MAGNON RELAXATION DUE TO SPIN-ORBIT INTERACTION IN ITINERANT ELECTRON FERROMAGNETS. APPLICATION TO NICKEL*

By J. Morkowski and W. Jaworski

Ferromagnetics Laboratory, Institute of Molecular Physics, Polish Academy of Sciences, Poznań**

(Received November 23, 1978)

The system of itinerant electrons of a ferromagnet is described by a model of multiple narrow bands. The intraatomic intra- and interband Coulomb interaction, the intraatomic exchange interaction and the spin-orbit coupling of the itinerant electrons are taken into account. An effective hamiltonian of magnons is introduced as an expansion of products of magnon operators up to four-magnon terms. The relaxation times are calculated for the three-magnon confluence and splitting processes due to the spin-orbit coupling and for the four-magnon scattering processes due to electrostatic and spin-orbit interactions. The wave-vector and temperature dependences of the relaxation times are computed for nickel using a model band structure simulating the real one in the vicinity of the Fermi level. The model structure takes the doubly degenerate bands in regions around ΓL and ΓX directions into account and assumes a single band in the rest of the volume of the Brillouin zone.

1. Introduction

The problem of magnon relaxation was extensively studied in the past decade (see e.g. [1] and [2] for a review). The theory was developed on the basis of the Heisenberg model, although in some cases the role of conduction electrons was also discussed in terms of the *s-d* interaction model. It is of interest to study the magnon relaxation in itinerant electron ferromagnets. The magnon relaxation due to the Coulomb and dipolar interactions was studied recently [3–5] for a one-band model of the itinerant electron ferromagnets. In these papers the method of effective magnon hamiltonian [3, 4] was used which is well suited to a systematic study of various scattering processes in the magnon system. The effective magnon hamiltonian is constructed from products of electron operators which corresponds to the magnon creation and annihilation operators, it has a form

* Work supported by the Project MR-I.9 of the Polish Academy of Sciences.

** Address: Instytut Fizyki Molekularnej PAN, Smoluchowskiego 17/19, 60-179 Poznań, Poland.

similar to the well-known boson representation of the spin hamiltonian of the Heisenberg model. This analogy enables us to use directly some results derived earlier for the Heisenberg model.

The method of the effective magnon hamiltonian was generalized for the case of several bands of itinerant electrons and the effect of spin-orbit coupling on the magnon energy spectrum was studied ([6], hereafter quoted as I). At present we extend the results of I calculating the three- and four-magnon interaction terms in the effective hamiltonian and we use these terms to calculate the magnon relaxation times due to the spin-orbit interaction. The present problem is pendant to the one of magnon relaxation due to the pseudo-dipolar coupling of spins in the Heisenberg model [7] as the pseudo-dipolar forces represent the effect of the spin-orbit coupling of magnetic electrons in localized electron systems [1].

2. *Hamiltonian*

In studying effects of the spin-orbit interaction in itinerant electron systems it is compulsory to use the multiple band model. We start with a general hamiltonian with spin--orbit and electrostatic, Coulomb and exchange interactions, given by matrix elements $E^{tt'}_{ss'}$ and $W(t_1, t_2; t_4, t_3)$, respectively

$$\mathscr{H} = \sum_{kts} \varepsilon_{kt} a^{\dagger}_{kts} a_{kts} + \sum_{\substack{ktt' \\ ss'}} E^{tt'}_{ss'} a^{\dagger}_{kts} a_{kt's'}$$

$$+ \frac{1}{2N} \sum_{\substack{kk'qss' \\ t_1 \ldots t_4}} W(t_1, t_2; t_4, t_3) a^{\dagger}_{k+qt_1s} a^{\dagger}_{k'-qt_2s'} a_{k't_3s'} a_{kt_4s}, \tag{1}$$

$a^{\dagger}_{kts}(a_{kts})$ are the creation (annihilation) operators for electrons of the wave-vector $\vec{k}$ from the band t and of the spin s, the Bloch energy is ε_{kt}. Only intraatomic Coulomb and exchange integrals $W(t_1, t_2; t_4, t_3)$ are retained in (1).

As pointed out in [9], the band structure of ferromagnetic nickel in the neighbourhood of the Fermi level can be approximately described in terms of a one-band model for nearly the whole Brillouin zone except for some regions of small volume around the ΓL and ΓX directions. In these regions, covering about 2–5% of the volume of the Brillouin zone as estimated in [8], two bands have to be considered.

In our calculations we simulate the band structure of nickel in the neighbourhood of the Fermi level by the following two-band model. Let the index $t = 1$ label the Bloch states which are of d-symmetry in the whole Brillouin zone, let $t = 2$ correspond to states which are of s-symmetry in nearly the whole Brillouin zone, except for regions of small volume around the ΓL and ΓX directions where $\varepsilon_{k1} = \varepsilon_{k2}$. Because of the low density of s-states at the Fermi level we neglect the Bloch energy of s-electrons altogether. We keep only the leading terms in the general hamiltonian (1), i.e. only the intraatomic intraband Coulomb interaction I, the intraatomic interband (between the d-states) Coulomb U and exchange J integrals are retained as well as the spin-orbit coupling elements $E^{tt'}_{ss'}$ between d-states.

The hamiltonian (1) for the above two-band model can be written as $\mathscr{H} = \mathscr{H}_0 + \mathscr{H}'$ where

$$\mathscr{H}_0 = \sum_{ks} \varepsilon_{k1} a^\dagger_{k1s} a_{k1s} + \frac{I}{N} \sum_{kk'q} a^\dagger_{k+q1+} a^\dagger_{k'-q1-} a_{k'1-} a_{k1+}, \tag{2}$$

and

$$\begin{aligned}\mathscr{H}' = {} & \sum_{ks}{}' \varepsilon_{k2} a^\dagger_{k2s} a_{k2s} + \sum_{ktt'ss'}{}' E^{tt'}_{ss'} a^\dagger_{kts} a_{kt's'} \\ & + \frac{I}{2N} \sum_{kk'q}{}' a^\dagger_{k+q2s} a^\dagger_{k'-q2s'} a_{k'2s'} a_{k2s} \\ & + \frac{U}{2N} \sum_{kk'qss't\neq t'}{}' a^\dagger_{k+qts} a^\dagger_{k'-qt's'} a_{k't's'} a_{kts} \\ & + \frac{J}{2N} \sum_{kk'qss't\neq t}{}' a^\dagger_{k+qts} a^\dagger_{k'-qt's'} a_{k'ts'} a_{kt's}.\end{aligned} \tag{3}$$

The summations $\sum'$ in (3) extend only on the regions around ΓL and ΓX directions in which the two d-bands are present. Since $\sum'$'s contain only a small fraction of the volume of the Brillouin zone we treat $\mathscr{H}'$ as a perturbation.

3. *Effective magnon hamiltonian*

The basic idea of the present paper is to introduce the effective magnon hamiltonian which describes the magnon interaction in the itinerant electron ferromagnets and then to study the magnon relaxation processes by the method of the kinetic equation, which is well developed for similar problem in Heisenberg systems.

The treatment is based on the Randon Phase Approximation (RPA). We are interested in low-energy acoustic magnons. Following paper I we define the creation operators for acoustic magnons

$$\beta^\dagger_q = \sum_{kt} b_t(q, k) a^\dagger_{k+qt+} a_{kt-}, \tag{4}$$

where, for the two-band model as formulated in the Section 2,

$$\begin{aligned} b_1(q, k) &= d_q(\varepsilon_{k+q1} - \varepsilon_{k1} + \Delta_1 - E^0_q)^{-1}, \\ b_2(q, k) &= \frac{1-(I-J)K_1}{1-(I-J)K_2} \frac{d_q}{\varepsilon_{k+q2} - \varepsilon_{k2} + \Delta_2 - E^0_q}. \end{aligned} \tag{5}$$

E^0_q denotes the magnon energy unperturbed by the spin-orbit coupling. Δ_t ($t = 1, 2$) are the exchange splitting parameters,

$$\Delta_t = N^{-1}(I \sum_{kt} n_{kt-} + J \sum_{kt'\neq t} n_{kt'-}) \tag{6}$$

and

$$K_t = N^{-1} \sum_k n_{kt-} (\varepsilon_{k+qt} - \varepsilon_{kt} + \Delta_t - E_q^0)^{-1}. \tag{7}$$

n_{kts} is the ground state expectation value of $a_{kts}^\dagger a_{kts}$. We assume here a strong ferromagnetic ground state. The normalization function d_q is determined by the condition $\sum_{kt} |b_t(q, k)|^2 n_{kt-} = 1$ and whence the operators $\beta_q^\dagger$ and their hermitian adjoint β_q satisfy in RPA the boson commutation rules.

The effective magnon hamiltonian has the form of a power expansion in terms of $\beta_q^\dagger$ and β_q (cf. [4])

$$\mathscr{H}_{\text{eff}} = \sum_q E_q \beta_q^\dagger \beta_q + \sum_{qq'} (C_{qq'} \beta_{q+q'}^\dagger \beta_q \beta_{q'} + \text{h.c.}) + \sum_{kk'q} G_{kk'}^q \beta_{k+q}^\dagger \beta_{k'-q}^\dagger \beta_k \beta_{k'}. \tag{8}$$

E_q is the magnon energy including corrections due to the spin-orbit interaction. As calculated in I for long-wavelength magnons it has the form $E_q = E_0 + Dq^2$, where E_0 is the gap in the magnon spectrum due to the spin-orbit interaction and D is the spin wave stiffness constant, only slightly influenced by the spin-orbit coupling.

The coefficients $C_{qq'}$ and $G_{kk'}^q$ determining the three- and four-magnon interactions are given by:

$$C_{qq'} = \tfrac{1}{2} \langle \phi | [\beta_{q+q'}, [[\mathscr{H}, \beta_q^\dagger], \beta_{q'}^\dagger]] | \phi \rangle, \tag{9}$$

$$G_{kk'}^q = \tfrac{1}{4} \langle \phi | [\beta_{k+q}, [\beta_{k'-q}, [[[\mathscr{H}, \beta_{k'}^\dagger], \beta_k^\dagger]]] | \phi \rangle. \tag{10}$$

The averages are taken with respect to the ground-state $|\phi\rangle$ of the system of itinerant electrons, determined by $\mathscr{H} = \mathscr{H}_0 + \mathscr{H}'$. $C_{qq'}$ and $G_{kk'}^q$ are expressed in terms of ground-state average values of products of electron operators. These averages do not factorize automatically, as it would be in the case of the absence of the spin-orbit interaction, because states with different spin and band indices undergo mixing. However, because the spin-orbit coupling parameter ξ is small, we can approximately factorize the averages as in I. We put

$$\langle \phi | a_{kts}^\dagger a_{kt's'} | \phi \rangle = \delta_{kk'} [\delta_{tt'} \delta_{ss'} n_{kts} + f_{ss'}^{tt'}(k)], \tag{11}$$

where the functions f are at least of the order of ξ.

The expressions for $C_{qq'}$ and $G_{kk'}^q$ are

$$C_{qq'} = \tfrac{1}{2} (\bar{C}_{qq'} + \bar{C}_{q'q}), \tag{12}$$

where

$$\bar{C}_{qq'} = \sum_{ktt'}{}' E_{+-}^{*tt'} [b_t^*(q+q', k) b_t(q, k+q') b_{t'}(q', k) f_{--}^{tt'}(k)$$

$$- b_t^*(q+q', k-q-q') b_t(q, k-q) b_{t'}(q', k-q-q') f_{++}^{tt'}(k)] \tag{13}$$

and

$$G^q_{kk'} = \tfrac{1}{4}(\bar{G}^q_{kk'} + \bar{G}^{k'-k-q}_{kk'} + \bar{G}^{k+q-k'}_{k'k} + \bar{G}^{-q}_{k'k}), \tag{14}$$

$$\bar{G}^q_{kk'} = \bar{G}^q_{0kk'} + \bar{G}^q_{1kk'}, \tag{15}$$

with

$$\bar{G}^q_{0kk'} = \frac{I}{N}\sum_{pp'} b_1^*(k+q, p)b_1^*(k'-q, p')b_1(k', p'-q)\,[b_1(k, p')-b_1(k, p)]n_{p1-}n_{p'1-}, \tag{16}$$

$$\bar{G}^q_{1kk'} = \sum_{ptt'}{}' b_t^*(k+q, p)b_{t'}^*(k'-q, p+q-k')b_t(k', p+k+q-k')$$

$$\times b_{t'}(k, p+q-k')E^{*tt'}_{+-}f^{t't}_{-+}(p). \tag{17}$$

Note that $C_{qq'}$ is proportional to ξ^4, $G^q_{1kk'}$ is proportional to ξ^2 whereas $G^q_{0kk'}$ is of Coulomb origin.

4. Correlation functions in the two-band model

The correlation functions $\langle a^\dagger_{kts}a_{kt's'}\rangle$ are determined by equations (22) and (24) of paper I

$$\langle a^\dagger_{kts}a_{kt's'}\rangle = i\lim_{0<\varepsilon\to 0}\int_0^{E_F} dE[G^{tt'}_{ss'}(E+i\varepsilon)-G^{tt'}_{ss'}(E-i\varepsilon)], \tag{18}$$

where $G^{tt'}_{ss'}(E) = \langle\langle a_{kt's'}; a^\dagger_{kts}\rangle\rangle_E$ are the one-electron Green functions determined by the equations:

$$(E-\varepsilon_{kts})G^{tt'}_{ss'}(E) - \sum_{r(\neq t),\sigma} E^{tr}_{s\sigma}G^{rt}_{\sigma s}(E) = \frac{1}{2\pi}\delta_{tt'}\delta_{ss'}. \tag{19}$$

The equations (19) are derived in the Hartree–Fock approximation. We assume that if the spin-orbit interaction is neglected, the two d bands of the two-band model introduced in Section 2 are degenerate in the regions around the ΓL and ΓX directions, i.e. $\varepsilon_{k2} = \varepsilon_{k1}$ within those regions, whereas $\varepsilon_{k2} = 0$ in the rest of the Brillouin zone. We assume also strong ferromagnetism for the ground state. The Hartree-Fock energy in (19) is thus $\varepsilon_{kts} = \varepsilon_{kt} + \Delta\delta_{s,+}$, where $\Delta = (\Delta_1+\Delta_2)/2$.

The matrix elements $E^{tt'}_{ss'}$ of the spin-orbit coupling are given in [10]; they are products of the spin-orbit parameter ξ and combinations of direction cosines of the magnetization with respect to the crystal axis. The following relations are useful in solving the equations (19) : $E^{tt'}_{ss'} = -E^{t't}_{ss'}$, $E^{tt'}_{++} = E^{*tt'}_{--}$, $E^{tt'}_{+-} = -E^{*tt'}_{-+}$. The solutions of (19) are (the Green functions listed below are sufficient to calculate the needed correlation functions)

$$G^{12}_{+-}(E) = \frac{E^{12}_{+-}}{2\pi R}\,[(E-\mathscr{E}_{k1})(E-\mathscr{E}_{k1}-\Delta)-F], \tag{20a}$$

$$G^{12}_{--}(E) = \frac{E^{12}_{--}}{2\pi R}\,[(E-\mathscr{E}_{k1}-\Delta)^2-F], \tag{20b}$$

$$G^{12}_{++}(E) = \frac{-E^{12}_{--}}{2\pi R}\,[(E-\mathscr{E}_{k1})^2-F], \tag{20c}$$

$$G^{11}_{--}(E) = \frac{1}{2\pi R}\,\{(E-\mathscr{E}_{k1})\,[(E-\mathscr{E}_{k1}-\Delta)^2-F]+\Delta|E^{12}_{+-}|^2\}, \tag{20d}$$

$$G^{11}_{+-}(E) = \frac{1}{2\pi R}\,\{(E-\mathscr{E}_{k1})\,[(E-\mathscr{E}_{k1})\,(E-\mathscr{E}_{k1}-\Delta)-F]+\Delta|E^{12}_{+-}|^2\}, \tag{20e}$$

where $R = (E-\mathscr{E}_{k1})(E-\mathscr{E}_{k2})(E-\mathscr{E}_{k3})(E-\mathscr{E}_{k4})$ and $F = |E^{12}_{--}|^2+|E^{12}_{+-}|^2$. The electron energies split by the spin-orbit interaction are given by

$$\mathscr{E}_{k1},\,\mathscr{E}_{k2} = \varepsilon_{k1}+\tfrac{1}{2}\,\Delta(1-S_{\pm}), \tag{21}$$

$$\mathscr{E}_{k3},\,\mathscr{E}_{k4} = \varepsilon_{k1}+\tfrac{1}{2}\,\Delta(1+S_{\mp}), \tag{22}$$

where

$$S_{\pm} = \left[1+4\left(\pm\frac{|E^{12}_{--}|}{\Delta}+\frac{|E^{12}_{--}|^2+|E^{12}_{+-}|^2}{\Delta^2}\right)\right]^{1/2}. \tag{23}$$

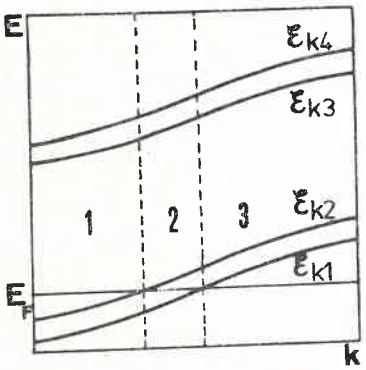

Fig. 1. Band structure near the Fermi level for the ΓL and ΓX directions, with exchange and spin-orbit splitting

Now we use formula (18) to calculate the correlation functions. For $\mathscr{E}_{k1},\,\mathscr{E}_{k2} < E_F$ i.e. for the region 1 of Fig. 1 we have

$$\langle a^{\dagger}_{k1+}a_{k1+}\rangle = \tfrac{1}{2}-\frac{1}{4S_+S_-}\left[(S_++S_-)-\frac{2}{\Delta}\,|E^{12}_{--}|\,(S_+-S_-)\right], \tag{24a}$$

$$\langle a^{\dagger}_{k2-}a_{k1-}\rangle = \frac{E^{12}_{--}}{4S_+S_-}\left[\frac{1}{|E^{12}_{--}|}\,(S_+-S_-)-\frac{2}{\Delta}\,(S_++S_-)\right], \tag{24b}$$

$$\langle a^{\dagger}_{k2-}a_{k1+}\rangle = -\frac{E^{12}_{+-}}{2\Delta S_+S_-}\,(S_++S_-), \tag{24c}$$

$$\langle a^{\dagger}_{k1-}a_{k1-}\rangle = 1-\langle a^{\dagger}_{k1+}a_{k1+}\rangle, \quad \langle a^{\dagger}_{k2+}a_{k1+}\rangle = -\langle a^{\dagger}_{k2-}a_{k1-}\rangle. \tag{24d}$$

whereas for $\mathscr{E}_{k1} < E_F < \mathscr{E}_{k2}$ i.e. for the region 2 of Fig. 1

$$\langle a^\dagger_{k1+} a_{k1+} \rangle = \frac{1}{4S_+}\left(S_+ - \frac{2|E^{12}_{--}|}{\Delta} - 1\right), \tag{25a}$$

$$\langle a^\dagger_{k2-} a_{k1-} \rangle = -\frac{E^{12}_{--}}{4S_+}\left(\frac{2}{\Delta} + \frac{S_+ + 1}{|E^{12}_{--}|}\right), \tag{25b}$$

$$\langle a^\dagger_{k2-} a_{k1+} \rangle = -\frac{E^{12}_{+-}}{2\Delta S_+}, \tag{25c}$$

$$\langle a^\dagger_{k2+} a_{k1+} \rangle = \frac{E^{12}_{--}}{4S_+}\left(\frac{2}{\Delta} - \frac{S_+ - 1}{|E^{12}_{--}|}\right), \tag{25d}$$

$$\langle a^\dagger_{k1-} a_{k1-} \rangle = -\langle a^\dagger_{k1+} a_{k1+} \rangle. \tag{25e}$$

In Fig. 1 in the region determined by $\mathscr{E}_{kt} > E_F$, $t = 1, \ldots 4$ all correlation functions vanish.

5. Matrix elements for the three- and four-magnon interaction

Having expressions for the correlation function we can calculate the three- and four--magnon matrix elements of the effective magnon hamiltonian. We shall calculate $C_{qq'}$ and $G^q_{kk'}$ as an expansion of the wave-vectors retaining only the leading terms.

We apply the formulae (12)–(17) to the model of band structure of ferromagnetic nickel, as described in the Section 2, with appropriate Bloch wave functions (see Section 7 of the paper I) being combinations of atomic d functions. In the presence of ferromagnetic ordering the four directions ΓL and the three directions ΓX are no longer equivalent, so matrix elements of spin-orbit interactions are different combinations of matrix elements between five atomic d-type functions for different directions (see I, [9], [10]) and have to be calculated separately. In expressions (13), (16) and (17) we expand the coefficients $b_i(q, k)$ in powers of the wave-vectors q up to quadratic terms and retain only leading contributions with respect to the small spin-orbit coupling parameter ξ.

The summation over the occupied part of the Brillouin zone in formulae (13), (16) and (17) is complicated because of the form of the correlation functions (21) and (22), which are different in regions 1 and 2, as depicted in Fig. 1. For summation over region 2 of Fig. 1, which is of small volume of the order of ξ^3, we use the same estimate as in I. For region 1 the following approximate Ansatz is used: the sum over wave-vectors from the region 1, $\sum'_k(\ldots)$ is replaced by $\sum'_k(\ldots) = \gamma \sum_k(\ldots)$ where γ is the fraction of the volume of region 1 of the total volume of the Brillouin zone and $\sum_k(\ldots)$ extends over all $\vec{k}$ from the Brillouin zone. The value of the parameter γ is estimated (cf. [8]) as $\gamma \cong 0.02 - 0.05$.

We get the following approximate expressions for $C_{qq'}$ and $G^q_{kk'}$:

$$C_{qq'} = 2(nN)^{-1/2}\Delta\xi_1^4\left\{\left[(\gamma-P)+(3n\Delta_1^2)^{-1}\frac{5}{N}\sum_p n_{p-}\left(\left(3+2\gamma\frac{\Delta_1}{\Delta_2}\right)\left(\tfrac{3}{2}\frac{\Delta_1}{\Delta_2}-1\right)\left(P+\gamma A_1\right)\right.\right.\right.$$
$$\left.\times|\nabla\varepsilon_p|^2+\Delta_1(\gamma A_2-P)\nabla^2\varepsilon_p\right)(q^2+q'^2+\vec{q}\cdot\vec{q}')+(3n\Delta_1^2)^{-1}$$
$$\left.\times(\gamma A_3(q^2+q'^2)+2\gamma\vec{q}\cdot\vec{q}')\frac{5}{N}\sum_p n_{p-}|\nabla\varepsilon_p|^2\right]\left(1+\frac{1}{6C^4}\right)F(\alpha_i)$$
$$+8\gamma A_4 Z(n\Delta_1^2)^{-1}[F_1(\alpha_i)\,(q+q')_x(q+q')_y+F_2(\alpha_i)\,(q+q')_x(q+q')_z$$
$$+F_3(\alpha_i)\,(q+q')_y(q+q')_z]+\tfrac{1}{2}A_4P(W\sin k_F/16\Delta_1)^2$$
$$\times[F_1'(\alpha_i)\,(q+q')_x(q+q')_y+F_2'(\alpha_i)\,(q+q')_x(q+q')_z$$
$$\left.+F_3'(\alpha_i)\,(q+q')_y(q+q')_z]\right\}, \tag{26}$$

where we denoted $(1+\gamma)n_{k1} = n_{k-}$, for short.

$$A_1 = \frac{\Delta_1}{\Delta_2}\left[\frac{\Delta_1}{\Delta_2}(\tfrac{5}{2}-4\gamma)+1+3\gamma\right]+\gamma\left(\frac{\Delta_1}{\Delta_2}\right)^2\frac{I-J}{J}-\tfrac{7}{2},$$
$$A_2 = \tfrac{1}{2}\left(1+\frac{\Delta_1}{\Delta_2}\right)-\frac{I-J}{J}\left(1-\gamma\frac{\Delta_1}{\Delta_2}\right),$$
$$A_3 = \tfrac{1}{4}\left[7-\frac{\Delta_1}{\Delta_2}\left(\frac{\Delta_1}{\Delta_2}+6\right)\right],$$
$$A_4 = \tfrac{1}{2}\left[5+3\left(\frac{\Delta_1}{\Delta_2}\right)^2+4\frac{\Delta_1}{\Delta_2}\right], \tag{26a}$$

$$(\Delta_1+\Delta_2)/2 = \Delta, \quad \Delta_1 = n(1-\gamma)\,(I+\gamma J), \quad \Delta_2 = n(1-\gamma)\,(\gamma I+J). \tag{26b}$$

$$G^q_{kk'} = -\gamma\Delta\xi_1^2 g(nN)^{-1}\left\{1+(3n\Delta_1)^{-1}\frac{5}{N}\sum_p n_{p-}[(A_1'\nabla^2\varepsilon_p+\Delta_1^{-1}A_2'|\nabla\varepsilon_p|^2)\right.$$
$$\times(k^2+k'^2+q^2+\vec{q}\cdot(\vec{k}-\vec{k}'))+\Delta_1^{-1}A_3'|\nabla\varepsilon_p|^2(k+k')^2]-8Z(ng\Delta_1^2)^{-1}$$
$$\times[H_{1xy}(\alpha_1\alpha_2-\xi_1^2(3\alpha_1\alpha_2-4\alpha_1\alpha_2\alpha_3^2))+H_{1xz}(\alpha_1\alpha_3-\xi_1^2(3\alpha_1\alpha_3-4\alpha_1\alpha_2^2\alpha_3))$$
$$\left.+H_{1yz}(\alpha_2\alpha_3-\xi_1^2(3\alpha_2\alpha_3-4\alpha_1^2\alpha_2\alpha_3))]\right\}+\tfrac{1}{5}(1-\gamma)I\frac{5}{N}\sum_p n_{p-}|\nabla\varepsilon_p|^2\vec{k}'\cdot\vec{k}', \tag{27}$$

where

$$A_1' = 1+(20J)^{-1}\left[10J\left(\frac{\Delta_1}{\Delta_2}-1\right)-2\left(1-\gamma\frac{\Delta_1}{\Delta_2}\right)(I-J)\right],$$

$$A_2' = \frac{\Delta_1}{\Delta_2}\left[\left(\frac{\Delta_1}{\Delta_2}-1\right)(1-3\gamma)+\gamma\frac{\Delta_1}{\Delta_2}\left(\frac{I-J}{5J}-1\right)\right]-1,$$

$$A_3' = \tfrac{1}{4}\left[\left(\frac{\Delta_1}{\Delta_2}\right)^2+8\frac{\Delta_1}{\Delta_2}+1\right]. \tag{27a}$$

$\alpha_1, \alpha_2, \alpha_3$ are the direction cosines of the magnetization with respect to the crystal axes, $n = N^{-1}\sum_k 5n_{k-}$ is the number of d holes per atom $\xi_1 = C\xi/\Delta$, where $C = (2\sqrt{2}-1)/2\sqrt{3}$. We put $P = 2(4\Delta/3\pi W \sin k_F a)^3/n$ where W is the band width, a is the lattice constant and k_F is the Fermi momentum determined from the Fermi energy E_F by the approximate relation $E_F = 3/4\ W(1-\cos^2(k_F a/2))+O(\xi)$. We denoted

$$Z = \frac{5}{N}\sum_{1/8k}\frac{\partial\varepsilon_{k1}}{\partial k_x}\frac{\partial\varepsilon_{k1}}{\partial k_y},$$

where the summation extends over an octant of the Brillouin zone containing the direction [111] as its symmetry axis. The following abbreviations were used

$$F(\alpha_i) = (1-\alpha_3^2)^{-1/2}\{\alpha_3[2\Gamma-(1-\alpha_3^2)]+i\alpha_1\alpha_2(\alpha_1^2-\alpha_2^2)\}, \tag{28a}$$

$$F_1(\alpha_i) = -(1-\alpha_3^2)^{1/2}\{4\alpha_1\alpha_2\alpha_3+i(\alpha_1^2-\alpha_2^2)\}, \tag{28b}$$

$$F_1'(\alpha_i) = (1-\alpha_3^2)^{-1/2}\{2\alpha_1\alpha_2\alpha_3(1-4\alpha_3^2)-i(\alpha_1^2-\alpha_2^2)(1+2\alpha_3^2)\}, \tag{28c}$$

$$F_2(\alpha_i) = (1-\alpha_3^2)^{-1/2}\{\alpha_1[1-\alpha_3^2-\alpha_2^2(1-4\alpha_3^2)]+i\alpha_2\alpha_3(\alpha_3^2+3\alpha_1^2)\}, \tag{28d}$$

$$F_2'(\alpha_i) = (1-\alpha_3^2)^{-1/2}\{\alpha_1[4\alpha_2^2(1-2\alpha_3^2)-(1-2\alpha_1^2)]-i\alpha_2\alpha_3[3(\alpha_1^2-\alpha_2^2)-\alpha_3^2]\}. \tag{28e}$$

$$F_3(\alpha_i) = (1-\alpha_3^2)^{-1/2}\{\alpha_2[1-\alpha_3^2-\alpha_1^2(1-4\alpha_3^2)]-i\alpha_1\alpha_3(\alpha_3^2+3\alpha_2^2)\}. \tag{28f}$$

$$F_3'(\alpha_i) = (1-\alpha_3^2)^{-1/2}\{\alpha_2[4\alpha_1^2(1-2\alpha_3^2)-(1-2\alpha_2^2)]-i\alpha_1\alpha_3[3(\alpha_1^2-\alpha_2^2)+\alpha_3^2]\}, \tag{28g}$$

$$\Gamma = \alpha_1^2\alpha_2^2+\alpha_1^2\alpha_3^2+\alpha_2^2\alpha_3^2. \tag{29}$$

$$g = 1+\frac{1}{3C^2}-3\xi_1^2\Gamma\left(1-\frac{1}{6C^4}\right)-\frac{\xi_1^2}{6C^4}, \tag{30}$$

and

$$H_{1x_ix_j} = \left(1+\left(\frac{\Delta_1}{\Delta_2}\right)^2\right)[6(\vec{k}+\vec{k}')_{x_i}(\vec{k}+\vec{k}')_{x_j}-2(k_{x_i}k'_{x_j}+k'_{x_i}k_{x_j})+q_{x_i}(\vec{k}+\vec{q}-\vec{k}')_{x_j}$$

$$+q_{x_j}(\vec{k}+\vec{q}-\vec{k}')_{x_i}]-\tfrac{1}{2}\left(7\left(\frac{\Delta_1}{\Delta_2}\right)^2-8\frac{\Delta_1}{\Delta_2}+7\right)(\vec{k}+\vec{k}')_{x_i}(\vec{k}+\vec{k}')_{x_j}, \tag{32}$$

where $x_i, x_j = x, y, z$.

For computing $C_{qq'}$ and $G^q_{kk'}$ it is convenient to use the function $M(E)$, introduced in [11]

$$M(E) = 5v/(2\pi)^3 \int_{\varepsilon_k = E} |\nabla \varepsilon_k| dS, \tag{33}$$

where the integration extends over the surface of constant energy $\varepsilon_k = E$ and v is the volume of the elementary cell. The sums in (26) and (27) can be expressed in terms of $M(E)$:

$$\frac{5}{N} \sum_k n_{k-} \nabla^2 \varepsilon_k = M(E_F), \tag{34a}$$

$$\frac{5}{N} \sum_k n_{k-} |\nabla \varepsilon_k|^2 = \int_0^{E_F} M(E) dE. \tag{34b}$$

The anisotropy of magnon relaxation times is not very pronounced, therefore for reasons of simplicity we shall confine the subsequent discussion to the special case of magnetization direction parallel to the z-axis, i.e. $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = 1$. For that case the expressions (26) and (27) are simplified considerably,

$$C_{qq'} = \{i(C_1 + C_2)[q_x + q'_x - (q_y + q'_y)] - C_2(q_x + q'_x + q_y + q'_y)\}(q_z + q'_z), \tag{35}$$

$$G^q_{kk'} = G_0 + G_1(k^2 + k'^2) + G_2 \vec{k} \cdot \vec{k}' + G_3[\vec{q} \cdot (\vec{k} - \vec{k}') + q^2], \tag{36}$$

where

$$C_1 = 16 A_4 (2nN)^{-1/2} \Delta (n \Delta_1^2)^{-1} Z \xi_1^4, \tag{37a}$$

$$C_2 = A_4 P (2nN)^{-1/2} \Delta (W \sin k_F a / 16 \Delta_1)^2 \xi_1^4, \tag{37b}$$

$$G_0 = -\gamma (nN)^{-1} \Delta g \xi_1^2, \tag{38a}$$

$$G_1 = (3n\Delta_1^2)^{-1} [A'_1 M(E_F) + \Delta_1^{-1}(A'_2 + A'_3) \int_0^{E_F} M(E) dE] G_0, \tag{38b}$$

$$G_2 = (3n\Delta_1^2)^{-1} \left[2A'_3 G_0 + \frac{1-\gamma}{5N} I \right] \int_0^{E_F} M(E) dE, \tag{38c}$$

$$G_3 = (3n\Delta_1^2)^{-1} [A'_1 \Delta_1 M(E_F) + A'_2 \int_0^{E_F} M(E) dE] G_0. \tag{38d}$$

6. *Three-magnon relaxation processes*

The effective magnon hamiltonian is convenient for studying separately various magnon relaxation mechanisms. The second term of the effective magnon hamiltonian describes two types of three-magnon processes. In the 3-magnon confluence process a given magnon $\vec{k}$ disappears with another magnon $\vec{q}$ to produce a third magnon $\vec{q}'$ whose wave-vector is $\vec{q}' = \vec{k} + \vec{q}$. In the 3-magnon splitting process a given magnon $\vec{k}$ disappears to

produce two other magnons, say $\vec{q}$ and $\vec{q}'$, whose wave-vectors satisfy the condition $\vec{q}+\vec{q}' = \vec{k}$. Because of the energy conservation requirement the splitting processes are allowed only if the magnon wave-vector exceeds some critical value k_c determined by the condition $k_c^2 = E_0/D$, where E_0 and D are the gap in magnon energy spectrum and spin-wave stiffness constant, respectively. For $k < k_c$ only the confluence processes are effective in relaxing magnon $\vec{k}$. Using the standard kinetic-equation approach developed in the traditional spin-wave theory we obtain the following expressions for the inverse relaxation times (see, e.g. [1])

$$\frac{1}{\tau_k^c} = \frac{2\pi}{\hbar}\sum_q 4|C_{kq}|^2(\bar{N}_q - \bar{N}_{k+q})\delta(E_k+E_q-E_{k+q}) \tag{39}$$

for the 3-magnon confluence process and

$$\frac{1}{\tau_k^s} = \frac{\pi}{\hbar}\sum_q 4|C_{q,k-q}|^2(\bar{N}_q+\bar{N}_{k-q}+1)\delta(E_k-E_q-E_{k-q}) \tag{40}$$

for the 3-magnon splitting process, where $\bar{N}_q = (\exp(E_q/k_BT)-1)^{-1}$ is the equilibrium value of the occupation number of magnons of energy E_q.

6.1. Three-magnon confluence processes

For simplicity we calculate the relaxation time for magnons propagating along the z-axis, i.e. for $\vec{k} = (0, 0, k)$. The summation in (39) is replaced by integration in the usual way. It is convenient to introduce spherical coordinates $q_x = q\sin\vartheta\cos\varphi$, $q_y = q\sin\vartheta\sin\varphi$, $q_z = q\cos\vartheta$. The angular integrations are easily performed using the relation $\delta(E_k+E_q-E_{k+q}) = (2Dkq)^{-1}\delta(d/2kq-\cos\vartheta)$, where we use the dispersion relation $E_q = E_0+Dq^2$ and we put $d = E_0/D$. Integration over the Brillouin zone is approximated by integration over a sphere with radius R (defined by the condition $4\pi R^3/3$ = volume of the Brillouin zone). For low temperatures the errors involved are negligible because of the Bose factors. In order to get an expression convenient for computations we perform a change of the variable q into $u = E_0/k_BT+(D/k_BT)q^2$. The final formula for the 3-magnon confluence relaxation time is

$$1/\tau_k^c = Q(k_BT/D)^2\frac{1}{k}(k+d/2k)^2(e^y-1)\theta(R-d/2k)$$

$$\times\int_{u_1}^{u_2} du[u-(k_BT/D)^{-1}(d+d^2/4k^2)]\,[(1-e^{-u})(e^{u+y}-1)]^{-1}. \tag{41}$$

The following notation was used

$$Q = V[(C_1+C_2)^2+C_2^2]/2\hbar D\pi, \qquad y = (E_0+Dk^2)/k_BT,$$

$$u_1 = (E_0/k_BT)(1+d/4k^2); \qquad u_2 = (E_0/k_BT)(1+d^{-1}R^2), \tag{42}$$

where V is the volume of the sample. $\theta(x)$ is the step function.

6.2. Three-magnon splitting processes

Momentum and energy conservation laws in (40) lead to the following conditions $\vec{q}+\vec{q}' = \vec{k}$ and $\vec{k}\cdot\vec{q} = q^2+d/2$, so the coefficient $|C_{qq'}|^2$ can be replaced by

$$|C_{qq'}|^2 = k_z^2[(C_1+C_2)^2(k_x-k_y)^2+C_2^2(k_x+k_y)^2].$$

Using similar procedure as for the confluence process we obtain the formula for the inverse relaxation time due to the 3-magnon splitting process:

$$\frac{1}{\tau^s_{\vec{k}}} = Vk_BT/(4\pi\hbar D^2k)k_z^2[(C_1+C_2)^2(k_x-k_y)^2+C_2^2(k_x+k_y)^2]$$
$$\times(e^y-1)\int_{u_3}^{u_4} du[(e^u-1)(e^{y-u}-1)]^{-1}, \tag{43}$$

where

$$u_3 = (D/k_BT)\{d+1/2[k(k-(k^2-2d)^{1/2})-d]\},$$
$$u_4 = (D/k_BT)\{d+1/2[k(k+(k^2-2d)^{1/2})-d]\}. \tag{44}$$

If we average (43) over directions of $\vec{k}$, the expression for $1/\tau^s_k = (1/4\pi)\int d\Omega_k/\tau^s_{\vec{k}}$ is

$$\frac{1}{\tau^s_k} = (1/15)(Q/D)k_BTk^3(e^y-1)\theta(k-2d)\int_{u_3}^{u_4} du[(e^u-1)(e^{y-u}-1)]^{-1}. \tag{45}$$

As we see, the 3-magnon splitting processes become effective if the wave-vector k exceeds the critical value given by $k_c^2 = 2d$. Olkhov and Lutovinov [12] pointed out that the kinetic equation for 3-magnon confluence processes is valid only above some critical wave-vector. For values smaller than the critical, the behaviour of the spin-wave system is not completely determined by one-particle distribution function, i.e., the correlation effects of spin waves should not be neglected. The interaction part of the magnon hamiltonian cause transitions between states and magnon correlations do not damp at all when $k \to 0$, but oscillate with the constant frequency $E_0/\hbar$. This means that in order to determine the magnon relaxation time we have to consider renormalization of 3-magnon relaxation by 4-magnon processes.

7. Four-magnon relaxation processes

The only 4-magnon scattering processes in the effective hamiltonian (8) for the system with spin-orbit interaction are the two-in and two-out ones determined by terms proportional to $G^q_{kk'}$. (In the case of systems with magnetic dipolar interaction of itinerant electrons also the 4-magnon one-in and three-out processes are possible, as shown in [4].) Applying formula (22) of paper [5] we have the following expression for the inverse relaxation time due to 4-magnon processes:

$$\frac{1}{\tau^{(2,2)}_k} = \frac{16\pi}{\hbar}\sum_{qq'}|G^{k-q}_{qq'}|^2\{\overline{N}_{q+q'-k}(\overline{N}_q+\overline{N}_{q'}+1)-\overline{N}_q\overline{N}_{q'}\}\delta(E_k+E_{q+q'-k}-E_q-E_{q'}). \tag{46}$$

The inverse relaxation time for magnons $\vec{k}$, averaged over directions of $\vec{k}$, is

$$\langle 1/\tau_k^{(2,2)}\rangle = (1/4\pi)\int d\Omega_k/\tau_k^{(2,2)} = 4V^2/[(2\pi)^5\hbar D]$$
$$\times \int d\vec{q}\int d\vec{q}' f_k(\beta E_q, \beta E_{q'})\,(k|\vec{q}+\vec{q}'|)^{-1}$$
$$\times\{G_0+G_1(q^2+q'^2)+G_2\vec{q}\cdot\vec{q}'\}^2\theta[k^2q^2+k^2q'^2-k^4-(\vec{q}\cdot\vec{q}')^2], \tag{47}$$

where

$$f_k(\beta E_q, \beta E_{q'}) = (\exp[\beta(E_q+E_{q'}-E_k)]-1)^{-1}[(\exp(\beta E_q)-1)^{-1}$$
$$+(\exp(\beta E_{q'})-1)^{-1}+1]-([\exp(\beta E_q)-1]\,[\exp(\beta E_{q'})-1])^{-1} \tag{48}$$

and $\beta = 1/k_BT$. We introduce spherical coordinates in $\vec{k}$ and $\vec{k}'$ spaces and integrate over a sphere of radius R, of the volume equal to the volume of the Brillouin zone. Having performed the angular integrations in (47) it is convenient to introduce new variables: $x = \beta D(q^2+q'^2)$, $y = \beta D(q'^2-q^2)$ and we get finally

$$\langle 1/\tau_k^{(2,2)}\rangle = Bt^{5/2}k^{-1}\int_z^{R^2/t} dx\int_0^x dy f_z(x,y)\{F_-(x)[(G_0+G_1tx)^2$$
$$-(2/3)G_2tx(G_0+G_1tx)+(1/5)G_2^2t^2(z(x-z)+(2/3)x^2)]$$
$$+(2/3)G_2tF_+(x)[z(x-z)]^{1/2}(G_0+G_1tx-(1/5)G_2tx)\} \tag{49}$$

where $t = k_BT/D$ is a reduced temperature, $z = Dk^2/k_BT$, $B = V^2/(2(2\pi)^3\hbar D)$, and the abbreviations are used

$$F_\pm(x) = \{x+2[z(x-z)]^{1/2}\}^{1/2}\pm\{x-2[z(x-z)]^{1/2}\}^{1/2}.$$

The function $f_z(x,y)$ corresponds to (48), its explicit expression in the new variables is

$$f_z(x,y) = [\exp(\varepsilon+x-z)-1]^{-1}\{(\exp[\varepsilon+\tfrac{1}{2}(x-y)]-1)^{-1}$$
$$+(\exp[\varepsilon+\tfrac{1}{2}(x+y)]-1)^{-1}+1\}-[(\exp[\varepsilon+\tfrac{1}{2}(x-y)]-1)(\exp[\varepsilon+\tfrac{1}{2}(x+y)]-1)]^{-1} \tag{48'}$$

where $\varepsilon = E_0/k_BT$.

In the limit $k \to 0$ formula (49) gives

$$\langle 1/\tau_0^{(2,2)}\rangle = 2Bt^2\int_0^{R^2/t} dx\int_0^x dy f_0(x,y)\,(G_0+G_1tx)^2 \tag{50}$$

so the 4-magnon relaxation time is finite for $k = 0$. This is due to the spin-orbit interaction, the coefficients G_0 and G_1 are proportional to the spin-orbit coupling parameter ξ^2. For a finite wave vector k the 4-magnon relaxation is a result of interplay of two mechanisms, the spin-orbit and the Coulomb interaction, as the coefficient G_2 is approximately proportional to the Coulomb integral I. With increasing k the Coulomb interaction begins to dominate and for large enough k the 4-magnon relaxation is predominantly of Coulomb origin.

8. *Numerical results for nickel*

The aim of the present paper is to study magnon relaxation due to the spin-orbit coupling in itinerant electron ferromagnets. The approach based on the effective magnon hamiltonian is quite general and can be applied to any magnetic systems of itinerant electrons. The method is applied to nickel which is a typical itinerant electron ferromagnet. The real band structure of nickel is approximated by the model being a modification of the model of Mori et al. [9] and the one used in I. In this model we take into account the existence of two bands (split due to the spin–orbit interaction) in small volume regions around ΓL and ΓX directions and we assume a single band elsewhere in the Brillouin zone. The band structure of ferromagnetic nickel [13] in the vicinity of the Fermi energy is sufficiently well simulated by this model. The following values of parameters for nickel are taken: the number of d-holes $n = 0.6$, the exchange splitting $\Delta = 0.56$ eV [14], the spin-orbit parameter $\xi = 0.12$ eV [14]. We assume the exchange parameter $J = 0.1$ eV and whence we determine the Coulomb integral I from given values of Δ and n i.e. from $\Delta = n(I+J)/2$, as $I = 1.8$ eV. The band-width parameter for d-holes is taken as $W = 1$ eV, in accordance with the calculated band structure [13]. The ratio of the bandwidth W to the Fermi energy

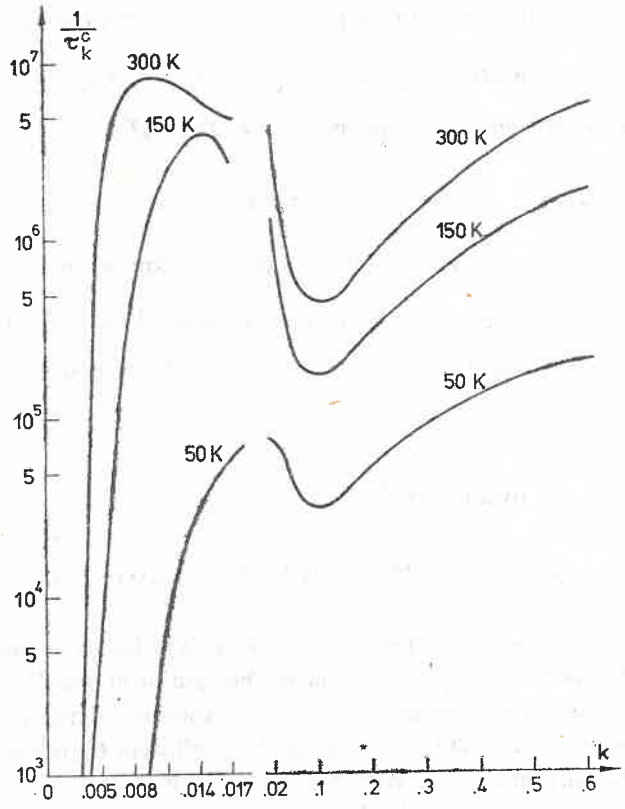

Fig. 2. The inverse relaxation time for 3-magnon confluence processes versus wave-vector, for various temperatures

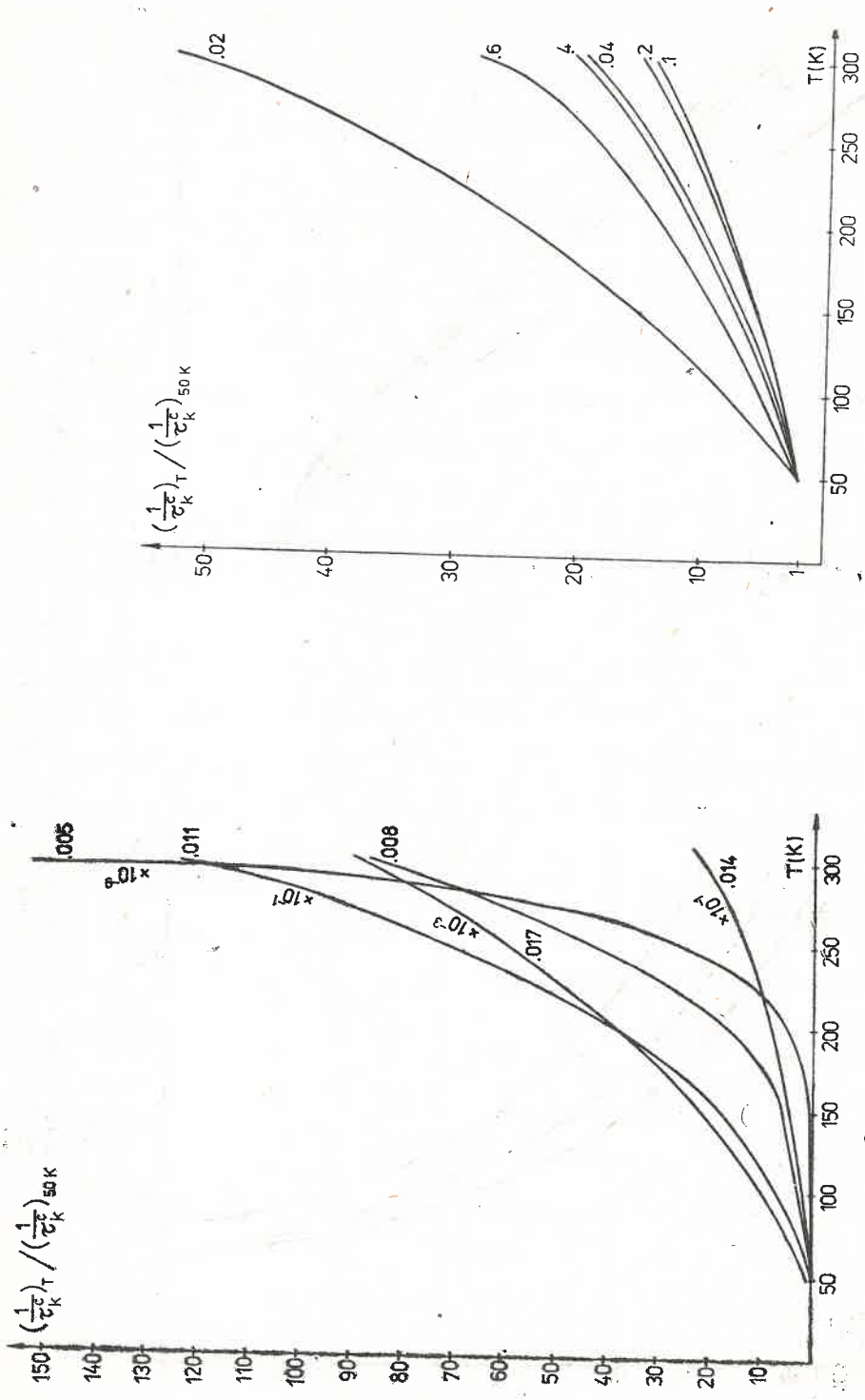

Fig. 3a, b. Temperature dependence of the 3-magnon confluence relaxation time for various wave-vectors

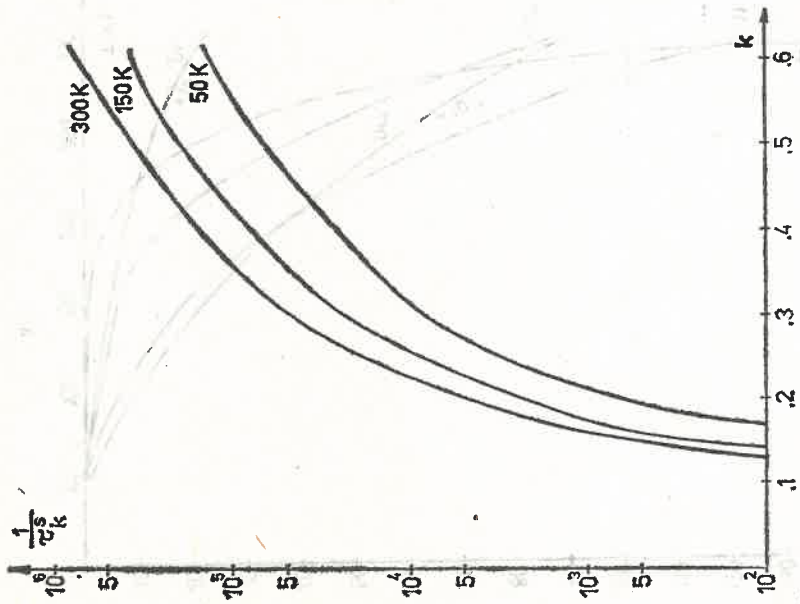

Fig. 4

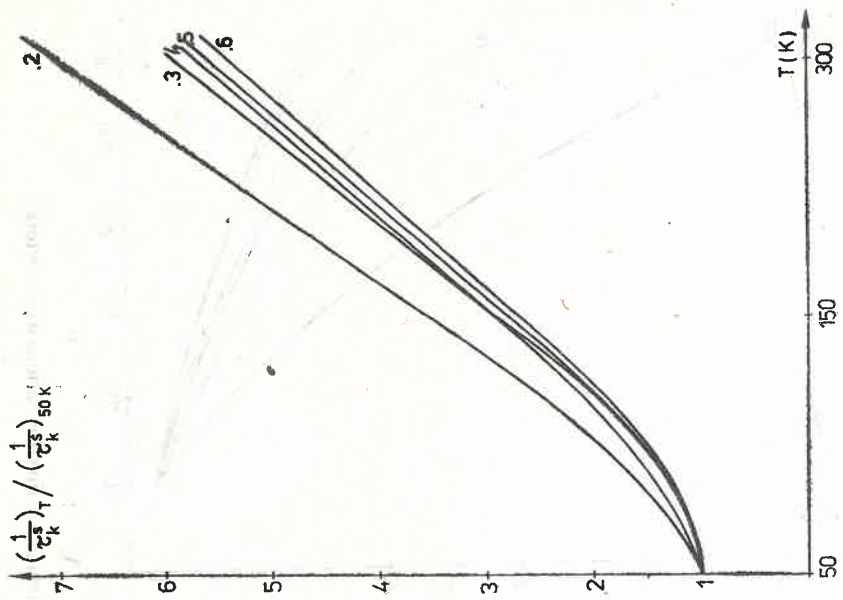

Fig. 5

Fig. 4. The inverse relaxation time for 3-magnon splitting processes versus wave-vector, for various temperatures
Fig. 5. Temperature dependence of the 3-magnon splitting relaxation time for various wave-vectors

E_F is computed from the relation $n = N^{-1}\sum_k 5n_{k-}$ where obviously $n_{k-} = \theta(E_F - \varepsilon_{k1})$. For the value $E_F = 0.49\ W$, determined in this way, we have $M(E_F) = 0.13\ Wa^2$, $\int_0^{E_F} M(E)dE = 0.03 W^2a^2$ and $Z = 7\times10^{-4}W^2a^2$, where a is the lattice constant. For the parameter γ determining the fraction of the volume of the Brillouin zone with double-degenerate energy levels we take $\gamma = 0.03$, in accordance with [8]. For these parameters the gap E_0 in the magnon spectrum is roughly 0.7 meV. For the spin-wave stiffness constant D we take the experimental value $Da^{-2} = 0.04$ eV, [15].

The results of computations of the relaxation times for various mechanisms are presented in Figs 2–9. For a very small wave-vector k, as is seen from Fig. 2, the inverse relaxation time for the 3-magnon confluence processes is roughly proportional to k and, since we neglected here the effect described in [12], vanishes in the limit $k \to 0$. For small wave-

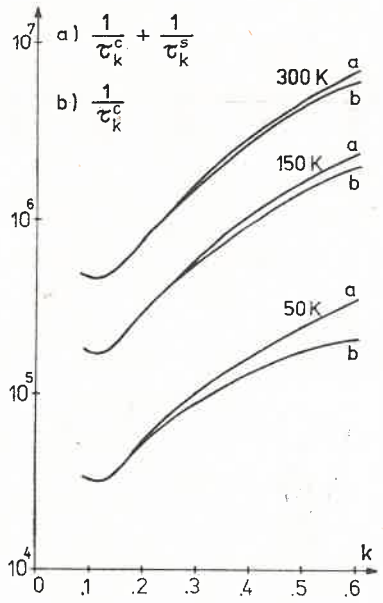

Fig. 6. The total inverse relaxation time for 3-magnon processes, $1/\tau_k^c + 1/\tau_k^s$, versus k (curves a), compared with the dominant contribution from confluence processes (curves b).

-vectors, say $k \lesssim 0.1$ (the wave-vector is measured in units of the inverse lattice constant), the wave-vector dependence of $1/\tau_k^c$ is similar to the one due to dipolar interaction (cf. [3]). For larger k, $1/\tau_k^c$ increases with increasing k, so the spin-orbit relaxation mechanism begins to dominate over the dipolar one.

The temperature dependence of the 3-magnon confluence relaxation time for different wave-vector is illustrated in Figs 3.

The 3-magnon splitting processes are effective above the critical wave-vector $k_c = (E_0/D)^{1/2} = 0.059$. The inverse relaxation time $1/\tau_k^s$, due to these processes, is depicted

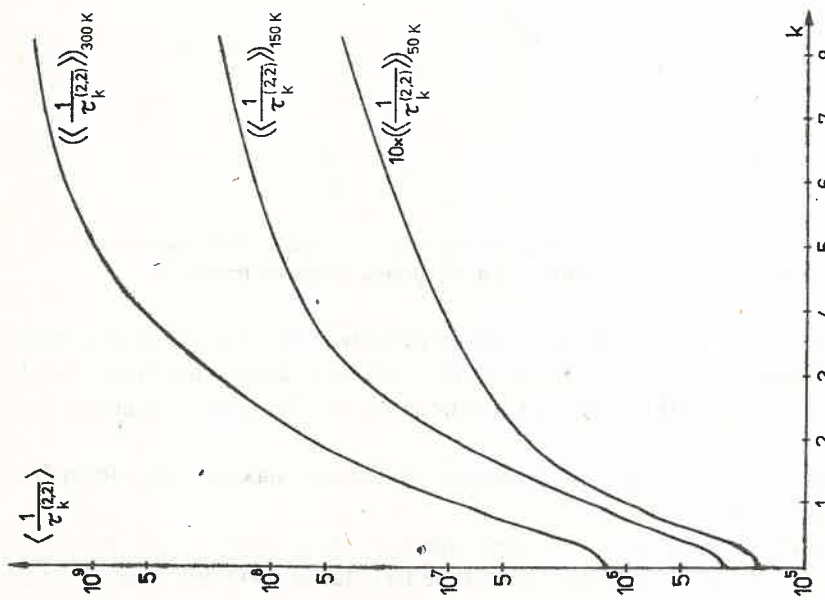

Fig. 7

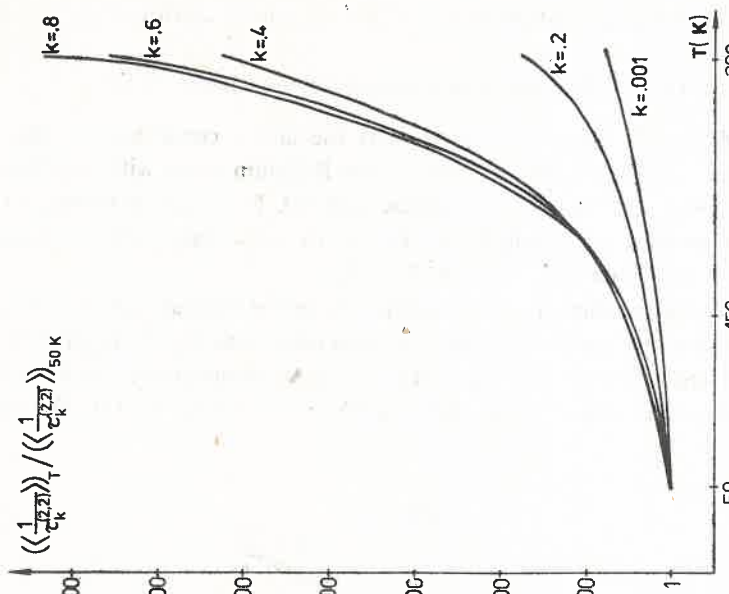

Fig. 8

Fig. 7. The inverse relaxation time for 4-magnon (two-in two-out) scattering processes versus k, for various temperatures
Fig. 8. Temperature dependence of 4-magnon relaxation time

in Fig. 4. The temperature dependence of the 3-magnon splitting relaxation time is presented in Fig. 5 for several values of the wave-vector. For a given wave-vector and temperature the inverse relaxation time for the splitting processes is much smaller than for the confluence ones, the contribution of the splitting processes to the 3-magnon inverse relaxation time is negligible for all but large wave-vectors as shown in Fig. 6.

The dependence of the 4-magnon relaxation time on wave-vector is illustrated in Fig. 7. The inverse relaxation time is finite for $k = 0$. The large values of $\langle 1/\tau_k^{(2,2)} \rangle$ for large k account for strong interaction of short-wavelengths magnons due to the Coulomb coupling. The temperature dependence of $\langle 1/\tau_k^{(2,2)} \rangle$ is shown in Fig. 8.

The total inverse relaxation time $1/\tau_k = 1/\tau_k^c + 1/\tau_k^s + \langle 1/\tau_k^{(2,2)} \rangle$ for the 3- and 4-magnon processes versus wave-vector k is plotted in Fig. 9. The peak for low k results from 3-magnon confluence processes.

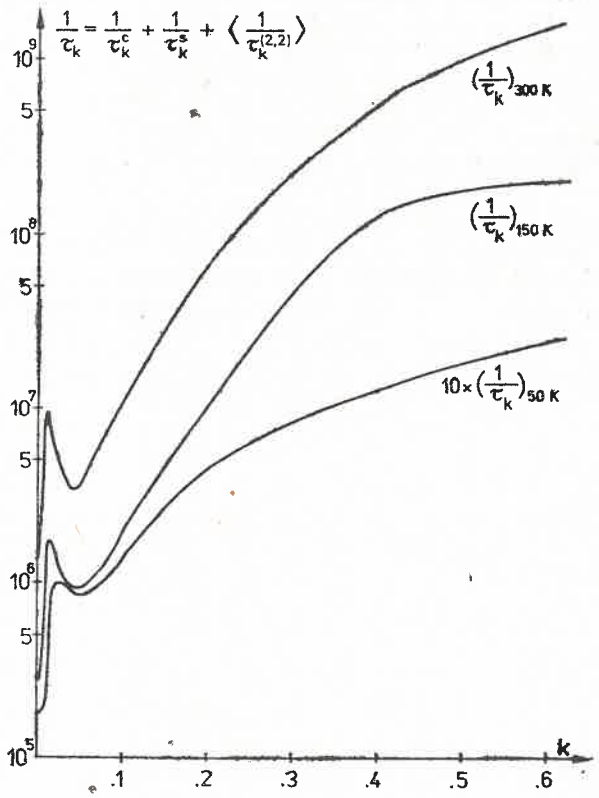

Fig. 9. The dependence of the total relaxation time, $1/\tau_k = 1/\tau_k^c + 1/\tau_k^s + \langle 1/\tau_k^{2,2} \rangle$ for 3- and 4-magnon processes on the wave-vector, for several temperatures

In estimating the numerical accuracy of the numbers obtained for the inverse relaxation time we have to remember that, apart from the fact that real band structure of nickel is approximated by a model one, the final results depend upon several parameters, notably ξ and γ, which can be determined only with rather low accuracy.

The qualitative aspects of the wave-vector and temperature dependence of the relaxation due to the spin-orbit interaction in itinerant electron ferromagnets are similar to the ones of the relaxation by pseudo-dipolar forces in the Heisenberg type ferromagnets, as discussed in [7], [1] and [2].

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