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A SOLITON PERTURBATION SCHEME FOR 3×3 INVERSE SCATTERING TRANSFORM

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A perturbation method for the soliton solutions of nonlinear equations tractable using 3×3 matrix IST formalism is discussed in detail. The corresponding changes in conservation laws are also considered.

Perturbation technique for soliton-like solution of nonlinear equations amenable to the scheme of Zhakharov and Shabat [1] and Ablowitz et al. [2] have been discussed in detail by Kaup [3], Keener and McLaughlin [4] and many others. In recent times there have been exhaustive studies of the inverse scattering framework in the treatment of Gardner et al. [5] and in the more recent review of Bullough and Dodd [6]. But many of the equations of physical interest are not tractable by the 2×2 formalism indicated above. For example, the three wave interaction in a plasma, interaction of an ion-acoustic wave with a Langmuir soliton, etc., are all treated in a IST frame work where the matrix structure is 3×3. Furthermore, the self-modulation of the waves in a three wave interaction are also very important in the stability consideration of the system. Such an analysis has been attempted recently by Franklin et al. [7]. On the other hand, only an approximate version of Sonic–Langmuir soliton equations are invertable. So there is a necessity for a perturbation formalism for the 3×3 system. In this note we have considered such a frame work. As a prototype we have considered the equations of three wave interaction but the calculations are similar in any 3×3 system.

Consider the 3×3 eigenvalue equation for the three wave interaction:

$$(-iI\partial x+V)\psi = A^{-1}\xi\psi, \tag{1}$$

where A and V are given by:

$$A = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 0 & V_{12} & V_{13} \\ V_{21} & 0 & V_{23} \\ V_{31} & V_{32} & 0 \end{pmatrix}.$$

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We have closely followed the notations in [8]. In a perturbation scheme one is essentially interested in the variation of the parameters of the theory as the nonlinear field becomes changed due to the effect of the perturbing term. Suppose the equations under consideration are:

$$\frac{\partial a_1}{\partial t} + \vartheta_1 \frac{\partial a_1}{\partial x} = a_2 a_3 + \varepsilon f_1(a_1, a_2, a_3),$$

$$\frac{\partial a_2}{\partial t} + \vartheta_2 \frac{\partial a_2}{\partial x} = a_3 a_1 + \varepsilon f_2(a_1, a_2, a_3),$$

$$\frac{\partial a_3}{\partial t} + \vartheta_3 \frac{\partial a_3}{\partial x} = a_1 a_2 + \varepsilon f_3(a_1, a_2, a_3), \qquad (2$$

where εf_i are perturbing terms.

Consider any functional $F(a_1, a_2, a_3)$ depending on the nonlinear fields a_i. Then the total change of F is given by:

$$\frac{dF}{dt} = \left(\frac{\partial F}{\partial t}\right)_0 + \int dx \left[\frac{\partial F(x)}{\partial a_1}\frac{\partial a_1}{\partial t} + \frac{\partial F(x)}{\partial a_2}\frac{\partial a_2}{\partial t} + \frac{\partial F(x)}{\partial a_3}\frac{\partial a_3}{\partial t}\right]$$

$$= \left(\frac{\partial F}{\partial t}\right)_0 + \varepsilon \int dx \left[\frac{\partial F(x)}{\partial a_1} f_1(x) + \frac{\partial F(x)}{\partial a_2} f_2(x) + \frac{\partial F(x)}{\partial a_3} f_3(x)\right], \qquad (3)$$

where $(\partial F/\partial t)_0$ are the time variations for the exactly solvable system. The most important quantities whose variations are essential are the eigenvalue ξ and the transmission reflection coefficients a_{ij}, defined through the linear relation between the Jost solutions ϕ_n^j and ψ_n^j:

$$\phi_n^j = \sum a_{jk} \psi_n^k. \qquad (4)$$

For the case under consideration we assume the matrix V to be anti-symmetric $V_{ij} = -V_{ji}$ and then one finds from the integral equations of the Jost functions that:

$$\frac{\delta a_{ij}(x)}{\delta V_{kl}(z)} = W^{-1}\left[ie^{i\xi(x-z)/\alpha k} \phi_l^i \varepsilon_{jab} \varepsilon_{k\lambda\mu} \psi_\lambda^a \psi_\mu^b + ie^{i\xi(x-z)/\alpha_l} \phi_k^i \varepsilon_{l\lambda\mu} \varepsilon_{jab} \psi_\lambda^a \psi_\mu^b\right]. \qquad (5)$$

Now the variation can be computed once the Jost functions are known. For this let us consider the case of one soliton solution.

Such a solution corresponds to a simple zero of $a_{11}(\xi)$ and $a_{33}(\xi)$ in their half planes. Let $\xi_1(\xi_3)$ to be simple zero of each $a_{11}(a_{33})$ in the lower (upper) half-plane. Then from equation (3.7a) and (3.7b) of [3] we have:

$$\bar{F}(x) = i\beta_{12} \bar{c} e^{-i\beta_{12}\xi_1 x},$$

$$F(x) = -i\beta_{23} c e^{+i\beta_{23}\xi_3 x}, \qquad (6)$$

where $\bar{F}$ and F are the kernel functions of the Marchenko equations (vide equation (3.6a) and (3.6b) of [6]). These equations when solved yield $K^{(1)}(x, y)$ and $K^{(3)}(x, y)$ which when used in the integral equations

$$\psi^{(1)}(\xi, x)e^{-i\xi x/\alpha_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \int_x^\infty K^{(1)}(x, s)e^{i\xi(s-x)\beta_{12}}ds,$$

$$\psi^{(3)}(\xi, x)e^{-i\xi x/\alpha_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \int_x^\infty K^{(3)}(x, s)e^{i\xi(s-x)\beta_{23}}ds, \tag{7}$$

yields:

$$K^{(1)}(x, y) = e^{-i\beta_{12}\bar{\xi}_1 y}f(x), \qquad K^{(3)}(x, y) = e^{+i\beta_{23}\xi_3 y}g(x), \tag{8}$$

$$f(x) = \frac{1}{M_2L_1 - L_2M_1} \cdot \begin{pmatrix} M_2C_1 - M_1C_1' \\ M_2C_2 - M_1C_2' \\ M_2C_3 - M_1C_3' \end{pmatrix}, \qquad g(x) = \frac{1}{M_1L_2 - M_2L_1} \cdot \begin{pmatrix} L_2C_1 - L_1C_1' \\ L_2C_2 - L_1C_2' \\ L_2C_3 - L_1C_3' \end{pmatrix},$$

with:

$$L_1 = \frac{\gamma_1\gamma_2\bar{C}^*\bar{C}}{2i\eta_1(\bar{\xi}_1^* - \bar{\xi}_1)} \cdot e^{i\beta_{12}(\bar{\xi}_1^* - \bar{\xi}_1)x},$$

$$L_2 = \frac{\gamma_1\gamma_2\beta_{23}\bar{C}^*C}{\beta_{12}(\bar{\xi}_1^* - \bar{\xi}_1)} \cdot \frac{e^{i\beta_{12}(\bar{\xi}_1^* - \bar{\xi}_1)x}}{(\xi_3 - \bar{\xi}_1^*)},$$

$$M_1 = \frac{\varepsilon_1\varepsilon_3\gamma_1\gamma_2\bar{C}C^*}{(\xi_3^* - \bar{\xi}_1)(\xi_3 - \xi_3^*)\beta_{23}} \cdot e^{i\beta_{23}(\xi_3 - \xi_3^*)x},$$

$$M_2 = 1 + \frac{\gamma_2\gamma_3C^*C}{2i\eta_3(\xi_3 - \xi_3^*)} \cdot e^{i\beta_{23}(\xi_3 - \xi_3^*)x},$$

$$C_1 = \frac{\gamma_1\gamma_2\beta_{12}\bar{C}^*\bar{C}}{2\eta_1} \cdot e^{i\beta_{12}\bar{\xi}_1^* x},$$

$$C_1' = \frac{i\gamma_1\gamma_2\beta_{23}\bar{C}^*C}{\xi_3 - \bar{\xi}_1^*} \cdot e^{i\beta_{12}\bar{\xi}_1^* x},$$

$$C_2 = -i\beta_{12}\bar{C}; \qquad C_2' = +i\beta_{23}C,$$

$$C_3 = \frac{i\varepsilon_1\varepsilon_3\gamma_1\gamma_2\bar{C}C^*}{(\xi_3^* - \bar{\xi}_1)} \cdot e^{-i\beta_{23}\xi_3^* x},$$

$$C_3' = \frac{\gamma_2\gamma_3\beta_{23}C^*C}{2\eta_3} \cdot e^{-i\beta_{23}\xi_3^* x}. \tag{9}$$

The choice of $\varepsilon_1, \varepsilon_2, \varepsilon_3$ depends on the particular sets of $\gamma_1, \gamma_2, \gamma_3$ which in turn depend upon the physical situation under consideration. Finally:

$$e^{-i\xi x/\alpha_1}\psi^{(1)}(\xi, x) = -\frac{e^{-i\beta_{12}\bar{\xi}_1 x}}{i\beta_{12}(\xi-\bar{\xi}_1)}\cdot f(x)$$

$$e^{-i\xi x/\alpha_3}\psi^{(3)}(\xi, x) = -\frac{e^{i\beta_{23}\xi_3 x}}{i\beta_{23}(\xi_3-\xi)}\cdot g(x).$$

$$e^{-i\xi x/\alpha_2}\psi^{(2)}(\xi, x) = \begin{bmatrix} -\dfrac{\varepsilon_1\varepsilon_3\bar{C}^*}{\xi_1^*-\xi}\, e^{i\xi_1^* x\beta_{12}} \\ 0 \\ \dfrac{\varepsilon_2\varepsilon_3 C^*}{\xi_3^*-\xi}\, e^{i\xi_3^* x\beta_{32}} \end{bmatrix}$$

$$+\frac{\varepsilon_1\varepsilon_2\bar{C}^* f(x)}{i\beta_{12}(\bar{\xi}_1^*-\xi)(\bar{\xi}_1^*-\bar{\xi}_1)}\cdot e^{i\beta_{12}(\bar{\xi}_1+\bar{\xi}_1^*)x}-\frac{\varepsilon_2\varepsilon_3 C^* g(x)}{i\beta_{23}(\xi_3^*-\xi)(\xi_3-\xi_3^*)}\cdot e^{i\beta_{23}(\xi_3+\xi_3^*)x}. \tag{10}$$

Next we consider the variation of the eigenvalue ξ for the corresponding change in the nonlinear field. For that consider the eigenvalue equation for the first kind of Jost function which is written in component form. When the nonlinear field ($p = V_{12} = -V_{21}$) has changed to $p+\delta p\cdot\delta(x-z)$

$$-i\partial x\psi_1^{(1)}+p\psi_2^{(1)}+\delta p\cdot\delta(x-z)\psi_2^{(1)}+V_{13}\psi_3^{(1)} = \frac{\xi}{\alpha_1}\psi_1^{(1)}$$

$$-i\partial x\psi_2^{(1)}-p\psi_1^{(1)}-\delta p\cdot\delta(x-z)\psi_1^{(1)}+V_{23}\psi_3^{(1)} = \frac{\xi}{\alpha_2}\psi_2^{(1)}$$

$$-i\partial x\psi_3^{(1)}-V_{31}\psi_1^{(1)}+V_{32}\psi_2^{(1)} = \frac{\xi}{\alpha_3}\psi_3^{(1)}. \tag{11}$$

Now except for the narrow zone $x = z$ these equations are the same as the original [1], so that we may set

$$\psi_1^{(1)} = \alpha\psi_1^{(1)} \quad \text{for} \quad x \leqslant z,$$

$$= \beta\phi_1^{(2)} \quad \text{for} \quad x > z.$$

Integrating (11) from ($z-\varepsilon$ to $z+\varepsilon$) and taking the limit $\varepsilon \to 0$ and using (12) we obtain:

$$(\phi_2^{(2)}\psi_3^{(1)}-\psi_2^{(1)}\phi_3^{(2)}) = \delta p\cdot\psi_1^{(1)}\phi_3^{(2)},$$

$$(\phi_1^{(2)}\psi_3^{(1)}-\psi_1^{(1)}\phi_3^{(2)}) = -\delta p\cdot\psi_2^{(1)}\phi_3^{(2)}.$$

Now let us recall the definition of $a_{11}(\xi+\delta\xi)$ which is

$$a_{11}(\xi+\delta\xi) = W^{-1}\begin{vmatrix} \psi_1^1 & \phi_1^2 & \phi_1^3 \\ \psi_2^1 & \phi_2^2 & \phi_2^3 \\ \psi_3^1 & \phi_3^2 & \phi_3^3 \end{vmatrix} = \dot{a}(\xi)\delta\xi \tag{14}$$

which yields in conjunction with (13):

$$\frac{\delta\xi}{\delta p} = \frac{1}{\dot{a}(\xi)}[\phi_2^3\psi_2^1-\phi_1^3\psi_1^1]W^{-1}, \tag{15}$$

where W is the Wronskian defined in [3].

When (15) is supplemented with the expressions for Jost functions (10), one can obtain $\delta\xi/\delta p$ explicitly. This is similar to the case of $\delta\xi/\delta q$ and $\delta\xi/\delta r$. Since the eigenvalue has a variation one should also expect that the infinite number of conservation laws of the 3×3 system will have a variation. For this we first consider the derivation of the conservation laws. Let us consider the eigenvalue equation for

$$\phi^{(1)} = Fe^{i\xi x/\alpha_1} \tag{16}$$

which can be written as:

$$\begin{aligned} F_{1x} &= -iV_{12}F_2-iV_{13}F_3, \\ F_{2x}+(i\xi\alpha_1^{-1}-i\xi\alpha_2^{-1})F_2 &= -iV_{21}F_1-iV_{23}F_3, \\ F_{3x}+(i\xi\alpha_1^{-1}-i\xi\alpha_3^{-1})F_3 &= -iV_{31}F_1-iV_{32}F_2. \end{aligned} \tag{17}$$

Substituting

$$F_1 = 1+\sum_1^\infty A_n\xi^{-n}, \quad F_2 = \sum_1^\infty B_n\xi^{-n}, \quad F_3 = \sum_1^\infty C_n\xi^{-n}$$

and equating different powers of ξ we get:

$$B_1 = \frac{V_{21}}{\beta_{21}}, \quad C_1 = \frac{V_{31}}{\beta_{31}},$$

$$\frac{dA_n}{dx} = -iV_{12}B_n-iV_{13}C_n,$$

$$\frac{dB_n}{dx} = -i\beta_{12}B_{n+1}-iV_{21}A_n-iV_{23}C_n,$$

$$\frac{dC_n}{dx} = -i\beta_{13}C_{n+1}-iV_{31}A_n-iV_{32}B_n.$$

Equation (18) can be solved iteratively for A_i's. Let us quote the first two:

$$A_1 = -i \int_{-\infty}^{x} \left[\frac{V_{12}V_{21}}{\beta_{21}} + \frac{V_{13}V_{31}}{\beta_{31}} \right] dx$$

$$A_2 = -\tfrac{1}{2}\left[\int_{-\infty}^{x} \left(\frac{V_{12}V_{21}}{\beta_{21}} + \frac{V_{13}V_{31}}{\beta_{31}} \right) dx \right]^2$$

$$- \int_{-\infty}^{x} \left[\frac{i}{\beta_{21}\beta_{31}} (V_{12}V_{23}V_{31} + V_{21}V_{32}V_{13}) + \frac{(V_{21})_x V_{12}}{(\beta_{21})^2} + \frac{(V_{31})_x V_{13}}{(\beta_{31})^2} \right] dx.$$

Now since in the exact situation $\dot{a}_{11}(\xi) = 0$ so that lim A_i are the conserved quantities and we first two integrals of motion:

$$I_1 = \int_{-\infty}^{\infty} \left(\frac{V_{12}V_{21}}{\beta_{21}} + \frac{V_{13}V_{31}}{\beta_{31}} \right) dx,$$

$$I_2 = \int_{-\infty}^{\infty} \left(\frac{i}{\beta_{21}\beta_{31}} (V_{12}V_{23}V_{31} + V_{21}V_{32}V_{13}) + \frac{(V_{21})_x V_{12}}{(\beta_{21})^2} + \frac{(V_{31})_x V_{13}}{(\beta_{31})^2} \right) dx.$$

Now in the perturbed system $\dot{a}_{11}(\xi) \neq 0$ and its change can be easily calculated from the nature of the Jost functions, and so can the subsequent change in I_1, I_2, etc.

Our above analysis clearly indicates a step by step procedure for evaluating the changes in a 3×3 inverse scattering framework when the original nonlinear equation has some correction term added.

The physical situations pertaining to our above analysis occurs in several domains of theoretical physics. To mention a few, one can consider the self interactions of the usual three wave interaction, the exact equations of the Sonic–Langmuir solitons, and coupled nonlinear Schrödinger equations describing a more exact theory of nonlinear self-focussing in liquids [9].

REFERENCES

[1] V. Zhakharov, A. Shabat, *Sov. Phys. JETP* **34**, 62 (1972).

[2] M. Ablowitz, D. J. Kaup, A. Newell, H. Segur, *Phys. Rev. Lett.* **30** 1262 (1973); *Stud. App. Math.* **LIII**, 249 (1974).

[3] D. J. Kaup, *Stud. App. Math.* **31**, 121 (1976).

[4] J. P. Keener, D. W. McLaughlin, *Phys. Rev.* **A16**, 777 (1977).

[5] C. S. Gardner, J. M. Greene, M. D. Kruskal, R. M. Miura, *Phys. Rev. Lett.* **19**, 1095 (1967).

[6] R. Bullough, R. Dodd, *Synergetics*, Ed. by H. Haken, Springer-Verlag 1977.

[7] R. N. Franklin et al., *Proc. R. Soc.* **A347**, 25 (1975).

[8] D. J. Kaup, *Stud. App. Math.* **55**, 9 (1976).

[9] S. V. Manakov, *Sov. Phys. JETP* **38**, 248 (1974).