ON THE BOUND STATES FOR HIGHER ANGULAR MOMENTA

By B. G. Sidharth*

International Centre for Theoretical Physics, Trieste, Italy

and Ahmed Abdel-Hafez**

Advanced School of Physics, University of Trieste, Italy

(Received June 23, 1978, revised version received February 26, 1979)

Using the fact that new bound states appear whenever the scattering length becomes infinite, we deduce a number λ_1 such that the potential $\lambda U(r)$ will certainly have bound states if λ is greater than λ_1, for higher angular momenta.

1. Introduction

If $\delta_l(K; \lambda)$ is the l-th order phase shift of the radial Schrödinger equation for the potential $\lambda U(r)$ and energy K^2, it is known from the theory of low-energy scattering that when the scattering length a_l, defined by

$$a_l(\lambda) = -\operatorname*{Lt}_{K\to 0} \frac{1}{K^{2l+1} \cot \delta_l(K; \lambda)} \tag{1}$$

becomes infinite for $\lambda = \lambda_0$, then, for $l > 0$, we have a zero-energy bound state and the potential $\lambda U(r)$ is just strong enough to produce a new bound state for $\lambda = \lambda_0$ [1, 2].

This fact has been used to estimate numerically the minimum strength λ_0 of a potential to produce bound states, at least for certain potentials [3].

2. Problem

We shall now obtain a value λ_1 such that the potential $\lambda U(r)$ will certainly have bound states when $\lambda \geqslant \lambda_1$. This inequality, as we shall see, supplements another inequality, which follows from Bargmann's inequality, viz. there are no bound states if $\lambda < \lambda_2$; λ_2 to be defined.

---

* Permanent address: Department of Mathematics, St. Xavier's College, 30, Park Street, Calcutta — 700016, India.

** Permanent address: Department of Mathematics, University of Assiut, Assiut, Egypt.

In the sequel we shall assume that $U(r) \leqslant 0$ only for all $r > 0$ and that $\lambda > 0$ (the potential is attractive everywhere). We shall also assume that $r^{2l+3}\ U(r) \to 0$ as $r \to \infty$, otherwise the scattering length defined by (1) does not exist.

We shall start with an inequality which is valid when $0 < |\delta_l| < \pi/2$ [4] (cf. Appendix for the proof):

$$\tan(\delta_l) \geqslant \frac{K\lambda B_l}{1+\lambda(C_l/B_l)}, \tag{2}$$

where

$$B_l = \int_0^\infty [rj_l(Kr)]^2 |U(r)| dr, \tag{3}$$

$$C_l = \int_0^\infty rj_l(Kr)\,|U(r)|\,\{Krn_l(Kr)\int_0^r [r'j_l(Kr')]^2 |U(r')| dr'$$

$$+Krj_l(Kr)\int_r^\infty r'^2 j_l(Kr')n_l(Kr')\,|U(r')| dr'\} dr. \tag{4}$$

Next we use the well-known relations

$$j_l(\varrho) = \frac{2^l(l!)}{(2l+1)!}\,\varrho^l + O(\varrho^{l+1}),$$

$$n_l(\varrho) = -\frac{(2l)!}{2^l(l!)}\,\varrho^{-(l+1)} + O(\varrho^{-l}). \tag{5}$$

If we substitute (5) in (3) and (4), we get

$$\operatorname*{Lt}_{K\to 0} \frac{B_l}{K^{2l}} = \left[\frac{2^l(l!)}{(2l+1)!}\right]^2 \int_0^\infty r^{2(l+1)} |U(r)| dr \equiv \left[\frac{2^l(l!)}{(2l+1)!}\right]^2 \bar{B}_l \tag{6}$$

and

$$\operatorname*{Lt}_{K\to 0} \frac{C_l}{K^{2l}} = -\left[\frac{2^l(l!)}{(2l+1)!}\right]^2 \cdot \frac{1}{(2l+1)} \left\{\int_0^\infty r^{l+1} |U(r)| \left[r^{-l} \int_0^r r'^{2(l+1)} |U(r')| dr'\right.\right.$$

$$\left.\left. + r^{l+1} \int_r^\infty r' |U(r')| dr'\right] dr\right\} \equiv \left[\frac{2^l(l!)}{(2l+1)!}\right]^2 \bar{C}_l. \tag{7}$$

Substituting (6) and (7) in (2) and remembering that for an everywhere attractive potential, $\delta_l > 0$, [5], so that $\tan|\delta_l| \equiv \tan\delta_l$, we get

$$-a_l(\lambda) \equiv \operatorname*{Lt}_{K\to 0} \frac{\tan|\delta_l|}{K^{2l+1}} \geqslant \frac{\left[\frac{2^l(l!)}{(2l+1)!}\right]^2 \lambda \bar{B}_l}{1+\lambda(\bar{C}_l/\bar{B}_l)}. \tag{8}$$

Now (8) shows that $|a_l(\lambda)| = \infty$ when

$$\lambda = \lambda_1 \equiv -\bar{B}_l/\bar{C}_l, \tag{9}$$

unless $\mathop{\mathrm{Lt}}_{K\to 0} \delta_l(K; \lambda)$ is already $> \pi/2$ for $\lambda < \lambda_1$, in which case (2) would no longer be valid.

In either case it follows that for a value λ_0 of λ, such that $\lambda_0 \leqslant \lambda_1$, $|a_l(\lambda_0)| = \infty$ so that bound states exist for $\lambda \geqslant \lambda_0$ and so bound states certainly exist for $\lambda \geqslant \lambda_1$.

Let us check this conclusion in the case of an attractive square well,

$$\begin{aligned} U(r) &= -V = \text{const} \quad &&\text{for } r < a, \\ U(r) &= 0 &&\text{for } r > a. \end{aligned} \tag{10}$$

Substitution of (10) in (9) after the definitions (6) and (7) of $\bar{B}_l$ and $\bar{C}_l$ are used, yields, after simple integration,

$$\lambda_1 = \frac{(2l+1)(2l+5)}{2Va^2},$$

i.e. certainly there are bound states if $\lambda Va^2 \geqslant \frac{1}{2}(2l+1)(2l+5) = 10.5$ and $l = 1$. On the other hand it is known from theory [6] that there are bound states if $\lambda Va^2 > \pi^2 \approx 9.86$.

From Bargmann's inequality, viz. [2],

$$n_l \leqslant \frac{1}{(2l+1)} \int_0^\infty r\lambda |U(r)| dr,$$

where n_l is the number of bound states, it follows that if

$$\frac{1}{(2l+1)} \int_0^\infty r\lambda |U(r)| dr < 1,$$

i.e.

$$\lambda < \frac{(2l+1)}{\int_0^\infty r|U(r)|dr} \equiv \lambda_2, \tag{11}$$

then there are no bound states.

So the inequalities (11) and $\lambda \geqslant \lambda_1$, λ_1 given by (9), supplement each other.

3. *Conclusion*

The potential $U(r)$ can certainly produce bound states if

$$\lambda \geqslant - \frac{(2l+1) \int_0^\infty r^{2(l+1)} U(r) dr}{\int_0^\infty r^{l+1} U(r) \left[r^{-l} \int_0^r r'^{2(l+1)} U(r') dr' + r^{l+1} \int_r^\infty r' U(r') dr' \right] dr}.$$

The authors are grateful to Professor L. Fonda for valuable discussions. Thanks are also due to Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

APPENDIX (cf. Ref. [4])

We wish to deduce the inequality,

$$\tan(\delta_l) \geqslant (K\lambda B_l)/[1+\lambda(C_l/B_l)].$$

For this we shall need the following (cf. Ref. [5]):

$$u = rj_l(Kr)\cos\delta_l + \lambda Krn_l(Kr)\int_0^r j_l(Kr')U(r')u(r')r'dr'$$

$$+\lambda Krj_l(Kr)\int_r^\infty n_l(Kr')U(r')u(r')r'dr', \tag{A1}$$

where

$$\sin\delta_l = -K\lambda\int_0^\infty rj_l(Kr)U(r)udr, \tag{A2}$$

$$\frac{\partial\delta_l}{\partial\lambda} = -K\int_0^\infty U(r)u^2dr. \tag{A3}$$

(Eq. (A3) is erroneously given in the reference quoted.)

Substitution of (A1) in (A2) gives

$$\tan\delta_l = K\lambda B_l - K\lambda^2 C_l + O(\lambda^3), \tag{A4}$$

where B_l and C_l are as defined in (3) and (4).

Applying the Cauchy-Schwarz inequality to (A2) and using (A3) we get,

$$\sin^2\delta_l(\lambda) \leqslant \lambda^2 KB_l\frac{\partial}{\partial\lambda}[\delta_l(\lambda)],$$

so that

$$\frac{1}{\lambda^2} \leqslant (-KB_l)\frac{\partial}{\partial\lambda}[\cot\delta_l(\lambda)].$$

On integration we get

$$\frac{1}{\varepsilon} - \frac{1}{\lambda} \leqslant (-KB_l)[\cot\delta_l(\lambda) - \cot\delta_l(\varepsilon)],$$

whence

$$\frac{1}{\lambda} - KB_l\cot\delta_l(\lambda) \geqslant \frac{1}{\varepsilon} - KB_l\cot\delta_l(\varepsilon).$$

If we use (A4) on the right side of the last inequality and take the limit $\varepsilon \to 0$, we get

$$\frac{1}{\lambda} - KB_l \cot \delta_l(\lambda) \geqslant - \frac{C_l}{B_l}.$$

Remembering that for attractive potentials $\delta_l > 0$, if in addition $\delta_l < \pi/2$ this gives the required inequality.

(In the special case $l = 0$, the inequality (2) can be sharpened, but this is beyond the scope of the present paper.)

REFERENCES

[1] N. F. Mott, H. S. W. Massey, *The Theory of Atomic Collisions*, Clarendon Press, Oxford 1965, p. 45.
[2] R. G. Newton, *Scattering Theory of Waves and Particles*, McGraw Hill Book Co., New York 1966, pp. 310–313, p. 377.
[3] H. M. Schey, J. L. Schwartz, *Phys. Rev.* **139**, B1428 (1965).
[4] B. G. Sidharth, to appear in *Bull. Cal. Math. Soc.*
[5] P. Roman, *Advanced Quantum Theory*, Addison-Wesley Publishing Co., Inc., Massachusetts 1965, p. 173.
[6] L. I. Schiff, *Quantum Mechanics*, McGraw Hill Book Co., New York 1949, p. 80.