here we will be following Ref. [4] for visualising the effect of finite relaxation time within the framework of soliton perturbation in conjunction with inverse scattering transform.

In the theoretical formulation of the self-induced transparency (SIT) the most important dynamical variables are given by five quantities. U, V, W, E and ϕ, all functions of the space-time variables (x, t). Of these five variables $E(x, t)$ and $\phi(x, t)$ represent the amplitude and phase of the applied electric field on the system, and U, V, W are important quantities describing the properties of the state of polarization and that of the atomic system. Here we essentially follow the notation of Diels and Hahn [4]. In their formulation U and V describe respectively the two states of polarization and $\Delta\omega$ is the particular off resonance frequency distance. These components are, respectively, in phase and 90° out of phase with the direction of the slowly varying modulas field $E(x, t)$, which rotates at the instantaneous frequency $\omega+\dot{\phi}(z, t)$. W represents the expectation value of the number density, N, of the two level system under consideration. In the governing equations which follow $g(\delta)$ denotes the non-homogeneous spectrum distribution function, chosen as:

$$g(\delta) = \left(\frac{T_2}{\sqrt{\pi}}\right) e^{-\delta^2 T_2{}^2}.$$

Then the equations describing the phenomenon of SIT when the material medium poses finite relaxation time reads [5]:

$$\frac{\partial U}{\partial t} = V(\Delta\omega-\dot{\phi}) - \frac{U}{T_1},$$

$$\frac{\partial V}{\partial t} = -(\Delta\omega-\dot{\phi})U - \frac{k^2}{\omega_0} EW - \frac{V}{T_2},$$

$$\frac{\partial W}{\partial t} = \omega_0 EV,$$

$$\frac{\partial E}{\partial z} + \frac{\eta}{c}\frac{\partial E}{\partial t} = -\frac{2\pi\omega}{\eta c}\int Vg(\delta)d\delta,$$

$$\varepsilon\frac{\partial \phi}{\partial z} + \varepsilon\frac{\eta}{c}\frac{\partial \phi}{\partial t} = -\frac{2\pi\omega}{\eta c}\int Ug(\delta)d\delta. \qquad (1)$$

It is important to note that for the application of the methodology expounded in [3] we require an evolution type equation of the form $q_t = G(q)$, where $G(q)$ is a nonlinear function of q, separable in the form $G_0(q)+\varepsilon G_1(q)$ where $G_0(q)$ is suitable for an exact

IST treatment and the effect of $\varepsilon G_1(q)$ is obtained by perturbation. In order to achieve this we eliminate the other dynamical variables from equations (1) and obtain:

$$E_t = -E_{xxt}X^{-1}+E_{xt}E_xE^{-1}X^{-1}+\mu^2E_xE^{-1}X^{-1}\int_0^x E_t dx$$

$$+\frac{1}{T}\left[E_tE_xE^{-1}X^{-1}-E_{xt}X^{-1}+\mu^2X^{-1}\int_0^x E_t dx-\mu^2xE_xE^{-1}X^{-1}\int_0^x E_t dx\right.$$

$$\left.+\mu^2E_xE^{-1}X^{-1}\int_0^x E_t x dx\right], \tag{2}$$

where

$$X = [E^2+\mu^2]$$

which is a nonlinear equation for the amplitude of the electric field E.

It is interesting that as $T \to \infty$ (we have assumed for simplicity that $T_1 = T_2$) the term in the (square parentheses) vanishes. So, it is quite pertinent to treat these as perturbation on the profile of the one soliton solution.

2. *Perturbation method*

The essential tools of the perturbation procedure have been already described in detail by Kaup [3]. The most important equations are those governing the time evolution of the soliton parameters written as:

$$i\zeta_{0,t} = I[\psi, \bar{\psi}; \zeta_0],$$

$$iC_{0,t} = \frac{a_0''}{(a_0')^3}I[\phi, \phi; \zeta_0]-\frac{1}{(a_0')^2}J[\phi(\zeta_0)], \tag{3}$$

where the symbols I, J etc. are defined as:

$$I[U, V; \zeta] \equiv \int_{-\infty}^{\infty}[iq_tU_2(\zeta, x)V_2(\zeta, x)-ir_tU_1(\zeta, x)V_1(\zeta, x)]dx,$$

$$J[U] \equiv \int_{-\infty}^{\infty}[iq_tU_2-ir_tU_1]dx. \tag{4}$$

To start with we chose the exact one soliton solution of equation (1) to be:

$$E(x, t) = 4iC_0\zeta_1 \operatorname{sech}\tfrac{1}{2}\left(E_{1x}-\frac{4E_1}{E_1^2+4\mu^2}t+\delta_1\right), \tag{5}$$

where

$$E_1 = 4i\zeta_1.$$

(U_1, U_2); (V_1, V_2) are the eigen functions of the associated inverse scattering eigen value problem formulated by Ablowitz, Kaup, Newell and Segar (AKNS). In equations (3) and (4) ζ stands for the eigenvalue of the associated inverse scattering equation. $(\phi, \bar{\phi})$ and $(\bar{\psi}, \psi)$ are two sets of Jost functions, each is a two component spinor defined through their asymptotic properties as $x \to \pm\infty$. In fact ζ_0 is connected to the velocity of the soliton and C_0 to the amplitude of it. r and q denotes the nonlinear fields for which the evolution equation is given. It should be noted that in general the eigenvalue ζ is complex and written as $\zeta = \xi + i\eta$. The influence of the spectrum has been collected into the a_0' and a_0'' terms, which are not independent terms and are completely determined by the eigenvalue spectrum. The evaluation of the right hand side of equation (4) involves on enormous number of highly transcendental integrals which when handled properly yields up to the first order in $1/T$ the following complicated sets of equations for the time dependence of the parameter of the one soliton solution of SIT. Therefore, our equations for the change of parameter become

$$\eta_{1t} = 0, \qquad \xi_{1t} = \frac{2\mu^2}{T\xi_1},$$

$$\bar{C}_{ORt} = \frac{1}{T} X(\eta_1)\xi_1, \qquad \bar{C}_{OIt} = \frac{1}{T} Y(\eta_1),$$

where

$$X(\eta_1) = \left[\frac{L\mu}{4\eta_1} - \frac{L\mu}{16\eta_1^2} - \frac{\pi^3\mu}{128\eta_1^2}\right],$$

$$Y(\eta_1) = \left[\frac{\pi^3\mu^2}{32\eta_1^2} + \frac{\pi^3\mu}{128\eta_1} + \frac{L\mu}{4}\right],$$

with a similar equation for $\frac{d\varrho}{dt}$.

3. *Conclusions*

From our above computation it is quite clear that in spite of the pressure of the perturbing term the imaginary part of the eigen value ζ does not change with time, while the real part varies as $\left(\frac{4\mu}{T}t + \xi_{10}\right)^{1/2}$. On the other hand, both C_{OR} and C_{OI} have the dependence given by

$$C_{OR} = \frac{1}{\mu^2} X(\eta_{10}) \left[\frac{4\mu^2}{T}t + \xi_{10}\right]^{3/2} + \bar{C}_{OR},$$

$$C_{OI} = \frac{1}{T} Y(\eta_{10})t + \bar{C}_{OI}.$$

The resultant effect is to change the amplitude and velocity of the soliton given by

$$v = \left| \frac{1}{\mu^2 - 4\zeta_1^2} \right| \approx \left[\alpha - \frac{4\mu^2}{T} t \right]^{-1}.$$

Therefore, as t increases $v \to 0$ and the soliton is slowed down or the time is kept fixed and if one makes the relaxation time of the medium T finite but large, then the effect is also the same. Furthermore, the amplitude of the soliton is given by

$$A = |C_0 \zeta_1| \approx (a + bt)^{1/2} (\alpha + \beta t + \gamma t^2 + \delta t^3)^{1/2},$$

where

$$\alpha, a \approx O(T^0), \quad \beta, b \approx O(T^{-1}),$$

and

$$\gamma \approx O(T^{-2}), \quad \delta \to O(T^{-3}).$$

Therefore, for a large but finite relaxation time the amplitude has the time variation given by $A \approx (C + dt)$, that is, it increases linearly with time. Finally the net effect of the perturbation is to make the amplitude grow indefinitely but it brings the soliton to rest. The important point to observe is that both the amplitude and velocity of the soliton are homogeneous functions of the variable t/T so essentially the growth or decay, acceleration or deceleration really depends on the scales of time and the magnitude of T. Finally some comments are in order. Numerous experiments have confirmed that in the scattering of two solitary pulses, the effect of perturbation is not only to change the shape and velocity but it also generates a cloud, which may be the effect of the continuous spectrum. Our analytical procedure till now is unable to account for such effects as creation and annihilation of soliton.

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