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REMARKS ON THE STABILITY CONDITIONS FOR A FERROMAGNETIC FERMI LIQUID

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The stability conditions for a normal ferromagnetic Fermi liquid are discussed. The simple model of band ferromagnetism with spherical Fermi surfaces is used. The inequalities guaranteeing the stability of the ferromagnetic Fermi liquid are derived. These inequalities are different from those obtained by Czerwonko using Leggett's method.

The problem of stability conditions for a normal paramagnetic Fermi liquid was considered originally by Pomeranchuk [1] using a phenomenological Landau approach. The microscopic proof of those conditions was given by Leggett [2]. Czerwonko [3] extended Leggett's results for the case of a normal ferromagnetic Fermi liquid. Recently the same author [4] in an independent microscopic approach got from the stability conditions for a paramagnetic liquid a sequence of inequalities different from those obtained by Leggett [2]. An analysis of these inequalities shows that they are fulfilled only when the Pomeranchuk inequalities hold. In paper [4] it was also suggested that the equivalence, from the point of view of the inequalities between Leggett's and the author's approaches, is related to the specific spin symmetry of the effective interaction for paramagnetic liquids. This symmetry causes the spin matrix of an effective interaction for different l to commute.

The purpose of the present paper is to examine the stability conditions for the normal ferromagnetic Fermi liquids. We consider a simple model of band ferromagnetism. It will be assumed that we have a single band and that the Fermi surfaces are spherical for both spins. In this case the autocorrelation functions transform similarly as in paper [4]. According to [4], the autocorrelation function, for an arbitrary vertex can be represented for $|\omega|$, $kv_\alpha \ll E_F$, where E_F denotes the Fermi energy and v_α — the velocity of particles with spin α on the Fermi sphere, as follows for details see e.g. [2] and [5]):

$$K_\xi(\vec{k}\omega) = K_\xi^\omega - \sum_{\alpha\beta} \sum_{\vec{p}} \xi^{\omega *}_{\vec{p},\alpha} \delta^{\alpha}_{\vec{p}}(\vec{k}\omega) \left[\delta^{\alpha\beta}_{\vec{p},\vec{p}'} - \sum_{\vec{p}'} f^{\alpha\beta}_{\vec{p},\vec{p}'}(\vec{k}\omega) \times \delta^{\beta}_{\vec{p}'}(\vec{k}\omega)\right] \xi^{\omega}_{\vec{p}',\beta}, \tag{1}$$

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where K_ξ^ω is the nonquasiparticle part of the autocorrelation function, which is equal to the (uncommutative) "ω limit" $\lim_{\omega\to 0}\lim_{k\to 0}$ of $K_\xi(\vec{k}\omega)$

$$\delta_{\vec{p}}^\alpha(\vec{k}\omega) = a_\alpha^2\delta(E_{\vec{p}}^\alpha - E_F)v_\alpha(\hat{\vec{p}}\cdot\vec{k})\,[v_\alpha(\hat{\vec{p}}\cdot\vec{k})-\omega]^{-1}.$$

$E_{\vec{p}}^\alpha$ — denotes energy of a particle with momentum $\vec{p}$ and spin α, $\hat{\vec{p}}$ — the unit vector directed along the momentum $\vec{p}$, δ — the Dirac delta function, $\xi_{\vec{p},\alpha}^\omega$ — the vertex ξ in the ω-limit, $f_{\vec{p},\vec{p}'}^{\alpha\beta}(\vec{k}\omega)$ — the four point function for the energy–momentum transfer ω, $\vec{k}$, a_α — the discontinuity of the density of particles with the spin α on the Fermi sphere.

Taking into account the relations between the functions $f_{\vec{p},\vec{p}'}^{\alpha\beta}$, and $\xi_{\vec{p},\alpha}$, and their "ω limit" respectively, which are analogous to those given in [4], and choosing ξ as a linear combination of normalized spherical functions $Y_{lm}(\vec{p})$:

$$\xi_{\vec{p},\alpha} = \sum_l y_l^\alpha f_l^\alpha(\vec{p})Y_{lm}(\vec{p}) \tag{2}$$

we obtain instead of (1) the following equation:

$$K_\xi(\vec{k},\omega) = K_\xi^\omega + \sum_{\alpha\beta}\sum_{ll'}\frac{v_\alpha^{\frac{1}{2}}v_\beta^{\frac{1}{2}}a_\alpha a_\beta y_l^\alpha y_{l'}^\beta}{a_l^\alpha a_{l'}^\beta}\{\langle Y_{lm}^*(\vec{p})Q(\vec{k}\omega)$$
$$\times[1-FQ(\vec{k},\omega)]^{-1}Y_{l'm}(\vec{p}')\rangle_{\vec{p}\vec{p}'}\}_{\alpha\beta}, \tag{3}$$

where v_α denotes a density of states on the Fermi surface of the α-th spin, $\langle g(\vec{p},\vec{p}')\rangle_{\vec{p}\vec{p}'} = \int\frac{d\Omega}{4\pi}\int\frac{d\Omega'}{4\pi}g(\vec{p},\vec{p}')$, and the operator $Q(\vec{k},\omega)$ is defined as follows:

$$Q(\vec{k},\omega)g_\alpha(\vec{p}) = \{v_\alpha(\hat{\vec{p}}\cdot\vec{k})\,[\omega - v_\alpha(\hat{\vec{p}}\cdot\vec{k})]^{-1}\}g_\alpha(\vec{p}).$$

The operator F describes an effective interaction between quasiparticles. For the considered model, F can be expanded into a series of the Legendre polynomials:

$$F(\vec{p},\vec{p}') = \begin{bmatrix} A & B \\ B & C\end{bmatrix} = \sum_l (2l+1)P_l(\hat{\vec{p}}\cdot\hat{\vec{p}}')\begin{bmatrix} A_l & B_l \\ B_l & C_l\end{bmatrix}. \tag{4}$$

A_l, B_l, C_l are called the Landau parameters for a ferromagnetic liquid. Without any loss of generality we can choose $\vec{k}$ along the z-th axis, in such a case $(\hat{\vec{p}}\cdot\vec{k}) = kz$. The multiplication by z has an operatorial character. Such a choice will be applied by us in the subsequent part of this paper.

Introducing the notion

$$\frac{v_\alpha^{\frac{1}{2}}a_\alpha y_l^\alpha}{a_l^\alpha} = x_l^\alpha, \quad Q[1-FQ]^{-1} = W, \tag{5}$$

Eq. (3) can be written as follows:

$$K_\xi(\vec{k},\omega) = K_\xi^\omega + \sum_{\alpha\beta}\sum_{ll'}x_l^\alpha x_{l'}^\beta\{\langle Y_{lm}^*(\vec{p})WY_{l'm}(\vec{p}')\rangle_{\vec{p}\vec{p}'}\}_{\alpha\beta}. \tag{6}$$

The operator W is a square matrix, with the elements:

$$W_{\alpha\beta} = \frac{Rv_\alpha z}{1-Rv_\alpha z}\left[1-F\,\frac{Rvz}{1-Rvz}\right]^{-1}_{\alpha\beta}, \tag{7}$$

where $R = k/\omega$. Expanding $W_{\alpha\beta}$ into a power series with respect to Rv_α we obtain

$$W_{\alpha\beta} = \sum_s R^{2s}\{\sum_{\beta_1,\beta_2,\ldots,\beta_{2s-1}} z^{2s}v_\alpha^{2s} + z^{2s-1}v_\alpha^{2s-1}Fzv_\beta + z^{2s-2}v_\alpha^{2s-2}Fzv_{\beta_1}(1+F)zv_\beta$$
$$+ \cdots + zv_\alpha Fzv_{\beta_{2s-1}}(1+F)zv_{\beta_{2s-2}}\cdots v_{\beta_1}(1+F)zv_\beta\} \equiv \sum_s R^{2s}W^{(s)}. \tag{8}$$

The odd terms with respect to R vanish as a result of the inversion invariance of an auto-correlation function. One can easily get the operator W for different s. Using (4) we have for $s = 1$ in matrix form

$$W^{(1)} = v_1^2 z\begin{bmatrix}1+A & qB\\ qB & q^2(1+C)\end{bmatrix}z, \quad \text{where} \quad q = \frac{v_{\text{I}}}{v_1}. \tag{9}$$

In the same way, after simple but long and tedious calculations, we obtain foı $s = 2$

$$W^{(2)} = v_1^4 z\begin{bmatrix}1+A & qB\\ qB & q^2(1+C)\end{bmatrix}z\begin{bmatrix}1+A & B\\ B & 1+C\end{bmatrix}z\begin{bmatrix}1+A & qB\\ qB & q^2(1+C)\end{bmatrix}z$$
$$\equiv v_1^4 zw^q zwzw^q z. \tag{10}$$

According to [4], and using (6), (8) the quadratic form

$$\sum_{\alpha\beta}\sum_{ll'} x_l^\alpha x_{l'}^\beta R^{2s}\{\langle Y_{lm}^*(\vec{p})W^{(s)}Y_{l'm}(\vec{p}')\rangle_{\vec{p}\vec{p}'}\}_{\alpha\beta} \tag{11}$$

have to be positive definite for every s, l, l' and m.

In further calculations the following recurrence formula [4, 6] will be used:

$$zY_{lm}(\hat{\vec{p}}) = b_{l,m}Y_{l+1,m}(\hat{\vec{p}}) + b_{l-1,m}Y_{l-1,m}(\hat{\vec{p}}), \tag{12}$$

where

$$b_{l,m} = \left[\frac{(l+1)^2-m^2}{(2l+1)(2l+3)}\right]^{\frac{1}{2}}. \tag{13}$$

It is seen from (11), (12) and (13) that m will occur in the inequalities only in the coefficients $b_{l,m}$ because these inequalities will be considered for fixed m, the index m in the subsequent formulae will be omitted. Taking into account (4), (12) and (14), and the addition theorem for spherical functions we obtain instead of (11) the following expression for $s = 1$:

$$\sum_l\sum_{\alpha\beta} v_1^2R^2\left\{b_{l-1}b_{l-2}x_l^\alpha x_{l-2}^\beta\begin{bmatrix}1+A_{l-1} & qB_{l-1}\\ qB_{l-1} & q^2(1+C_{l-1})\end{bmatrix}\right.$$
$$+x_l^\alpha x_l^\beta\left[b_l^2\begin{bmatrix}1+A_{l+1} & qB_{l+1}\\ qB_{l+1} & q^2(1+C_{l+1})\end{bmatrix}+b_{l-1}^2\begin{bmatrix}1+A_{l-1} & qB_{l-1}\\ qB_{l-1} & q^2(1+C_{l-1})\end{bmatrix}\right]$$
$$\left.+b_{l+1}b_{l+2}x_l^\alpha x_{l+2}^\beta\begin{bmatrix}1+A_{l+1} & qB_{l+1}\\ qB_{l+1} & q^2(1+C_{l+1})\end{bmatrix}\right\}. \tag{14}$$

After simple calculations we find that the quadratic form (14) is positive definite if the following inequalities are fulfilled:

$$1+A_l > 0; \quad (1+A_l)(1+C_l)-B_l^2 > 0. \tag{15}$$

These conditions are identical with those obtained by Czerwonko in [3].

In a similar way we can find for $s = 2$:

$$\begin{aligned} R^4 v_1^4 \sum_l \sum_{\alpha\beta} & x_l^\alpha x_{l-4}^\beta \{b_{l-1} b_{l-2} b_{l-3} b_{l-4} w_{l-3}^q w_{l-2} w_{l-1}^q\}_{\alpha\beta} \\ & + x_l^\alpha x_{l-2}^\beta \{b_{l-1} b_{l-2} [b_l^2 w_{l+1}^q w_l w_{l-1}^q + b_{l-1}^2 w_{l-1}^q w_l w_{l-1}^q + b_{l-2}^2 w_{l-1}^q w_{l-2} w_{l-1}^q \\ & + b_{l-3}^2 w_{l-3}^q w_{l-2} w_{l-1}^q]\}_{\alpha\beta} + x_l^\alpha x_l^\beta \{b_l^4 w_{l+1}^q w_l w_{l+1}^q + b_l^2 b_{l+1}^2 w_{l+1}^q w_{l+2} w_{l+1}^q \\ & + b_l^2 b_{l-1}^2 [w_{l-1}^q w_l w_{l+1}^q + w_{l+1}^q w_l w_{l-1}^q] + b_{l-2}^2 b_{l-1}^2 w_{l-2}^q w_{l-1} w_{l-2}^q + b_{l-1}^4 w_{l-1}^q w_l w_{l-1}^q\}_{\alpha\beta} \\ & + x_l^\alpha x_{l+2}^\beta \{b_l b_{l+1} [b_{l-1}^2 w_{l-1}^q w_l w_{l+1}^q + b_l^2 w_{l+1}^q w_l w_{l+1}^q + b_{l+1}^2 w_{l+1}^q w_{l+2} w_{l+1}^q \\ & + b_{l+2}^2 w_{l+1}^q w_{l+2} w_{l+3}^q]\}_{\alpha\beta} + x_l^\alpha x_{l+4}^\beta \{b_l b_{l+1} b_{l+2} b_{l+3} w_{l+1}^q w_{l+2} w_{l+3}^q\}_{\alpha\beta}, \end{aligned} \tag{16}$$

where

$$w_l = \begin{bmatrix} 1+A_l & B_l \\ B_l & 1+C_l \end{bmatrix}, \quad w_l^q = \begin{bmatrix} 1+A_l & qB_l \\ qB_l & q^2(1+C_l) \end{bmatrix}. \tag{17}$$

Let us consider the case $l = m = 0$. Then all terms containing b_{l-1} vanish and instead of (16) we get:

$$\begin{aligned} R^4 v_1^4 \sum_\alpha \sum_\beta & x_0^\alpha x_0^\beta b_0^2 \{b_0^2 w_1^q w_0 w_1^q + b_1^2 w_1^q w_2 w_1^q\}_{\alpha\beta} \\ & + x_0^\alpha x_2^\beta b_0 b_1 \{b_0^2 w_1^q w_0 w_1^q + b_1^2 w_1^q w_2 w_1^q + b_2^2 w_1^q w_2 w_3^q\}_{\alpha\beta} + x_0^\alpha x_4^\beta b_0 b_1 b_2 b_3 \{w_1^q w_2 w_3^q\}_{\alpha\beta}. \end{aligned} \tag{18}$$

The quadratic form (18) is positive definite if and only if all its principal minors are strictly positive. This leads to a variety of inequalities which are equivalent to the stability conditions obtained in [3], however, there appears also the inequality, for which the stability conditions [3] are insufficient. Namely, the positivity requirement of the matrix determinant in the second term of expression (18) after introducing the numerical values of the coefficients b_l and after using (17) the following inequality holds:

$$\begin{aligned} q^4[(1+A_1)(1+C_1)-B_1^2] \{ & [\tfrac{1}{3}(1+C_0)+\tfrac{4}{15}(1+C_2)] [\tfrac{1}{3}(1+A_0)+\tfrac{4}{15}(1+A_2)] \\ & -(\tfrac{1}{3} B_0+\tfrac{4}{15} B_2)^2+(\tfrac{9}{35})^2[(1+A_2)(1+C_2)-B_2^2] [(1+A_3)(1+C_3)-B_3^2] \\ & +\tfrac{12}{175} [(1+A_2)(1+C_2)-B_2^2] [(1+A_1)(1+C_3)+(1+A_3)(1+C_1)-2B_1B_3] \\ & +\tfrac{3}{35} [(1+A_1)B_3-(1+A_3)B_1] [(1+A_0)B_2-(1+A_2)B_0] \\ & +\tfrac{3}{35} [(1+C_2)B_0-(1+C_0)B_2] [(1+C_3)B_1-(1+C_1)B_3] \\ & +\tfrac{3}{35} [(1+A_0)(1+C_2)-B_0B_2] [(1+A_1)(1+C_3)-B_1B_3] \\ & +\tfrac{3}{35} [(1+A_2)(1+C_0)-B_0B_2] [(1+A_3)(1+C_1)-B_1B_3]\} > 0. \end{aligned} \tag{19}$$

Let

$$\begin{aligned} 1+A_0 &= 1-\eta, & B_0 &= \eta, & 1+C_0 &= \eta, \\ 1+A_1 &= \eta, & B_1 &= \eta, & 1+C_1 &= 2\eta, \\ 1+A_2 &= \eta, & B_2 &= \eta, & 1+C_2 &= 2\eta, \\ 1+A_3 &= 1-\eta, & B_3 &= \eta, & 1+C_3 &= \eta. \end{aligned} \tag{20}$$

From the stability conditions [3] we have

$$\left.\begin{aligned} &(1-\eta)\eta-\eta^2 > 0 \\ &2\eta^2-\eta^2 > 0 \\ &2\eta^2-\eta^2 > 0 \\ &(1-\eta)\eta-\eta^2 > 0 \\ &1-\eta > 0 \\ &\eta > 0 \end{aligned}\right\} \Rightarrow 0 < \eta < \tfrac{1}{2}. \tag{21}$$

That means that for $0 < \eta < \frac{1}{2}$ the stability conditions [3] are fulfilled. Substituting (20) into (19) we get the following inequality for η:

$$-\left[\tfrac{94}{225}+2\left(\tfrac{9}{35}\right)^2+\tfrac{36}{175}+\tfrac{3}{7}\right]\eta^2+\left[\tfrac{13}{45}+\tfrac{24}{175}+\left(\tfrac{9}{35}\right)^2+\tfrac{12}{35}\right]\eta-\tfrac{3}{25} > 0. \tag{22}$$

This inequality is fulfilled for $\frac{1}{8} < \eta < \frac{3}{5}$. Comparing this fact with (21) we see that for $0 < \eta < \frac{1}{8}$ the fulfilment of the conditions [3] does not guarantee the stability of the ferromagnetic Fermi liquid. This fact is connected with a lower symmetry of the considered system and confirms Czerwonko's suggestions [4], that the equivalence of the inequalities given in [2] and [4] arises from the specific spin symmetry of the effective interaction for a paramagnetic liquid. It is easy to prove that when we substitute the interaction (4) with the effective interaction for the paramagnetic liquid, i.e.

$$F = \sum_l (2l+1)P_l(\hat{p}\cdot\hat{p}')\begin{bmatrix} A_l-B_l & B_l \\ B_l & A_l-B_l \end{bmatrix}. \tag{23}$$

We can get inequalities identical to those obtained by Leggett [2].

It is seen from (19) that for $s = 2$ the restriction only to the terms with $l = 0$ leads to a very complicated inequality, from which we cannot obtain any direct restrictions on the Landau parameters. For a higher l, as well as for higher s, the inequalities become more complicated and so does their discussion. On the other hand, it is rather clear that these inequalities lead to restrictions on Landau parameters of our problem independent of the usual Pomeranchuk type inequalities.

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APPENDIX

We will see that for all s the operator $W^{(s)}$ (see Eq. (8)) has the form:

$$W^{(s)} = v_1^{2s} zw^q zwzw^q zw \cdots zwzw^q z, \tag{A.1}$$

where z occurs $2s$ times. The proof exploits the method of mathematical induction:

(a) For $s = 1$ and $s = 2$ it was proved (see Eqs. (9) and (10)),

(b) The element $W_{\alpha\beta}^{(s+1)}$ of the matrix $W^{(s+1)}$ can be obtained in the following way

$$W_{\alpha\beta}^{(s+1)} = v_1^2 \sum_{\gamma,\delta} (z^2 v_\alpha^2 + z^2 v_\alpha^2 F_{\alpha\delta} + zv_\alpha F_{\alpha\delta} zv_\delta + zv_\alpha F_{\alpha\gamma} zv_\gamma F_{\gamma\delta}) W_{\delta\beta}^{(s)}, \tag{A.2}$$

where $\gamma, \delta = 1$ or $\bar{1}$. It is easy to see that we can get the following relation between $W^{(s+1)}$ and $W^{(s)}$:

$$W^{(s+1)} = v_1^2 zw^q zw W^{(s)}. \tag{A.3}$$

In order to check it one has to find from (A.2) all elements of the matiix $W^{(s+1)}$. This ends our proof.

The form (A.1) of the operator $W^{(s)}$ is very convenient. Taking into account (12) and (13) it is clear that expression (11), for every s, is a sum of products of matrices. In each of the products we have $2s-1$ matrices of the type (10) and Landau parameters with suitable l appear in these matrices. Each of the products appear with a certain coefficient. Such a shape of the quadratic form (11) allows us to obtain immediately the necessary conditions of the stability of the ferromagnetic liquid. Namely, from the condition that for every s, l and l' (11) have to be positive definite, we can get that each of the matrices has to be positive definite, which means

$$1 + A_l > 0; \quad (1 + A_l)(1 + C_l) - B_l^2 > 0 \tag{A.4}$$

for every l. These conditions are identical to those obtained by Czerwonko in [3]. However, as was shown above, these conditions are not sufficient. It is a consequence of the fact that the sum of positive definite matrices does not have to be a positive definite matrix.

The form (A.1) of the operator $W^{(s)}$ allow an immediate proof of the Pomeranchuk inequalities. This proof is a generalization of Czerwonko's proof [4] for $s = 1$ and $s = 2$. We consider a paramagnetic liquid. Let $\tilde{F}$ denote exchange or the direct part of the dimensionless effective interaction. The operator $W^{(s)}$ has the form:

$$W^{(s)} = v^{2s} z(1 + \tilde{F}) z(1 + \tilde{F}) z \cdots z(1 + \tilde{F}) z. \tag{A.5}$$

After similar considerations, as for the ferromagnetic liquid, one can obtain that a necessary and sufficient condition of the stability of the paramagnetic Fermi liquid is the fulfilment of the inequality

$$1 + \tilde{F}_l > 0 \tag{A.6}$$

for every l. It is just the Pomeranchuk inequality. The sufficiency of the conditions (A.6) results from the fact that a sum of positive terms is also a positive expression.

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