ANGULAR PHOTOELECTRON DISTRIBUTIONS IN ONE-QUANTUM ATOMIC PHOTOEFFECT DUE TO ELLIPTICALLY POLARIZED RADIATION*

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A formula for the differential photoionisation cross section for elliptically polarized light is derived and shown to lead, in particular cases, to all the hitherto known formulae for linearly and circularly polarized as well as unpolarized light. The angular photoelectron distributions in the one-quantum photoeffect are discussed in their dependence on the polarisation state of the electromagnetic radiation. In particular, those directions of photoelectron emission are determined for which the differential cross section is the same for all states of light polarisation. An interesting theorem is proved, enouncing that the sum of electrons ejected in two mutually perpendicular directions, lying in the plane perpendicular to the photon propagation direction, is a quantity independent of the state of light polarisation. Three methods are proposed for measurements of the asymmetry parameter β versus the photoelectron energy W using elliptically polarized light. It is shown how photoionisation experiments can be applied to determine the ellipticity parameter and principal axes orientation of the polarisation ellipse of elliptically polarized radiation. Five new summation formulae, fulfilled by spherical harmonics, are proved.

1. *Introduction*

Since 1972, the electron synchrotron has come into frequent use as the source of electromagnetic radiation in studies of the one-quantum photoeffect, especially in determinations of the asymmetry parameter $\beta(W)$ [1–10]. Although synchrotron radiation, emitted in the plane of the synchrotron (electron orbit) is completely linearly polarized in that plane, the radiation emitted at an angle to the plane of the synchrotron exhibits completely elliptical polarisation. The ellipticity parameter κ (the ratio of the small and large semi-axes of the polarisation ellipse) is a function of the emission angle measured with respect to the synchrotron plane. This state of polarisation of synchrotron radiation

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had been predicted theoretically by Sokolov and Ternov [11] and was first observed experimentally by Joos [12] at the Cornell synchrotron. It is important [12] that synchrotron radiation, emitted at an angle of as little as several milliradian with respect to the synchrotron plane, is perceptibly polarized elliptically. With regard to the rapidly developing studies on one-quantum photoeffect by synchrotron radiation, the problem of the angular distributions of photoelectrons, ejected from atoms by elliptically polarized radiation, is steadily gaining in importance. Though the problem has already been dealt with by Schmidt [13] in 1973, we believe ours is the first complete analysis there of; in addition, we propose three methods for the measurement of the asymmetry parameter $\beta(W)$.

2. *Differential photoionisation cross section*

The golden rule of Fermi [14] can be taken as the starting point of the theory of angular distributions of photoelectrons, emitted from a sub-shell nl of isolated, randomly oriented (unpolarized) atoms under the action of elliptically polarized radiation with photon energy $\hbar\omega$ at least equal to the bonding energy of electron in the sub-shell:

$$\frac{d\sigma_{nl}}{d\Omega_k} = \frac{\omega}{2\pi c}(ka_0)a_0^2\langle|M_W^{nlm}|^2\rangle_m. \tag{1}$$

Above, ω is the circular frequency of the incident electromagnetic radiation, c — the velocity of light, $k = |\boldsymbol{k}|$ the wave number of the photoelectron related to its energy W as follows: $W = \hbar^2k^2/2m$, whereas $a_0 = 5.29\times10^{-9}$ cm is the Bohr radius and $\langle|M_W^{nlm}|^2\rangle_m$ the squared module of the matrix element of the quantum transition averaged over the magnetic quantum numbers m of the light-absorbing electron. In the non-relativistic, electric dipole approximation, the one-quantum ionisation transition matrix element is:

$$M_W^{nlm} = \int \phi_W^*(\boldsymbol{r})\boldsymbol{e}_\pm\cdot\boldsymbol{r}\phi_{nlm}(\boldsymbol{r})d\boldsymbol{r}, \tag{2}$$

where $\phi_{nlm}(\boldsymbol{r})$ and $\phi_W(\boldsymbol{r})$ are the wave functions of the electron in the atom and, respectively, detached therefrom (photoelectron), $\boldsymbol{r}$ the radius vector of the electron with origin at the nucleus, and $\boldsymbol{e}_\pm$ the unit (in the meaning that $\boldsymbol{e}_\pm\cdot\boldsymbol{e}_\pm^* = 1$), in general complex polarisation vector of the electromagnetic radiation. On the central potential model, the one-particle wave functions of combining quantum states of the electron can be written in the form [15]:

$$\phi_{nlm}(\boldsymbol{r}) = R_{nl}(r)Y_{l,m}(\vartheta, \varphi), \tag{3}$$

$$\phi_W(\boldsymbol{r}) = (8\pi^3/k)^{1/2}\sum_{l_f,m_f} i^{l_f}e^{i\delta_{l_f}}Y_{l_fm_f}^*(\vartheta_k, \varphi_k)Y_{l_fm_f}(\vartheta, \varphi)R_{Wl_f}, \tag{4}$$

where $r = |\boldsymbol{r}|$, ϑ and φ are the polar and azimuthal angles defining the orientation of $\boldsymbol{r}$, ϑ_k and φ_k are the angles defining that of the wave vector $\boldsymbol{k}$ of the photoelectron, δ_{l_f} is the shift in phase of the l_f-th partial wave of the latter, Y_{lm} are spherical harmonics, and $R_{nl}(r)$, $R_{Wl_f}(r)$ are the radial parts of the complete wave functions of the bound and emitted electron, respectively. We now assume the Z-axis of Cartesian coordinates to coincide with the propagation direction and the axes X and Y, respectively, with the large and small semi-axes of the polarisation ellipse of the elliptically polarized incident radiation, causing the photoeffect (Fig. 1).

For this geometry, the unit polarisation vector of the radiation takes the form:

$$e_{\pm} = \frac{\boldsymbol{i} \pm i\kappa \boldsymbol{j}}{(1+\kappa^2)^{1/2}}, \tag{5}$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vector along X and Y, and $\kappa = b/a$ the ellipticity parameter (the ratio of b, the small and a the large semi-axis of the polarisation ellipse), which lies in the interval $0 \leqslant \kappa \leqslant 1$, the value 0 corresponding to complete linear polarisation along X and the value 1 to circular polarisation. For $\kappa \neq 0$, the sign "+" at the imaginary unit

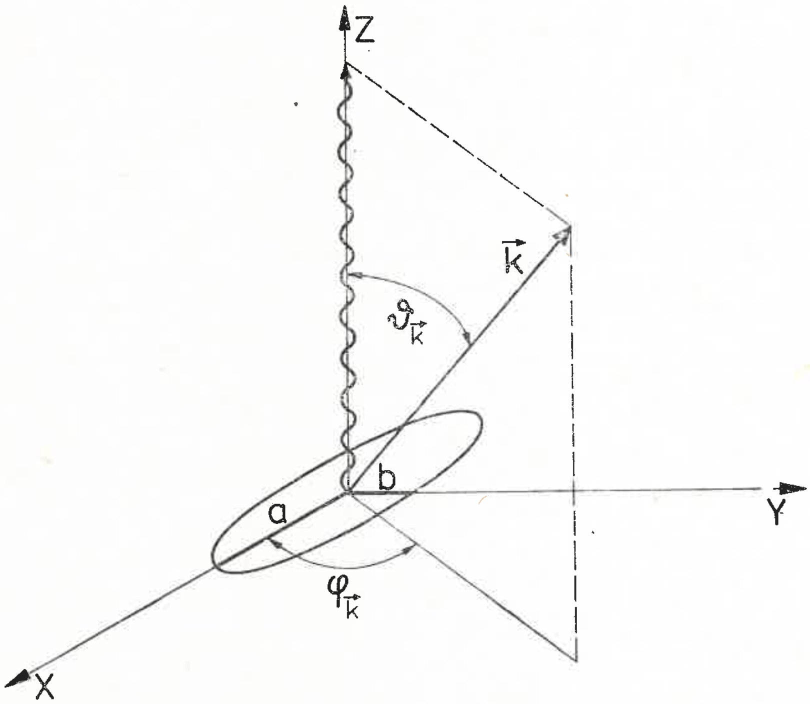

Fig. 1. Coordinate system, applied for calculating the differential cross section for ionisation by elliptically polarized light. The Z-axis is chosen to coincide with the photon propagation direction; the X-axis ——— with the direction of the large semi-axis a of the vibration ellipse of light; and the Y-axis ——— with the small semi-axis b. The wave vector of the photoelectron $\boldsymbol{k} = \boldsymbol{k}(k, \vartheta_k, \varphi_k)$

corresponds to right and "−" to left polarisation, defined according to the convention for angular momentum. The scalar product $\boldsymbol{e}_{\pm} \cdot \boldsymbol{r}$ in the volume integral (2) can now be expressed, in a form well adapted to our aims, in terms of first-order spherical harmonics:

$$\boldsymbol{e}_{\pm} \cdot \boldsymbol{r} = \frac{(2\pi/3)r}{(1+\kappa^2)^{1/2}} [(1 \mp \kappa) Y_{1,-1}(\vartheta, \varphi) - (1 \pm \kappa) Y_{1,+1}(\vartheta, \varphi)]. \tag{6}$$

With regard to the relation

$$\int Y^*_{l_1 m_1}(\vartheta, \varphi) Y_{l_2 m_2}(\vartheta, \varphi) Y_{l_3 m_3}(\vartheta, \varphi) d\Omega$$

$$= (-1)^{m_1}[(2l_1+1)(2l_2+1)(2l_3+1)/4\pi]^{1/2} \begin{pmatrix} l_1 & l_2 & l_3 \\ -m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (7)$$

permitting the expression of an angular integral over the product of three spherical harmonics in terms of Wigner 3*j*-coefficients [16], we obtain the squared module of the matrix element M_W^{nlm} in the following form:

$$|M_W^{nlm}|^2 = \frac{4\pi^3}{k(2l+1)} \Big\{ S_1(m)(R^{nl}_{W,l+1})^2 + S_2(m)(R^{nl}_{W,l-1})^2$$

$$+2S_3(m) R^{nl}_{W,l+1} R^{nl}_{W,l-1} \cos(\delta_{l+1} - \delta_{l-1})$$

$$- \frac{1-\kappa^2}{1+\kappa^2} \cos 2\varphi_k [S_4(m)(R^{nl}_{W,l+1})^2 + S_5(m)(R^{nl}_{W,l-1})^2$$

$$+2S_6(m) R^{nl}_{W,l+1} R^{nl}_{W,l-1} \cos(\delta_{l+1} - \delta_{l-1})] \Big\}, \qquad (8)$$

where, for brevity, we have introduced the notation:

$$S_1(m) = (l+1-m)(l+2-m)(2l+3)^{-1} |Y_{l+1,m-1}(\vartheta_k, \varphi_k)|^2, \qquad (9)$$

$$S_2(m) = (l-1-m)(l-m)(2l-1)^{-1} |Y_{l-1,m+1}(\vartheta_k, \varphi)_k|^2, \qquad (10)$$

$$S_3(m) = \left[\frac{[l^2-(m-1)^2](l-m+2)(l+m)}{(2l-1)(2l+3)} \right]^{1/2}$$

$$\times Y_{l+1,m-1}(\vartheta_k, \varphi_k) Y^*_{l-1,m-1}(\vartheta_k, \varphi_k), \qquad (11)$$

$$S_4(m) = (2l+3)^{-1} \{[(l+1)^2 - m^2][(l+2)^2 - m^2]\}^{1/2}$$

$$\times Y_{l+1,m-1}(\vartheta_k, 0) Y^*_{l+1,m+1}(\vartheta_k, 0), \qquad (12)$$

$$S_5(m) = (2l-1)^{-1} \{[(l-1)^2 - m^2](l^2 - m^2)\}^{1/2}$$

$$\times Y_{l-1,m+1}(\vartheta_k, 0) Y^*_{l-1,m-1}(\vartheta_k, 0), \qquad (13)$$

$$S_6(m) = \left[\frac{[(l-m)^2-1](l-m+2)(l-m)}{(2l-1)(2l+3)} \right]^{1/2}$$

$$\times Y_{l+1,m-1}(\vartheta_k, 0) Y^*_{l-1,m+1}(\vartheta_k, 0), \qquad (14)$$

$$\cos(\delta_{l+1} - \delta_{l-1}) = \mathrm{Re}(e^{-i\delta_{l+1}} e^{i\delta_{l-1}}). \qquad (15)$$

Above, $R^{nl}_{Wl\pm1}$ is the usual dipolar radial matrix element, defined as follows:

$$R^{nl}_{W,l\pm1} = \int_0^\infty R_{W,l\pm1}(r) r R_{nl}(r) r^2 dr. \qquad (16)$$

In accordance with the following definition of the isotropic average of the function $f(m)$ over the magnetic quantum number m:

$$\langle f(m)\rangle_m = (2l+1)^{-1} \sum_{m=-l}^{+l} f(m), \tag{17}$$

we now have to calculate the six sums $\mathfrak{S}_j = \sum_m S_j(m)$, where the quantities $S_j(m)$ ($j = 1, 2, \ldots, 6$), are given by Eqs (9)—(14). By way of the formal interchange $m \to m+1$, the sum $\mathfrak{S}_3$ can be put in the form in which it occurs in the book of Varshalovich et al. [16]. The result taken directly from Ref. [16] is:

$$\mathfrak{S}_3 = \frac{1}{8\pi} l(l+1)(2-3\sin^2 \vartheta_k). \tag{18}$$

The other five sums $\mathfrak{S}_1$, $\mathfrak{S}_2$, $\mathfrak{S}_4$, $\mathfrak{S}_5$ and $\mathfrak{S}_6$ remain to be calculated in the present paper. Applying certain summation and recurrential formulae fulfilled by spherical harmonics and tabulated in [16], we prove the five novel summation relations:

$$\mathfrak{S}_1 = \frac{1}{8\pi}(l+1)[2l+(l+2)\sin^2 \vartheta_k], \tag{19}$$

$$\mathfrak{S}_2 = \frac{1}{8\pi} l[2(l+1)+(l-1)\sin^2 \vartheta_k], \tag{20}$$

$$\mathfrak{S}_4 = -\frac{1}{8\pi}(l+1)(l+2)\sin^2 \vartheta_k, \tag{21}$$

$$\mathfrak{S}_5 = -\frac{1}{8\pi} l(l-1)\sin^2 \vartheta_k, \tag{22}$$

$$\mathfrak{S}_6 = \frac{3}{8\pi} l(l+1)\sin^2 \vartheta_k. \tag{23}$$

With regard to these summation relations, as well as Eq. (8), we obtain the following expression for $\langle |M_W^{nlm}|^2\rangle_m$:

$$\langle |M_W^{nlm}|^2\rangle_m = \frac{\pi^2}{k(2l+1)^2}\left[B+C\sin^2 \vartheta_k\left(1+\frac{1-\kappa^2}{1+\kappa^2}\cos 2\varphi_k\right)\right], \tag{24}$$

where

$$B = l(l+1)[(R_{W,l+1}^{nl})^2+(R_{W,l-1}^{nl})^2+2R_{W,l+1}^{nl}R_{W,l-1}^{nl}\cos(\delta_{l+1}-\delta_{l-1})], \tag{25}$$

and

$$C = \tfrac{1}{2}(l+1)(l+2)(R_{W,l+1}^{nl})^2+\tfrac{1}{2}l(l-1)(R_{W,l-1}^{nl})^2 - 3l(l+1)R_{W,l+1}^{nl}R_{W,l-1}^{nl}\cos(\delta_{l+1}-\delta_{l-1}). \tag{26}$$

Insertion of Eq. (24) into Fermi's golden rule (1) leads to the following general formula of the angular distributions of photoelectrons emitted from the nl subshell of randomly oriented atoms due to elliptically polarized electromagnetic radiation:

$$\frac{d\sigma_{nl}}{d\Omega_k} = \frac{\sigma_{nl}}{4\pi}\left\{1+\beta(W)\left[\tfrac{3}{4}\left(1+\frac{1-\kappa^2}{1+\kappa^2}\cos 2\varphi_k\right)\sin^2\vartheta_k-\tfrac{1}{2}\right]\right\}. \tag{27}$$

In this, the final formula of the theory proposed by us, $\beta(W)$ is the asymmetry parameter defined as:

$$\begin{aligned}\beta(W) = {} & [l(l-1)(R^{nl}_{W,l-1})^2+(l+1)(l+2)(R^{nl}_{W,l+1})^2\\ & -6l(l+1)R^{nl}_{W,l+1}R^{nl}_{W,l-1}\cos(\delta_{l+1}-\delta_{l-1})]\\ & \times\{(2l+1)[l(R^{nl}_{W,l-1})^2+(l+1)(R^{nl}_{W,l+1})^2]\}^{-1},\end{aligned} \tag{28}$$

whereas σ_{nl} is the total photoionisation cross section, given as follows:

$$\sigma_{nl} = \frac{4\pi^2}{3(2l+1)}\frac{\omega a_0^3}{c}[l(R^{nl}_{W,l-1})^2+(l+1)(R^{nl}_{W,l+1})^2]. \tag{29}$$

If more electrons than one are present in the sub-shell nl, the right-hand term of (29) has to be multiplied by the number of these electrons. We note that neither the differential or total cross-sections depend on whether the radiation is right or left polarized. However, the total (contrary to the differential) cross section is moreover independent of the ellipticity parameter κ and, thus, of the state of polarisation of the radiation causing the one-quantum photoeffect.

This, among others, is a feature distinguishing the one-quantum photoeffect from multi-photon ionisation, where both the differential and total cross sections are rather strongly polarisation-dependent [17]. Thus, in the case of the one-quantum photoeffect, the so-called polarisation effects are related with the differential cross section only. In fact, Eq. (27) derived above should also describe correctly the angular distributions of photoelectrons emitted by randomly oriented molecules as well as those of dissociation products of molecules and nuclear reactions due to elliptically polarized radiation with a wavelength much in excess of the linear dimensions of the objects taking part in the effect. With regard to the condition of non-negativity of the differential cross section (27), the asymmetry parameter has to be contained within the interval $-1 \leqslant \beta \leqslant 2$. It may be worth stating once again that, in (27), ϑ_k and φ_k are polar and azimuthal angles in a Cartesian coordinate system the Z-axis of which coincides with the propagation direction of the incident photons and the X-axis with that of the large semi-axis of the polarisation ellipse (see, Fig. 1).

The formula (27), derived here, is the most general expression describing angular photoelectron distributions in one-quantum effect; for various particular cases, it reduces to hitherto known formulae for linearly polarized, circularly polarized and unpolarized light. On putting e.g. $\kappa = 0$, which corresponds to complete linear polarization along the X-axis (Fig. 1), and having recourse to the following formula of spherical trigonometry:

$$\cos\gamma = \cos\vartheta_1\cos\vartheta_2+\sin\vartheta_1\sin\vartheta_2\cos(\varphi_1-\varphi_2), \tag{30}$$

where γ is the angle subtended by two vectors with orientations given by the angles ϑ_1, φ_1 and ϑ_2, φ_2 we obtain in particular:

$$\frac{d\sigma_{nl}}{d\Omega_k} = \frac{\sigma_{nl}}{4\pi}[1+\beta(W)P_2(\cos\theta)] \text{ (linear polarisation).} \quad (31)$$

This is Bethe's well known formula [18] for linear polarisation, where $P_2(x) = \frac{3}{2}x^2-\frac{1}{2}$ is a Legendre polynomial and θ the angle between the direction of the electric vector of incident radiation and the ejection direction of the electron. Whereas on putting in (27) $\kappa = 1$, corresponding to completely circular polarisation, we obtain the previously known formula:

$$\frac{d\sigma_{nl}}{d\Omega_k} = \frac{\sigma_{nl}}{4\pi}[1-\tfrac{1}{2}\beta(W)P_2(\cos\vartheta_k)] \text{ (circular polarisation and unpolarized),} \quad (32)$$

where ϑ_k is the angle between the propagation direction of the radiation and the direction of ejection of the electron. Eq. (32) holds as well for unpolarized radiation [19]. The general expression (27) also remains valid if the incident radiation is partly polarized linearly. In this case, we have but to carry out in (27) the formal interchange:

$$\frac{1-\kappa^2}{1+\kappa^2} \to p = \frac{I_{\max}-I_{\min}}{I_{\max}+I_{\min}}, \quad (33)$$

where p is the degree of polarisation, and $I_{\max}$, $I_{\min}$ the maximal and minimal intensity, corresponding to the two mutually perpendicular electric vector components of the partly linearly polarized beam, whereas the azimuthal angle φ_k has to be measured with respect to the most probable vibration direction of the electric vector of light.

If the photoionisation originates in the subshell $n0$ ($l = 0$), then $\beta(W) = 2$ has to be put in all equations, in accordance with (28).

3. *Polarisation effects in one-quantum photoeffect*

On comparing Eqs (27) and (29) we draw the following conclusion: In the one-quantum photoeffect, the so-called polarisation effects commonly understood to denote a dependence of the differential and/or total cross section on the state of polarisation of the electromagnetic radiation bear solely on $d\sigma_{nl}/d\Omega_k$ but not on σ_{nl}.

Let us consider the angular photoelectron distributions in the plane perpendicular to the light propagation direction (the XY-plane of Fig. 1) versus the state of light polarisation. In this case $\vartheta_k = 90°$, and

$$\frac{d\sigma_{nl}}{d\Omega_k} = \frac{\sigma_{nl}}{4\pi}\left\{1+\beta(W)\left[\tfrac{1}{4}+\tfrac{3}{4}\frac{1-\kappa^2}{1+\kappa^2}\cos 2\varphi_k\right]\right\}. \quad (34)$$

The unsophisticated analysis of (34) taking into consideration the condition $-1 \leqslant \beta(W) \leqslant 2$ — that of non-negativity of the differential cross section — leads to the following conclusions:

(1) For unpolarized and circularly polarized light, contrary to the case of elliptically or linearly polarized light, the distribution in the XY-plane is isotropic. Such a distribution is directly related with the circumstance that, in unpolarized and circularly polarized light, no direction in the plane perpendicular to the light beam is favoured;

(2) For $\varphi_k = 45°$, the differential cross section is independent of κ and thus is the same as for unpolarized and circularly polarized light. Hence the angular distribution curves for the various states of light polarisation intersect in the point $\varphi_k = 45°$;

(3) The total number of electrons ejected in any two, mutually perpendicular directions lying in the XY-plane is independent of κ and thus is independent of the state of light

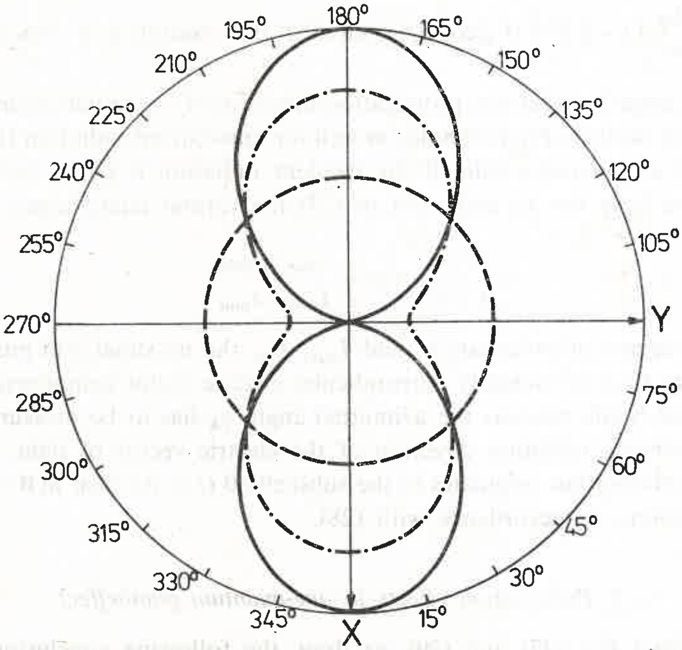

Fig. 2. Angular distributions of the photoelectrons (in relative units) in the plane perpendicular to the light propagation direction, for various states of incident light polarisation. Curve ——— is for linear polarisation; curve –·–·– for elliptical polarisation with $\kappa = 1/2$; curve — — — for circular polarisation and unpolarized light. All the curves are plotted assuming $\beta = 2$

polarisation, and is proportional to $(\sigma_{nl}/4\pi)\,[2+\frac{1}{2}\,\beta(W)]$. In particular, if the light wave is elliptically polarized, these directions can be those of the large and small semi-axes of the vibration ellipse;

(4) If $\beta(W)$ is positive and the incident light elliptically polarized, the number of electrons ejected in the direction of the large semi-axes of the ellipse is always larger than that of the electrons ejected in the direction of the small semi-axis;

(5) If, however, $\beta(W)$ is negative, fewer electrons are always ejected in the direction of the large semi-axis than in that of the small semi-axis;

(6) If $\beta(W) > 0$, the number of electrons ejected in the direction $\varphi_k = 0°$ by linearly polarized light is always larger than that of the electrons ejected by elliptically polarized light of the same intensity. In either case, the number of electrons ejected in the direction $\varphi_k = 0°$ is larger than for circularly polarized and unpolarized light;

(7) If $\beta(W) > 0$, elliptically polarized light ejects more electrons, in the direction $\varphi_k = 90°$, than linearly polarized and less than circularly polarized and unpolarized light;

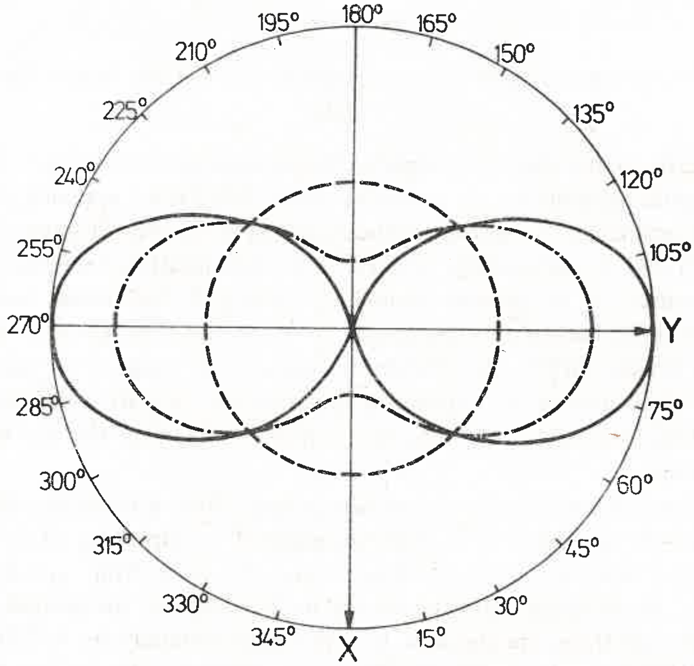

Fig. 3. Angular distributions of photoelectrons (in relative units) in the plane perpendicular to the photon propagation direction; Curve ——— is for linear polarisation; curve –·–·– for elliptical polarisation with $\kappa = 1/2$; curve – – – for circular polarisation and unpolarized light. All the curves are plotted on the assumption $\beta = -1$

(8) If $\beta(W) < 0$, the number of electrons ejected in the direction $\varphi_k = 0°$ is the largest for circularly polarized and unpolarized light and the smallest for linearly polarized light;

(9) If $\beta(W) < 0$, the number of electrons ejected in the direction $\varphi_k = 90°$ is the largest for linearly polarized and the smallest for circularly polarized and unpolarized light.

Figs 2 and 3 adequately illustrate these properties of the angular photoelectron distributions in the plane perpendicular to the propagation direction of the light wave. The graphs are plotted in polar coordinates, for Eq. (34), assuming the limiting values $\beta(W) = 2$ in Fig. 2 and $\beta(W) = -1$ in Fig. 3. The number of electrons, ejected from the irradiated volume of the atomic gas at the angle φ_k to the light propagation direction in the perpendicular plane, is rendered by the length of the radius vector connecting the origin of axes X, Y and the point on the distribution curve. It is worth noting that for values of $\beta(W)$ other than 2 and -1 the curves are similar, with the significant difference that the curve for linear polarisation does not traverse the origin of coordinates but intersects the Y-axis (Fig. 2), or X-axis (Fig. 3), in a point situated between the origin and the point in which the distribution curve for elliptical polarisation intersects the Y-axis (Fig. 2), or X-axis (Fig. 3).

4. *Methods for the measurement of the asymmetry parameter using elliptically polarized light*

On the basis of the theory of angular photoelectron distributions of Chapter 2, we shall now propose methods for the measurement of $\beta(W)$ when applying elliptically polarized light. The problem is of especial practical interest at the present stage when the electron synchrotron [1–10] is increasingly applied in photoionisation experiments as a highly appreciated source of in general completely elliptically polarized radiation [11, 12]. Measurements of the asymmetry parameter β versus the photoelectron energy $W \sim \hbar\omega$ are important because $\beta(W)$ characterizes completely the anisotropic part of the angular photoelectron distributions and conveys information on the matrix elements of the photoionisation transition as well as the difference in phase shifts of the two partial photoelectron waves at inference.

Let us assume the photoelectrons as being ejected from a point source and the photoelectron detector (with regard to its finite dimensions) as capturing all the photoelectrons ejected into the interval from $\vartheta_k - \Delta\vartheta_k$ to $\vartheta_k + \Delta\vartheta_k$ and from $\varphi_k - \Delta\varphi_k$ to $\varphi_k + \Delta\varphi_k$. Denoting by d the distance from the source to the detector, the surface of the detector on which those electrons are incident is given approximately by $4d^2 \sin\vartheta_k \sin\Delta\vartheta_k \Delta\varphi_k$. Integration of the general expression (27) over ϑ_k from $\vartheta_k - \Delta\vartheta_k$ to $\vartheta_k + \Delta\vartheta_k$ and over φ_k from $\varphi_k - \Delta\varphi_k$ to $\varphi_k + \Delta\varphi_k$ now yields:

$$N(\vartheta_k \varphi_k) = \frac{1}{d^2} \frac{N_t}{4\pi} \Bigg[1 - \tfrac{1}{2}\beta(W) + \tfrac{3}{4}\beta(W) \sin^2\vartheta_k \left(1 + \frac{\sin^2 \Delta\vartheta_k}{\sin^2\vartheta_k} - \tfrac{4}{3}\sin^2\Delta\vartheta_k \right) \times \left(1 + \frac{1-\kappa^2}{1+\kappa^2} \cos 2\varphi_k \sin(2\Delta\varphi_k)/2\Delta\varphi_k \right) \Bigg], \tag{35}$$

where $N(\vartheta_k, \varphi_k)$ is the flux of photoelectrons emitted in the direction ϑ_k, φ_k i.e. their number, incident per unit time per unit surface area of the detector, and N_t the number of those emitted per unit time into the body angle 4π. On the assumption that the detector records photoelectrons from a small body angle only ($\Delta\vartheta_k \ll 1$, $\Delta\varphi_k \ll 1$), Eq. (35) takes the form

$$N(\vartheta_k, \varphi_k) = \frac{N_t}{4\pi d^2}$$

$$\times \left\{1+\beta(W)\left[\tfrac{3}{4}\left(1+\frac{1-\kappa^2}{1+\kappa^2}\cos 2\varphi_k\right)\sin^2\vartheta_k-\tfrac{1}{2}\right]\right\}. \quad (36)$$

4.1. Method I

A non-trivial method for the measurement of $\beta(W)$ using elliptically polarized radiation can well consist in measuring the ratio η of the fluxes of photoelectrons for two directions of ejection ϑ_k', φ_k' and ϑ_k'', ϑ_k'' since, with regard to the general equation (36), we have

$$\beta(W) = \frac{1-\eta}{\eta f(\vartheta_k'', \varphi_k'')-f(\vartheta_k', \varphi_k')}, \quad (37)$$

with

$$\eta = \frac{N(\vartheta_k', \varphi_k')}{N(\vartheta_k'', \varphi_k'')}, \quad (38)$$

$$f(\vartheta_k, \varphi_k) = \tfrac{3}{4}\left(1+\frac{1-\kappa^2}{1+\kappa^2}\cos 2\varphi_k\right)\sin^2\vartheta_k-\tfrac{1}{2}. \quad (39)$$

Above, $N(\vartheta_k', \varphi_k')$ and $N(\vartheta_k'', \varphi_k'')$ are the fluxes of photoelectrons ejected in the directions ϑ_k', φ_k' and ϑ_k'', φ_k'', respectively. If one of these two directions is made to coincide with that of the large semi-axis of the polarisation ellipse ($\vartheta_k' = 90°$, $\varphi_k' = 0°$) and the other with that of the small semi-axis ($\vartheta_k'' = 90°$, $\varphi_k'' = 90°$) — see, Fig. 1, Eq. (37) becomes:

$$\beta = 2\frac{(1+\kappa^2)(\eta_1-1)}{(\eta_1+2)-\kappa^2(2\eta_1+1)}; \quad \eta_1 = \frac{N(\vartheta_k = 90°, \varphi_k = 0°)}{N(\vartheta_k = 90°, \varphi_k = 90°)}. \quad (40)$$

In the above formula, η_1 is the ratio of fluxes of photoelectrons emitted respectively in the directions of the large and small semi-axis. Our choice of these directions is essentially advantageous in that η now differs the most strongly from 1 (cf. the general equation (36) and Figs 2, 3). In order to determine β from (40), the orientation of the principal axes of the polarisation ellipse as well as the ellipticity parameter κ of the electromagnetic wave have to be measured in a separate experiment (cf. Section 4.4). We stress that Eq. (40) is not applicable to circularly polarized ($\kappa = 1$) or unpolarized light.

4.2. Method II

Another method of measuring $\beta(W)$ with elliptically polarized light can have recourse to the important fact that, for the azimuthal angle $\varphi_k = 45°$, the differential photoionisation cross section is independent of κ and, thus, of the state of polarisation. Putting $\vartheta_k = 90°$

and $\varphi_k = 45°$ in Eq. (36), we obtain the following simple formula for the asymmetry parameter $\beta(W)$:

$$\frac{4\pi d^2}{N_t} N(\vartheta_k = 90°, \varphi_k = 45°) = 1+\tfrac{1}{4}\beta(W). \tag{41}$$

The measurement of the value N_t can, however, be reduced to that of the flux of photoelectrons for the "magic" direction $\vartheta_k = 54.73°$, $\varphi_k = 45°$, for which the following relation holds:

$$N(\vartheta_k = 54.73°, \varphi_k = 45°) = \frac{N_t}{4\pi d^2}. \tag{42}$$

With regard to the preceding two relations, we finally obtain:

$$\beta(W) = 4(\eta_2 - 1), \tag{43}$$

where

$$\eta_2 = \frac{N(\vartheta_k = 90°, \varphi_k = 45°)}{N(\vartheta_k = 54.73°, \varphi_k = 45°)} \tag{44}$$

and $N(\vartheta_k, \varphi_k)$ is the flux of photoelectrons, emitted in the direction ϑ_k, φ_k. Thus, by measuring experimentally the ratio η_2, one can determine from Eq. (43) the asymmetry parameter β for a given value of the photoelectron energy $W \sim \hbar\omega$. This method, like method I, requires a supplementary experiment to have the orientation of the principal axes of the polarisation ellipse, but there is no need to determine κ. Eq. (43) remains valid for circularly polarized and unpolarized light, albeit in this case $\eta_2 = N(\vartheta_k = 90°) / N(\vartheta_k = 54.73°)$.

4.3. Method III

Yet another method of measuring $\beta(W)$ using elliptically polarized radiation suggests itself when one takes into account that the total number of electrons ejected in any two, mutually perpendicular directions in the plane perpendicular to the light propagation direction is independent of the state of light polarisation. In fact, by Eq. (36), we have:

$$N(\vartheta_k = 90°, \varphi_k) + N(\vartheta_k = 90°, \varphi_k + 90°) = \frac{N_t}{4\pi d^2}[2+\tfrac{1}{2}\beta(W)], \tag{45}$$

whence

$$\beta(W) = \frac{8\pi d^2}{N_t}[N(\vartheta_k = 90°, \varphi_k) + N(\vartheta_k = 90°, \varphi_k + 90°)] - 4. \tag{46}$$

Above, N_t is the total number of all electrons produced per unit time in the photoionisation process, and $N(\vartheta_k = 90°, \varphi_k)$ and $N(\vartheta_k = 90°, \varphi_k + 90°)$ are the fluxes of electrons emitted in the directions φ_k and $\varphi_k + 90°$, both lying in the plane perpendicular to the light beam. Eq. (46) from which $\beta(W)$ is to be determined experimentally by Method III, does not require a separate determination of κ or the principal axes orientation (contrary to Methods

I and II). In Method III, a separate determination of the principal axes orientation is necessary only if wishing, in accordance with Eq. (42), to reduce the measurement of N_t, for technical reasons, to that of $N(\vartheta_k = 54.73°,\ \varphi_k = 45°)$ the flux of photoelectrons ejected at the "magical" direction. Method III is applicable to elliptically, circularly and linearly polarized as well as unpolarized radiation.

4.4. Determination of the ellipticity parameter κ and principal axes directions from photoionisation studies

The ellipticity parameter κ and principal axes of the polarisation ellipse can be determined by current methods of classical optics. They can, as well, be determined in photoionisation experiments on atoms for which the asymmetry parameter $\beta(W)$ is already available. Atoms of helium and some other gases, for which β is constant and equals 2 for all frequencies of electromagnetic radiation, appear especially well adapted to this purpose. Once β is available, κ can be determined from the following relation:

$$\kappa^2 = \frac{\eta_1(\beta-2)+2(\beta+1)}{2\eta_1(\beta+1)+(\beta-2)}, \tag{47}$$

resulting from (40). Above, η_1 is the ratio of fluxes of photoelectrons ejected from atoms with known $\beta(W)$ in the directions of the large and small semi-axes of the polarisation ellipse of light. If $\beta = 2$, Eq. (47) reduces to:

$$\kappa^2 = \frac{1}{\eta_1} = \frac{N(\vartheta_k = 90°,\ \varphi_k = 90°)}{N(\vartheta_k = 90°,\ \varphi_k = 0°)}, \tag{48}$$

whence one sees that, in order to determine κ experimentally, it suffices to measure the fluxes of electrons, ejected from the $S(l = 0)$-state of e.g. helium in the directions of the two principal axes. If the effect bears on s-electrons, the direction of the large semi-axis is that, perpendicular to the light propagation direction, in which the most electrons are ejected, whereas the fewest are ejected along the small semi-axis (cf. Fig. 2 and the discussion following Eq. (34)).

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