THE LINEAR COSSERATS THEORY DESCRIPTION OF THE MAGNETOELASTIC INTERACTIONS IN UNIAXIAL CRYSTALS

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The potential energy density, equations of motion, strain and stress tensor definitions, constitutive equations, and boundary conditions have been formulated for deformable ferromagnetic crystals in the linear Cosserats theory terms. The uniaxial material tensors dependence on the magnetic field and the magnetization of the equilibrium state has been shown. The coupled magnetoelastic wave propagation has been considered and the dispersion relation and the resonance conditions have been obtained.

1. Basic equations

The coupled magnetoelastic equations of motion are usually written in the following way [1]

$$\dot{p}_i = T_{ki,k},$$

$$\frac{1}{g}\dot{m}_i = \varepsilon_{ijk} M_j^0 \tilde{h}_k, \tag{1.1}$$

where p_i is a momentum density, the stress tensor T_{ki} is a sum of the mechanical and Maxwell stress tensors

$$T_{ki} = t_{ki} + M_k^0 h_i,$$

$$t_{ki} = \frac{\partial F^*}{\partial u_{i,k}}\,. \tag{1.2}$$

g is a magnetomechanical coefficient, M_j^0 — magnetization of equilibrium state, $m_i = M_i - M_i^0$ and $\tilde{h}_i = \tilde{H}_i - \tilde{H}_i^0$ are small deviations of magnetization and effective

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magnetic field, respectively, and F^* is a potential energy of ferromagnetic crystal. The field $\tilde{h}_i$ consists of two parts

$$\tilde{h}_i = h_i - \frac{\delta F^*}{\delta m_i}, \tag{1.3}$$

where $h_i = H_i - H_i^0$ is a deviation of internal magnetic field. The constitutive equations i. e.

$$t_{ki} = t_{ki}(u_{i,k};\, m_i;\, m_{i,k}),$$
$$\tilde{h}_i = \tilde{h}_i(h_i;\, u_{i,k};\, m_i;\, m_{i,k}),$$

are rather complicated. This is the reason to present the other form of motion and constitutive equations. We are going to make use of new variable θ_i in place of m_i, where θ_i is an angle describing small deviations of magnetization:

$$m_i = \varepsilon_{ijk}\theta_j M_k^0. \tag{1.4}$$

In this manner the basic equations will have the linear Cosserats theory form [2]. Let us start from the deformable ferromagnetic crystal energy density which depends on the magnetization, magnetization gradients and displacement gradients

$$F = F(M_i;\, M_{i,j};\, u_{i,j}). \tag{1.5}$$

If we admit the small deviation of displacement and magnetization, we can express the energy in shape of Taylor series

$$F = F^0 - b_k^0 m_k + t_{ki}^0 u_{i,k} + F^*, \tag{1.6}$$

where

$$F^* = \tfrac{1}{2}\, a_{kilj} u_{i,k} u_{j,l} + c_{klj} m_k u_{j,l} + \tfrac{1}{2}\, d_{kl} m_k m_l + \tfrac{1}{2}\, \alpha_{kilj} m_{i,k} m_{j,l}. \tag{1.6a}$$

We assume that in the equilibrium state the displacement gradients are constants and the magnetization gradients are equal to zero. The partial derivative $(\partial F/\partial u_{i,k})_0 = t_{ki}^0$ is an initial stress tensor and derivative $-(\partial F/\partial m_i)_0 = b_i^0$ is part of the effective field $\tilde{H}_i^0$, i. e.

$$b_i^0 = \tilde{H}_i^0 - H_i^0. \tag{1.7}$$

The pseudotensors which are equal to zero for crystals with central symmetry have been omitted. The rigid rotation of crystal implies that the generalized forces $\partial F/\partial u_{i,k}$ and $-\partial F/\partial m_k$ are equal to the initial stress tensor t_{ki}^0 and vector b_k^0, respectively, i. e.

$$\begin{aligned} a_{kilj} u_{j,l} + c_{jki} m_j + t_{ki}^0 &= t_{ki}^0 \\ -d_{kl} m_l - c_{klj} u_{j,l} + b_i^0 &= b_i^0, \end{aligned} \tag{1.8}$$

for

$$m_i = \varepsilon_{irk}\alpha_r M_k^0, \qquad u_i = \varepsilon_{irk}\alpha_r x_k, \tag{1.9}$$

where α_r is a small rigid rotation pseudovector. By means of (1.8) we can obtain two constraints between antisymmetric parts of material tensors

$$a_{ki\langle lj\rangle} = c_{lj\langle ki\rangle}, \quad d_{ki\langle lj\rangle} = c_{ki\langle lj\rangle}, \tag{1.10}$$

where

$$c_{kilj} = c_{klj}M_i^0, \quad d_{kilj} = d_{kl}M_i^0 M_j^0. \tag{1.11}$$

By means of these constraints we can get the following form of potential energy F^*

$$F^* = \tfrac{1}{2}\, a_{kilj}\varepsilon_{ki}\varepsilon_{lj} + \tfrac{1}{2}\, b_{kilj}\kappa_{ki}\kappa_{lj}, \tag{1.12}$$

where

$$\varepsilon_{ki} = u_{i,k} - \varepsilon_{kir}\theta_r, \quad \kappa_{ki} = \theta_{i,k} \tag{1.13}$$

are nonsymmetric generalized strain tensors. After taking into account that

$$\varepsilon_{ijk}M_j^0 \frac{\delta F^*}{\delta m_k} = \frac{\partial F^*}{\partial \theta_i} - \left(\frac{\partial F^*}{\partial \theta_{i,k}}\right)_{,k}, \tag{1.14}$$

we can present equations (1.1) in the following manner

$$\dot{p}_i = T_{ki,k}, \quad \frac{1}{g}\dot{m}_i = \varepsilon_{ijk}T_{jk} + N_{ki,k}. \tag{1.15}$$

The stress tensor

$$T_{ki} = t_{ki} + M_k^0 h_i, \tag{1.16}$$

where

$$t_{ki} = \frac{\partial F^*}{\partial u_{i,k}} = a_{kilj}\varepsilon_{lj}, \tag{1.16a}$$

and the couple stress tensor

$$N_{ki} = \frac{\partial F^*}{\partial \kappa_{ki}} = b_{kilj}\kappa_{lj} \tag{1.17}$$

are nonsymmetric tensors. The two equations of motion should be supplied by Maxwell equations (magnetostatic case)

$$h_{i,i} + 4\pi\varepsilon_{ijk}\kappa_{ij}M_k^0 = 0, \quad h_i = -\varphi_{,i}. \tag{1.18}$$

The scalar φ is a magnetostatic potential. The following boundary conditions should be in force on the free crystals surfaces S

$$n_k' h_k|_{S_+} = n_k'(h_k + 4\pi\varepsilon_{ijk}\theta_j M_k^0)|_{S_-},$$

$$h_{\text{tang}}|_{S_+} = h_{\text{tang}}|_{S_-}, \quad n_k' T_{ki}|_S = 0, \quad n_k' N_{ki}|_S = 0, \tag{1.19}$$

where n'_k is a unit vector, perpendicular to the surface S. The last condition is equivalent to [1]

$$n'_k \varepsilon_{krs} \frac{\partial F^*}{\partial m_{i,s}} M^0_r|_S = 0.$$

2. *The material tensors for uniaxial crystals*

The energy definition (1.6a) implies the following symmetry of tensor a_{kilj}

$$a_{kilj} = \left(\frac{\partial^2 F}{\partial u_{i,k} \partial u_{j,l}}\right)_{u=u^0} = a_{ljki}. \tag{2.1}$$

We assume that for uniaxial crystals this tensor is a function of the symmetry axis unit vector n_i and the magnetization pseudovector M^0_i. We also assume that this tensor is made of two parts

$$a_{kilj}(\vec{n}, \vec{M}^0) = \lambda_{kilj}(\vec{n}) + \gamma_{kilj}(\vec{n}, \vec{M}^0), \tag{2.2}$$

where $\lambda_{kilj} = \lambda_{ljki} = \lambda_{jlki}$ is a pure elastic symmetric material tensor and $\gamma_{kilj} = \gamma_{ljki}$ is an anisotropy and magnetoelastic material tensor. After taking into account the symmetry relations we can obtain the following forms of these tensors

$$\begin{aligned} \lambda_{kilj} = {} & \lambda_1(n_k n_i \delta_{lj} + n_l n_j \delta_{ki}) \\ & + \lambda_2(n_k n_l \delta_{ij} + n_k n_j \delta_{il} + n_i n_l \delta_{kj} + n_i n_j \delta_{kl}) \\ & + \lambda_3 \delta_{ki} \delta_{lj} \\ & + \lambda_4(\delta_{kl}\delta_{ij} + \delta_{il}\delta_{kj}) \\ & + \lambda_5 n_k n_i n_l n_j, \\ \gamma_{kilj} = {} & \gamma_1(\delta_{kl} M^0_i M^0_j - \delta_{ij} M^0_k M^0_l) \\ & + \gamma_2(n_k n_l M^0_i M^0_j - n_i n_j M^0_k M^0_l) \\ & + \gamma_3(\delta_{lj} M^0_k M^0_i + \delta_{ki} M^0_l M^0_j) \\ & + \gamma_4(n_l n_j M^0_k M^0_i + n_k n_i M^0_l M^0_j) \\ & + \beta_1 \delta_{kl} M^0_i M^0_j \\ & + \beta_2 n_k n_l M^0_i M^0_j \\ & + \beta_3 M^0_k M^0_i M^0_l M^0_j. \end{aligned} \tag{2.3}$$

Since the terms as $n_k M^0_i M^0_l M^0_j$ or $M^0_k n_i n_l n_j$ are pseudotensors they must vanish for crystals with central symmetry. The coefficients $\beta_1, \beta_2, \beta_3$ are those of expansion of tensor (see: (1.10))

$$d_{kl} = \beta_1 \delta_{kl} + \beta_2 n_k n_l + \beta_3 M^0_k M^0_l. \tag{2.4}$$

If we adopt a typical assumption [1] that the energy F^0 is a sum of the anisotropy term F^a and isotropy term $f[(M^0)^2]$, and if we take the anisotropy term in the form

$$F^a = -\tfrac{1}{2}\beta(\vec{M}^0\cdot\vec{n})^2, \tag{2.5}$$

we find that

$$\frac{\partial^2 F^0}{\partial M_k^0 \partial M_l^0} = 2f'\delta_{kl} - \beta n_k n_l + 4f'' M_k^0 M_l^0. \tag{2.6}$$

Since

$$d_{kl} = \left(\frac{\partial^2 F}{\partial m_k \partial m_l}\right)_0 = \frac{\partial^2 F^0}{\partial M_k^0 \partial M_l^0}, \tag{2.7}$$

hence, from (2.4) we can obtain

$$\beta_1 = 2f', \quad \beta_2 = -\beta, \quad \beta_3 = 4f'', \tag{2.8}$$

where $f' = df/d(M^0)^2$ and β is an anisotropy constant. The constant β_1 can be calculated from the equilibrium condition [1]

$$\tilde{H}_i^0 = H_i^0 - \frac{\delta F^0}{\delta M_i^0} = H_i^0 + \beta M_k^0 n_k n_i - 2f' M_i^0 = 0. \tag{2.9}$$

Hence,

$$\beta_1 = 2f' = \frac{\beta(\vec{M}^0\cdot\vec{n}) + \vec{H}^0\cdot\vec{M}^0}{(M^0)^2}. \tag{2.10}$$

The second material tensor

$$b_{kilj} = \varepsilon_{irq}\varepsilon_{jsp} M_r^0 M_s^0 \alpha_{kqlp} \tag{2.11}$$

characterizes the isotropic exchange energy. It means that the tensor α_{kqlp} ought to be in the form

$$\alpha_{kqlp} = \alpha_{kl}\delta_{qp}. \tag{2.12}$$

Hence,

$$b_{kilj} = [(M^0)^2\delta_{ij} - M_i^0 M_j^0]\alpha_{kl}. \tag{2.13}$$

The symmetric tensor α_{kl} should have two independent coefficients for uniaxial crystals

$$\alpha_{kl} = \alpha\delta_{kl} + \alpha' n_k n_l. \tag{2.14}$$

The aproximative assumption that $\alpha \cong \alpha'$ is usually made.

3. Magnetoelastic wave propagation

Let us consider the coupled magnetoelastic wave propagation in the (x_1, x_3) plane. Let us assume that the anisotropy axis has x_3 direction and the vectors $\vec{M}^0$ and $\vec{H}^0$ point in the x_2 direction. In this case we have three displacement equations of motion

$$\varrho\ddot{u}_1 = t_{11,1}+t_{31,3}, \quad \varrho\ddot{u}_2 = t_{12,1}+t_{32,3}, \quad \varrho\ddot{u}_3 = t_{13,1}+t_{33,3}, \tag{3.1}$$

two angle equations of motion

$$-\frac{M^0}{g}\dot{\theta}_3 = -M^0\varphi_{,3}+t_{23}-t_{32}+N_{11,1}+N_{31,3},$$

$$\frac{M^0}{g}\dot{\theta}_1 = M^0\varphi_{,1}+t_{12}-t_{21}+N_{13,1}+N_{33,3}, \tag{3.2}$$

and one Maxwell equation

$$\varphi_{,11}+\varphi_{,33} = 4\pi M^0(\theta_{1,3}-\theta_{3,1}). \tag{3.3}$$

The stress tensor components have the form

$$\begin{aligned}
t_{11} &= (\lambda_3+2\lambda_4)\varepsilon_{11}+(\lambda_3+\gamma_3M_0^2)\varepsilon_{22}+(\lambda_1+\lambda_3)\varepsilon_{33},\\
t_{12} &= \lambda_4(\varepsilon_{12}+\varepsilon_{21})+(\gamma_1+M_0^2\beta_1)\varepsilon_{12},\\
t_{13} &= (\lambda_2+\lambda_4)(\varepsilon_{13}+\varepsilon_{31}),\\
t_{21} &= \lambda_4(\varepsilon_{12}+\varepsilon_{21})-\gamma_1M_0^2\varepsilon_{21},\\
t_{22} &= (\lambda_3+\gamma_3M_0^2)\varepsilon_{11}+[\lambda_3+2\lambda_4+(2\gamma_3+\beta_1+\beta_3)M_0^2]\varepsilon_{22}\\
&\quad +[\lambda_1+\lambda_3+(\gamma_3+\gamma_4)M_0^2]\varepsilon_{33},\\
t_{23} &= (\lambda_2+\lambda_4)(\varepsilon_{23}+\varepsilon_{32})-(\gamma_1+\gamma_2)M_0^2\varepsilon_{23},\\
t_{31} &= (\lambda_2+\lambda_4)(\varepsilon_{31}+\varepsilon_{13}),\\
t_{32} &= (\lambda_2+\lambda_4)(\varepsilon_{32}+\varepsilon_{23})+(\gamma_1+\beta_1+\gamma_2+\beta_2)M_0^2\varepsilon_{32},\\
t_{33} &= (\lambda_1+\lambda_3)\varepsilon_{11}+[\lambda_1+\lambda_3+(\gamma_3+\gamma_4)M_0^2]\varepsilon_{22}\\
&\quad +(2\lambda_1+4\lambda_2+\lambda_3+2\lambda_4+\lambda_5)\varepsilon_{33},
\end{aligned} \tag{3.4}$$

and the couple stress tensor have the form

$$\begin{aligned}
N_{11} &= \alpha M_0^2\theta_{1,1}, & N_{12} &= 0, & N_{13} &= \alpha M_0^2\theta_{3,1},\\
N_{21} &= \alpha M_0^2\theta_{1,2}, & N_{22} &= 0, & N_{23} &= \alpha M_0^2\theta_{3,2},\\
N_{31} &= \alpha M_0^2\theta_{1,3}, & N_{32} &= 0, & N_{33} &= \alpha M_0^2\theta_{3,3}.
\end{aligned} \tag{3.5}$$

If we assume the following form of solutions

$$\begin{pmatrix} u_j \\ \theta_l \\ \varphi \end{pmatrix} = \begin{pmatrix} u_j^0 \\ \theta_l^0 \\ \varphi^0 \end{pmatrix} e^{i(K_1x_1+K_3x_3-\omega t)}, \quad j = 1, 2, 3, \quad l = 1, 3 \tag{3.6}$$

and if we eliminate the potential φ from Eqs (3.2) we can obtain two sets of equations

$$[-\omega^2\varrho+K_1^2(\lambda_3+2\lambda_4)+K_3^2(\lambda_2+\lambda_4)]u_1+K_1K_3(\lambda_1+\lambda_2+\lambda_3+\lambda_4)u_3 = 0$$

$$K_1K_3(\lambda_1+\lambda_2+\lambda_3+\lambda_4)u_1+[-\omega^2\varrho+K_1^2(\lambda_2+\lambda_4)$$
$$+K_3^2(2\lambda_1+4\lambda_2+\lambda_3+2\lambda_4+\lambda_5)]u_3 = 0, \tag{3.7}$$

$$(-\omega^2\varrho+\omega'^2\varrho)u_2+iK_1BM_0^2\theta_3-iK_3AM_0^2\theta_1 = 0,$$

$$iK_3AM_0^2u_2+\left(\frac{i\omega M_0}{g}-\frac{4\pi M_0^2K_1K_3}{K^2}\right)\theta_3$$
$$+\left(D+\alpha K^2+\frac{4\pi K_3^2}{K^2}\right)M_0^2\theta_1 = 0,$$

$$iK_1BM_0^2u_2-\left(\beta_1+\alpha K^2+\frac{4\pi K_1^2}{K^2}\right)M_0^2\theta_3$$
$$+\left(\frac{i\omega M_0}{g}+\frac{4\pi M_0^2K_1K_3}{K^2}\right)\theta_1 = 0, \tag{3.8}$$

where

$$\omega'^2\varrho = K_1^2(\lambda_4+BM_0^2)+K_3^2(\lambda_2+\lambda_4+AM_0^2),$$

$$A = \gamma_1+\gamma_2+\beta_1+\beta_2, \quad B = \gamma_1+\beta_1,$$

$$D = 2\gamma_1+2\gamma_2+\beta_1+\beta_2, \quad \beta_1 = \frac{H_0}{M_0}, \quad \beta_2 = -\beta, \quad K^2 = K_1^2+K_3^2. \tag{3.9}$$

The spin wave is coupled with elastic wave u_2 only. If the coupling terms A and B were equal to zero the set of Eqs (3.8) would describe the noninteracting waves u_2 and (θ_1, θ_3). The spin wave frequency would be in the form [3]

$$\omega_s^2 = \frac{g^2M_0^2}{K^2}\{4\pi[K_3^2(\beta_1+\alpha K^2)+K_1^2(D+\alpha K^2)+(\beta_1+\alpha K^2)(D+\alpha K^2)K^2\}. \tag{3.10}$$

The system (3.8) describing the coupled magnetoelastic waves has the solutions if the determinant

$$\begin{vmatrix} -\omega^2\varrho+\omega'^2\varrho, & iK_1BM_0^2, & -iK_1AM_0^2 \\ iK_3AM_0^2, & \dfrac{i\omega M_0}{g}-\dfrac{4\pi M_0^2K_1K_3}{K^2}, & \left(D+\alpha K^2+\dfrac{4\pi K_3^2}{K^2}\right)M_0^2 \\ iK_1BM_0^2, & -\left(\beta_1+\alpha K^2+\dfrac{4\pi K_1^2}{K^2}\right)M_0^2, & \dfrac{i\omega M_0}{g}+\dfrac{4\pi M_0^2K_1K_3}{K^2} \end{vmatrix} \tag{3.11}$$

is equal to zero. Hence we obtain the following dispersion relation

$$\omega^4-(\omega_s^2+\omega'^2)\omega^2+\omega_s^2\omega'^2-E = 0, \tag{3.12}$$

where

$$E = \frac{g^2M_0^2}{\varrho}\left[4\pi K_1^2K_3^2\frac{(A-B)^2}{K^2}+B^2(\alpha K^2+D)+A^2(\alpha K^2+\beta_1)\right]. \tag{3.13}$$

The bisquare solutions of (3.12) are as follows

$$\omega_\pm^2 = \tfrac{1}{2}\{\omega_s^2+\omega'^2\pm[(\omega_s^2-\omega'^2)^2+4E]^{1/2}\}. \tag{3.14}$$

From Eqs (3.8) one can also obtain the following resonance constraints between the displacement and spin waves amplitudes

$$|u_2| = \frac{[A^2M_0^2\omega^2K_3^2/g^2P^2+(BM_0^2K_1-4\pi AM_0^2K_3^2K_1/K^2P)^2]^{1/2}}{-\omega^2\varrho+\omega'^2\varrho-A^2M_0^2K_3^2/P}|\theta_3|,$$

$$|u_2| = \frac{[B^2M_0^2\omega^2K_1^2/g^2Q^2+(AM_0^2K_3-4\pi BM_0^2K_1^2K_3/K^2Q)^2]^{1/2}}{-\omega^2\varrho+\omega'^2\varrho-B^2M_0^2K_1^2/Q}|\theta_1|,$$

$$|\theta_1| = \frac{M_0^5[B^2\omega^2K_1^2/g^2+M_0^2(4\pi BK_1^2K_3/K^2-AQK_3)^2]^{1/2}}{g^2(\omega^2-\omega_s^2)}|u_2|,$$

$$|\theta_3| = \frac{M_0^5[A^2\omega^2K_3^2/g^2+M_0^2(-4\pi AK_3^2K_1/K^2+BPK_1)^2]^{1/2}}{g^2(\omega^2-\omega_s^2)}|u_2|, \tag{3.15}$$

where

$$P = D+\alpha K^2+4\pi K_3^2/K^2,$$

$$Q = \beta_1+\alpha K^2+4\pi K_1^2/K^2. \tag{3.16}$$

The dispersion relation (3.12) and constraints (3.15) can be used in discussion of the parametric magnetoelastic resonance excitation in thin deformable ferromagnetic film. We are going to present this phenomenon in the next paper.

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