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ASYMPTOTIC SOLUTION TO THE REFLECTED CRITICAL SLAB

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The principle of invariant embedding is used with the asymptotic method to solve the critical slab problem. The formalism is carried out for both finite and infinite reflected slabs with general anisotropy. Calculations are only made for an infinite reflector slab with linearly anisotropic scattering.

1. Introduction

In a recent note by Siewert et al. [1–3], the principle of invariant embedding, as developed by Chandrasekhar [4], is used effectively to analyse the critical reactor problem. In their method they used the elementary solution method [5] or the P-L method for solving the one-speed neutron transport equation in the core.

In this work we use a technique blending the treatment of the Chandrasekhar function with the construction of an asymptotic expansion solution of the core equation with respect to a small parameter. The leading term of such an asymptotic solution gives results for the criticality dimensions of the reflected slab. The formalism is developed for a finite reflector which tends to infinity; calculations are made for an infinite reflector slab.

To be specific, consider a slab of multiplying media which extends from $x = -a$ to $x = a$ and is characterized by the mean number of secondaries per collision $c_1 > 1$. This uniform slab is surrounded by uniform finite reflectors of another material characterized by $c_2 < 1$ and extended to b and $-b$.

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For anisotropic scattering of mono-energetic neutrons in a sourceless medium, the neutron angular flux $\psi^\alpha(x, \mu)$ satisfies the equation

$$\left(\mu \frac{\partial}{\partial x} + 1\right) \psi^\alpha(x, \mu) = \frac{c_\alpha}{4\pi} \sum_{n=0}^{\infty} (2n+1) f_n^\alpha P_n(\mu) \int_{-1}^{1} d\mu' P_n(\mu') \psi_n^\alpha(x, \mu'), \quad \alpha = 1, 2, \tag{1}$$

where the distance is measured in terms of mean free paths.

We seek the value of c_1 when c_2 is given for which there exists a real non-negative solution of Eq. (1), subject to the boundary conditions:

$$\psi^\alpha(x, \mu) = \psi^\alpha(-x, -\mu), \tag{2}$$

$$\psi^2(b+z_0, \mu) = 0, \tag{3}$$

where z_0 is the extrapolated distance and

$$\psi'(a, \mu) = \psi^2(a, \mu). \tag{4}$$

2. *Analysis*

For the reflector $c_2 < 1$ we can use the Chandrasekhar results [4] for the reflected and trasmitted distributions:

$$\psi^2(a, -\mu; \tau) = \frac{1}{2\mu} \int_0^1 S(\tau; \mu', \mu) \psi^2(a, \mu'; \tau) d\mu', \quad \mu > 0, \tag{5}$$

and

$$\psi^2(\tau, \mu; \tau) = \psi^2(a, \mu; \tau) e^{-\tau/\mu} + \frac{1}{2\mu} \int_0^1 T(\tau; \mu', \mu) \psi^2(a, \mu'; \tau) d\mu', \quad \mu > 0, \tag{6}$$

where

$$S(\tau; \mu', \mu) = \frac{\mu\mu'}{\mu+\mu'} \sum_{l=0}^{N} (-1)^l c_2 f_l^2 [\psi_l(\tau; \mu') \psi_l(\tau; \mu) - \varphi_l(\tau; \mu') \varphi_l(\tau; \mu)] \tag{7}$$

and

$$T(\tau; \mu', \mu) = \frac{\mu\mu'}{\mu-\mu'} \sum_{l=0}^{N} c_2 f_l^2 [\varphi_l(\tau; \mu) \psi_l(\tau; \mu') - \psi_l(\tau; \mu) \varphi_l(\tau; \mu')], \tag{8}$$

with

$$\psi_l(\tau; \mu) = P_l(\mu) + \frac{\mu}{2} \sum_{k=0}^{N} (-1)^{l+k} c_2 f_k^2 \int_0^1 \frac{d\mu' P_l(\mu')}{\mu+\mu'} [\psi_k(\tau; \mu) \psi_k(\tau; \mu') - \varphi_k(\tau; \mu) \varphi_k(\tau; \mu')] \tag{9}$$

and

$$\varphi_l(\tau;\mu) = e^{-\tau/\mu}P_l(\mu) + \frac{\mu}{2}\sum_{k=0}^{N} c_2 f_k^2 \int_0^1 \frac{d\mu' P_l(\mu')}{\mu-\mu'}[\varphi_k(\tau;\mu)\psi_k(\tau;\mu') - \psi_k(\tau;\mu)\varphi_k(\tau;\mu')]. \tag{10}$$

In the case of an infinite reflector, Eqs. (5)–(10) tend to the equations for half space, which are conformally obtained by limiting τ to infinity [6] and writing $\lim S(\tau;\mu,\mu') = S(\mu;\mu')$, $\lim T(\tau;\mu,\mu') = 0$, $\lim \varphi_l(\tau;\mu) = 0$, and

$$\lim \psi_l(\tau;\mu) = \psi_l(\mu) = P_l(\mu) + \frac{\mu}{2}\sum_{k=0}^{N}(-1)^{l+k}c_2 f_k^2 \int_0^1 \frac{d\mu' P_k(\mu')}{\mu+\mu'}\psi_k(\mu)\psi_l(\mu'), \tag{11}$$

and

$$\psi^2(a,-\mu) = \frac{1}{2\mu}\int_0^1 S(\mu;\mu')\psi^2(a,\mu')d\mu', \qquad \mu\in(0,1). \tag{12}$$

For region 1 a rigorous solution is reported in Ref. [5], but here the asymptotic method is used. This method is developed by Larsen and Keller [7] and by Boffi et al. [8]. The latter is a direct method of calculation. However, their asymptotic series show oscillatory behaviour plus the complexity of the inhomogeneous term of the diffusion equation obtained which increases with higher-order corrections. In order to improve the Boffi method we adopt the method of constrained co-ordinates [9]. By this method the coefficients of the asymptotic expansion are chosen such that the secular terms of the inhomogeneous equations are omitted. Les us introduce the dimensionless parameter ε and dimensionless spatial variable defined by

$$\varepsilon = \frac{1}{2a}, \qquad 0 \leqslant \varepsilon \leqslant 1, \tag{13}$$

$$x = \frac{x}{2a}, \tag{14}$$

into Eq. (1) to get the scaled transport equation in the core as follows:

$$\varepsilon\mu\frac{\partial}{\partial x}F^1(x,\mu) + F^1(x,\mu) = \frac{c_1}{4\pi}\sum_{n=0}^{N}(2n+1)f_n^1 P_n(\mu)\varphi_n^1(x)$$

$$(-\tfrac{1}{2} \leqslant x \leqslant \tfrac{1}{2},\ -1 \leqslant \mu \leqslant 1), \tag{15}$$

which is to be integrated using condition (2). By substituting

$$F^1(x,\mu) = \sum_{m=0}^{\infty}\varepsilon^m F_m^1(x,\mu), \quad c_1 = \sum_{m=0}^{\infty}\varepsilon^m d_m, \quad \varphi_n^1(x) = \sum_{m=0}^{\infty}\varepsilon^m \varphi_{nm}^1(x) \tag{16}$$

in Eq. (15), then the resulting equation can be formally solved for $F_m^1(x, \mu)$ to give

$$F_m^1(x, \mu) = \frac{1}{4\pi} \sum_{l=0}^{m} (-1)^l A_{ml}(x, \mu)\mu^l, \tag{17}$$

where

$$A_{ml}(x, \mu) = \sum_{j=0}^{m-l} d_j \sum_{n=0}^{N} (2n+1) f_n^1 P_n(\mu) \frac{d^l \varphi^1_{n,m-l-j}}{dx^l}. \tag{18}$$

From which one can find that

$$d_0 = 1, \quad \varphi^1_{00}(x) = 0, \quad \varphi^1_{10}(x) = \varphi^1_{20}(x) = \ldots = \varphi^1_{N0}(x) = 0$$

and

$$d_1 = 0, \quad \varphi^1_{01}(x), \quad \varphi^1_{11}(x) \neq 0, \quad \varphi^1_{21}(x) = \ldots = \varphi^1_{N1}(x) = 0, \tag{19}$$

while for $m = 2$, φ^1_{0k} $(k = 0, 1, \ldots, m)$ one gets

$$\frac{d^2 \varphi^1_{0k}(x)}{dx^2} + 3d_2(1-f_1^1)\varphi^1_{0k}(x) = G_k(x), \tag{20}$$

where $G_k(x)$ depends on $d_2, d_3, \ldots, d_{k+2}$ and $\varphi^1_{00}(x)$, $\varphi^1_{01}(x)$, ..., $\varphi^1_{0,k-1}(x)$ with

$$G_0(x) = 0, \quad G_1(x) = -3d_3(1-f_1^1)\varphi^1_{00}(x). \tag{21}$$

The general solution of Eq. (20) is

$$\varphi^1_{0k}(x) = A_k \cos \gamma x + G_k^*(x; \beta), \tag{22}$$

with $\gamma = 3d_2(1-f_1^1)$ and $G_k^*(x; \beta)$ denoting the particular (even) solution to (20).

In order to find d_k, appearing in Eq. (22), we apply the continuity condition (4) to Eqs. (5) and (6) for a finite reflector and to Eq. (12) for an infinite one. The result for an infinite reflector with dimensionless spatial variables is

$$F_m^1(\tfrac{1}{2}, -\mu) = \frac{1}{2\mu} \int_0^1 S(\mu; \mu') F^1(\tfrac{1}{2}, \mu') d\mu', \tag{23}$$

or, by expansions (16),

$$F_m^1(\tfrac{1}{2}, -\mu) = \frac{1}{2\mu} \int_0^1 S(\mu; \mu') F_m^1(\tfrac{1}{2}, \mu') d\mu'. \tag{24}$$

For $m = k = 0$, Eq. (22) together with Eq. (24) form an eigenvalue problem to determine d_2, i.e. from Eqs. (22) and (24) one has

$$\cos \frac{\gamma}{2} = \frac{1}{2\mu} \int_0^1 S(\mu; \mu') \cos \frac{\gamma}{2} d\mu', \tag{25}$$

which gives $\gamma^2 = \pi^2$ or

$$d_2 = \frac{\pi^2}{3(1-f_1^1)} \tag{26}$$

since

$$1-\frac{1}{2\mu}\int_0^1 S(\mu;\mu')d\mu' \neq 0. \tag{27}$$

For $m = 1$, the general solution (22) is

$$\varphi_{01}^1(x) = A_1 \cos \pi x - \frac{3d_3}{2\pi}(1-f_1^1)x \sin \pi x. \tag{28}$$

Then using Eq. (28) in Eqs. (18), (17) and (24) and multiplying the resulting equation by μ and integrating over the interval $\mu \in (0, 1)$, one gets

$$d_3 = \frac{-8\pi^2}{9[1-f_1^1]^2}\,\frac{B}{D}, \tag{29}$$

where

$$B = 1+\tfrac{3}{2}\int_0^1\int_0^1 \mu' S(\mu';\mu)d\mu d\mu'$$

and

$$D = 1-\int_0^1\int_0^1 S(\mu';\mu)d\mu d\mu'. \tag{30}$$

With these values of B and D, d_3 can be determined. The higher-order coefficient d_m, $m \geqslant 4$, can also be evaluated. The values of d_2 and d_3 for $B/D = 1$, which is the case of the non--reflected slab, are the same as given by Boffi [8].

3. *Numerical applications*

To assess the method, a numerical application is made for the linearly anisotropic critical case and for an approximation of order $L \geqslant 0$ in which

$$F^1(x,\mu) = \sum_{m=0}^{L} \varepsilon^m F_m^1(x,\mu)$$

and

$$c_1^L = \sum_{m=0}^{L+2} \varepsilon^m d_m. \tag{31}$$

We restrict ourselves to the approximation $L = 1$, in which

$$c_1^1 = 1+\varepsilon^2 d_2+\varepsilon^3 d_3, \tag{32}$$

where d_2 and d_3 are given by Eqs. (25) and (28).

The values of c_1^1 are listed in Tables I, II and III for a given c_2 as a function of the half thickness a and different anisotropy. The values of $\psi_0(\mu)$ and $\psi_1(\mu)$ which appeared in $S(\mu, \mu')$ are equivalent to the tabulated Chandrasekhar function $\psi(\mu)$ and $\varphi(\mu)$, respectively. In the case when $c_2 = 0$, the ratio $B/D = 1$ and the Boffi [8] result for the slab is obtained.

TABLE I

Values of critical c_1 for $f_1^1 = f_1^2 = 0$

c_2 \ a	5	10	20	40	100
0.2	1.023932	1.007104	1.001916	1.000497	1.000081
0.5	1.023452	1.007044	1.0019089	1.000496	1.000081
0.9	1.020564	1.006683	1.001863	1.00049	1.000081

TABLE II

Values of critical c_1 for $f_1^1 = 0$ and $f_1^2 = 0.333$

c_2 \ a	5	10	20	40	100
0.2	1.028327	1.007653	1.001985	1.000505	1.000082
0.5	1.022128	1.006878	1.001888	1.000493	1.000081
0.9	1.013148	1.005631	1.001732	1.000474	1.000081

TABLE III

Values of critical c_1 for $f_1^1 = f_1^2 = 0.333$

c_2 \ a	5	10	20	40	100
0.2	1.039048	1.011046	1.002922	1.000751	1.000122
0.5	1.025113	1.009305	1.002704	1.000723	1.000120
0.9	1.00268	1.0065	1.002354	1.00068	1.000117

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