DENSITY EVOLUTION OF A TWO-COMPONENT SYSTEM IN A NONLOCAL POTENTIAL*

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We present a detailed study of the exact time evolution in a two-component ideal gas in which at time $t = 0$ a short-ranged external potential of arbitrary strength is switched on that can transform a into b particles. The redistribution of matter is followed via the space- and time-dependent a particle mass density $\varrho_a(r, t)$ of which we present a series of perspective views over the (r, t)-plane.

1. Introduction

In a series of papers [1–4] we have proposed and studied a number of microscopic statistical models in which the non-equilibrium time evolution of macroscopically large multi-component systems can be calculated exactly. In Ref. [1] we considered a two-component ideal gas of a and b particles in which at some initial time, $t = 0$ say, a constant external potential is switched on which will transform a into b particles. The Hamiltonian of such a system is given in second quantization by

$$H = \sum_{\vec{k}} (\varepsilon_{\vec{k}}^{(a)} a_{\vec{k}}^{+} a_{\vec{k}} + \varepsilon_{\vec{k}}^{(b)} b_{\vec{k}}^{+} b_{\vec{k}}) + \theta(t)\Omega^{-1} \sum_{\vec{k},\vec{k}'} V_{\vec{k}\vec{k}'}(a_{\vec{k}}^{+} b_{\vec{k}'} + b_{\vec{k}'}^{+} a_{\vec{k}}) = H_0 + \theta(t)V \qquad (1)$$

where $a_{\vec{k}}^{+}(a_{\vec{k}})$ and $b_{\vec{k}}^{+}(b_{\vec{k}})$ are creation (annihilation) operators for a and b particles (Fermions or Bosons) in momentum states $\vec{k}$, respectively. Putting $\hbar = 1$, the single particle energies are given by $\varepsilon_{\vec{k}}^{(a)} = k^2/2m_a$ and $\varepsilon_{\vec{k}}^{(b)} = k^2/2m_b + \varepsilon_b$ where ε_b is the threshold energy for the formation of a b particle. (For illustration, consider a spin 1/2 system: a and b, then refer to the two spin orientations, $V_{\vec{k}\vec{k}'}$ induces spin-flip scattering, and ε_b is the (relative) Zeeman energy in an additional external magnetic field.)

Let us assume that for times $t < 0$ we have prepared an ideal gas of a and b particles in thermal equilibrium distributed uniformly throughout the volume Ω. Its statistical operator (for a canonical ensemble) is then given by

$$\hat{\varrho}_0 = \hat{\varrho}(t = 0) = \hat{\varrho}_a \hat{\varrho}_b = \exp(-\beta H_0). \qquad (2)$$

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At time $t = 0$ we switch on an external potential that is constant in configuration space and thus diagonal in momentum space, i.e. $V_{\vec{k}\vec{k}'} = V_0\delta_{\vec{k}\vec{k}'}$, which, according to the Hamiltonian equation (1), will transform a into b particles. We call this the diagonal model. We can follow its effect on the system by calculating the resulting time evolution of the average a particle density, given by

$$\varrho_a(t) = \Omega^{-1}\sum_{\vec{k}} n_{\vec{k}}(t) = \Omega^{-1}\sum_{\vec{k}} \mathrm{Tr}\,(a_{\vec{k}}^{+}a_{\vec{k}}\hat{\varrho}_t)/\mathrm{Tr}\,\hat{\varrho}_t \tag{3}$$

where

$$\hat{\varrho}_t = e^{iHt}\hat{\varrho}_0 e^{-iHt}. \tag{4}$$

Shifting the time evolution operators from $\hat{\varrho}_0$ onto $a_{\vec{k}}^{+}$ and $a_{\vec{k}}$, we have to evaluate

$$\varrho_a(t) = \Omega^{-1}\sum_{\vec{k}} \mathrm{Tr}\,(a_{\vec{k}}^{+}(t)a_{\vec{k}}(t)\hat{\varrho}_0)/\mathrm{Tr}\,\hat{\varrho}_0, \tag{5}$$

where the $a_{\vec{k}}^{+}(t)$ and $a_{\vec{k}}(t)$ are subject to the equation-of-motion

$$i\dot{a}_{\vec{k}} = [a_{\vec{k}}, H] \tag{6}$$

which in Ref. [1] was solved by diagonalizing the Hamiltonian, Eq. (1), explicitly. The result is given by

$$\varrho_a(t) = \varrho_a(0) + \frac{1}{(2\pi)^2}\int_0^\infty k^2 dk V_0^2/(V_0^2 + \tfrac{1}{4}(\varepsilon_{\vec{k}}^{(a)} - \varepsilon_{\vec{k}}^{(b)}))$$

$$\times\left[1 - \cos\left(2t(V_0^2 + \tfrac{1}{4}(\varepsilon_{\vec{k}}^{(a)} - \varepsilon_{\vec{k}}^{(b)})^2)^{1/2}\right)\right](n_{\vec{k}}^{(b)}(0) - n_{\vec{k}}^{(a)}(0)). \tag{7}$$

It should be stressed that in our approach the dynamics of the system, i.e. its time evolution, is calculated completely at the operator, i.e. single-particle level, whereas the statistics only enters in the final averaging over the initial conditions, i.e. via the equilibrium distribution functions $n_{\vec{k}}^{(a)}(t = 0)$ and $n_{\vec{k}}^{(b)}(t = 0)$ which can refer to any particle statistics. The density $\varrho_a(t)$, Eq. (7), will, in general, exhibit damped oscillations as illustrated in Refs [1] and [2]. The source of this irreversible behavior can be found in the thermodynamic limit involved by replacing sums $\Omega^{-1}\sum_{\vec{k}}$ by integrals $(2\pi)^{-3}\int d^3k$ in going e.g. from Eq. (5) to Eq. (7). This leads physically to a randomization of the phases of the various momentum components $n_k(t)$ as $t \to \infty$ as already recognized by Schrödinger [5] in a much simpler model, namely a classical harmonic chain of mass points with nearest neighbor coupling.

The diagonal model was subsequently [2] studied in great numerical detail for three- and four-component systems and also for various initial conditions, i.e. for Maxwell-Boltzmann gases, weakly degenerate Fermi-Dirac and Bose-Einstein gases and for Fermi-Dirac gases at zero temperature. A discussion of the equilibrium properties of the diagonal model can be found in Ref. [6].

Although the diagonal model exhibits irreversible behavior (damped oscillations) in the thermodynamic (large volume) limit, it shows no approach to equilibrium as $t \to \infty$

due to the lack of momentum mixing in the constant external potential. Thus the final steady state reached by the system will in general depend on the initial conditions. To remove this, in most circumstances [7] unwanted feature, a momentum mixing mechanism has to be present in $V_{\vec{k}\vec{k}'}$. This we have done in a subsequent paper [3] in which it was shown that the diagonal model is, in some sense, the lowest order approximation of the more general problem of a two-component system which can undergo "catalytic" reactions in a random collection of static scattering centers ("impurities"). The Hamiltonian of such a system is given by

$$H = \sum_{\vec{k}} (\varepsilon_{\vec{k}}^{(a)} a_{\vec{k}}^{+} a_{\vec{k}} + \varepsilon_{\vec{k}}^{(b)} b_{\vec{k}}^{+} b_{\vec{k}}) + \theta(t)\Omega^{-1} \sum_{\vec{k},\vec{k}'} \tilde{V}_{\vec{k}\vec{k}'} (a_{\vec{k}}^{+} b_{\vec{k}'} + b_{\vec{k}}^{+} a_{\vec{k}}) \tag{8}$$

with

$$\tilde{V}_{\vec{k}\vec{k}'} = V_{\vec{k}\vec{k}'} \sum_{i} e^{-i(\vec{k}-\vec{k}')\cdot\vec{R}_i} \tag{9}$$

where $\vec{R}_i$ are the random positions of the scattering centers and $V_{\vec{k}\vec{k}'}$ is, in most cases, a short-ranged potential mediating the transformation of an a into a b particle during a scattering event. Choosing for $V_{\vec{k}\vec{k}'}$ a separable potential, we were able to calculate the exact time evolution of the a particle density $\varrho_a(t)$, Eq. (3), in a system subject to the Hamiltonian Eqs (8) and (9), without, however, giving any numerical results. In this paper we want to consider the much simpler problem of the time evolution of a two-component system in which "catalytic" transformation of a into b particles can occur in a momentum mixing scattering process at one isolated, static impurity, described by a separable potential. The dynamics of the system is thus controlled by a Hamiltonian like Eq. (1), but one in which the external interaction

$$V_{\vec{k}\vec{k}'} = g v_{\vec{k}} v_{\vec{k}'} \tag{10}$$

is chosen to be a single-term separable potential acting in s-waves only, i.e. to be spherically symmetric in configuration space. In our numerical examples we will choose the potential form factor $v_k = (k^2+\gamma^2)^{-1}$ to be of Yamaguchi type [8] though our methods of solution are independent of this choice. In coordinate space this form factor shows exponential damping of range $\lambda = \gamma^{-1}$.

Because the effect of a potential of finite range on the time response of an averaged quantity like $\varrho_a(t)$, Eq. (5), averaged over an infinite system, is zero, we have to study in this problem the time evolution of local macroscopic quantities like the space- and time-dependent local density

$$\varrho_a(\vec{r}, t) = \mathrm{Tr}\,(\Psi_a^{+}(\vec{r}, t)\Psi_a(\vec{r}, t)\hat{\varrho}_0)/\mathrm{Tr}\,\hat{\varrho}_0 \tag{11}$$

where

$$\Psi_a(\vec{r}, t) = \Omega^{-1/2} \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} a_{\vec{k}}(t) \tag{12}$$

are field operators expressed in terms of particle annihilation operators $a_{\vec{k}}(t)$.

In the next Section we will present the exact calculation of $\varrho_a(\vec{r}, t)$ and continue in Section 3, with a detailed numerical study in several examples. The results will be discussed with the help of three-dimensional plots in which perspective views of the density surface

$\varrho_a(\vec{r}, t)$ are presented over the space-time plane. This will be done not only for weak potentials but also for strong interactions, strong enough, indeed, to cause ideal gas particles to be trapped in resonances and boundstates around the potential center.

Similar pictures have been presented elsewhere [4] for the time evolution of a one-component ideal gas in which at time $t = 0$ an external separable potential is switched on. The main results from the numerical work in the one-component system can be summarized as follows: for the initial time period after the potential has been switched on, we observe a very complex behavior in the local density within the range of the potential. After some time has elapsed highly dispersive density waves emerge that travel outward, i.e. away from the "impurity", and eventually build up, for strong potentials, into collisionless shockwaves. In this last stage of the time evolution, a simple hydrodynamic description seems appropriate.

We would like to mention here that in the one-component system [4] as well as in the two-component system to be studied in this paper, one can easily show that the system will settle for large times ($t \to \infty$) in the new steady state appropriate for an ideal gas in such an external potential. To understand this, we have to note that in these systems irreversibility is again due to the thermodynamic (or large volume) limit: let us, for the moment, imagine that the whole system is divided into two subsystems: a finite part extending over the region of space where the external potential affects the motion of the gas particles and the (infinite) rest of the system. The latter part will then obviously act as an infinite particle and energy reservoir for the former, and will force it isothermally into equilibrium as time progresses. Let us also note that in a truly finite system (taken to be spherical of radius R for simplicity) the Poincaré recurrence time is of the order $t_P \sim 2R/v_s$ which is the time needed for a typical wave emitted in the switch-on process to travel with a speed v_s to the boundary and be reflected back to the origin, very reminiscent of spin-echo experiments [7].

The interest in the time evolution of a two-component system in an external potential center stems mainly from its larger complexity as compared to one-component systems due to at least three features: the potential center will act as a particle sink for one kind of particles due to the fact that it transforms a into b particles in a scattering event. Secondly, in a two-component system a single separable potential can produce both a resonance and a boundstate. (This feature can be most easily demonstrated in the generalized optical potential approach, see e.g. Refs [8] and [9].) As a third bonus, we have in a two-component system an additional parameter at our display, namely the mass ratio m_a/m_b of the two kinds of particles involved. All these features will be demonstrated and discussed at length in Section 3. In the next Section, we will briefly outline the calculational methods employed to find the exact time evolution of $\varrho_a(r, t)$.

2. *Some calculational details*

It is our objective in this paper to study the time evolution in a two-component system of a and b particles which is initially, for times $t < 0$, prepared in such a way that an ideal gas of only a particles is uniformly distributed throughout a volume Ω at a constant

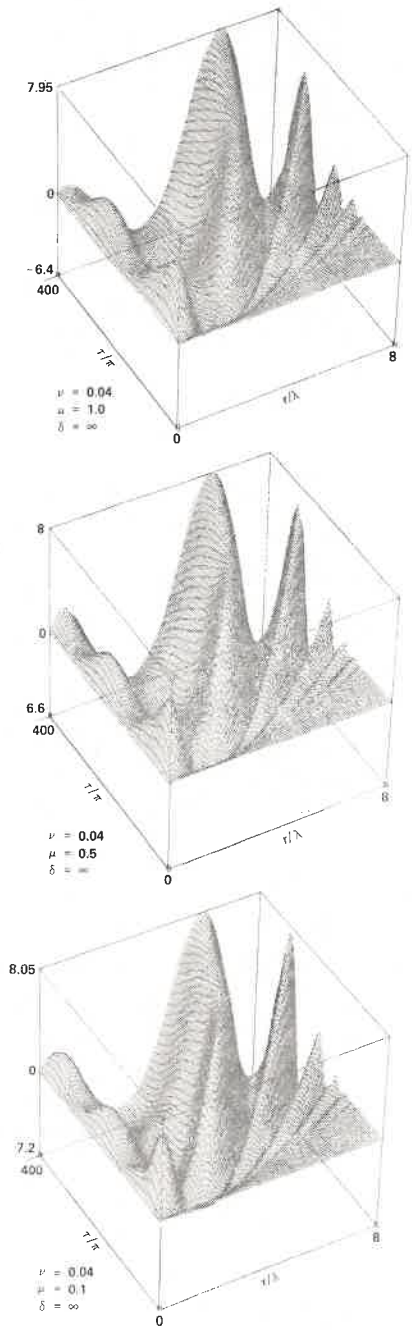

Fig. 1. Perspective view of the surface $(r/\lambda)^2(\varrho_a(r,\tau)/\varrho_a(0)-1)$ over the (r,τ)-plane. Yaw $= -30°$, Pitch $= 45°$ at a distance of the observe $= 10\times$ diagonal of the cube. For meaning of parameters see text below Eq. (31)

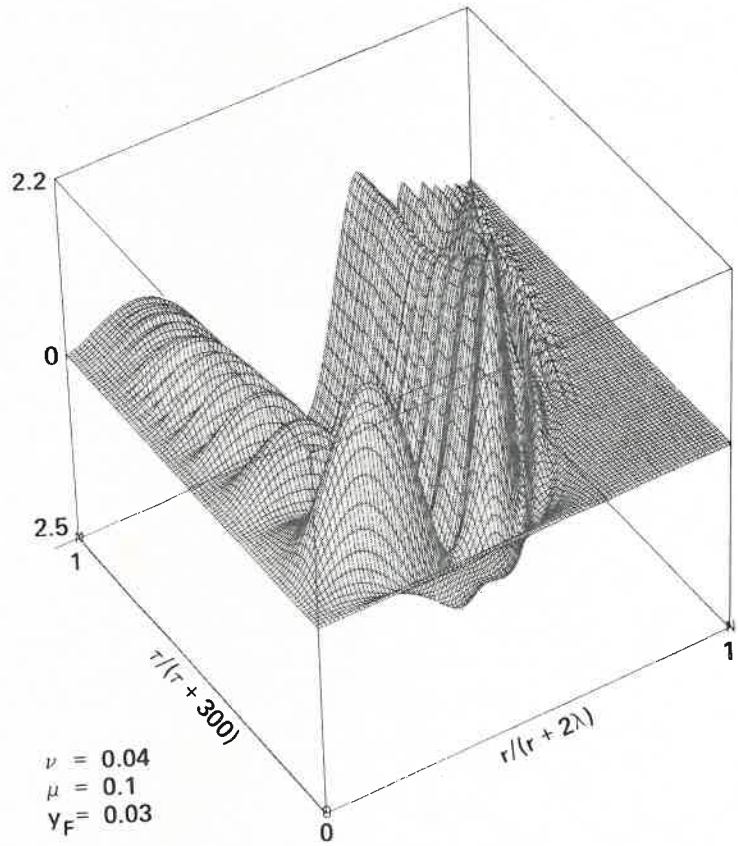

Fig. 2. See Fig. 1, but note the rescaling of the plane. y_F is the dimensionless Fermi-momentum, Eq. (44)

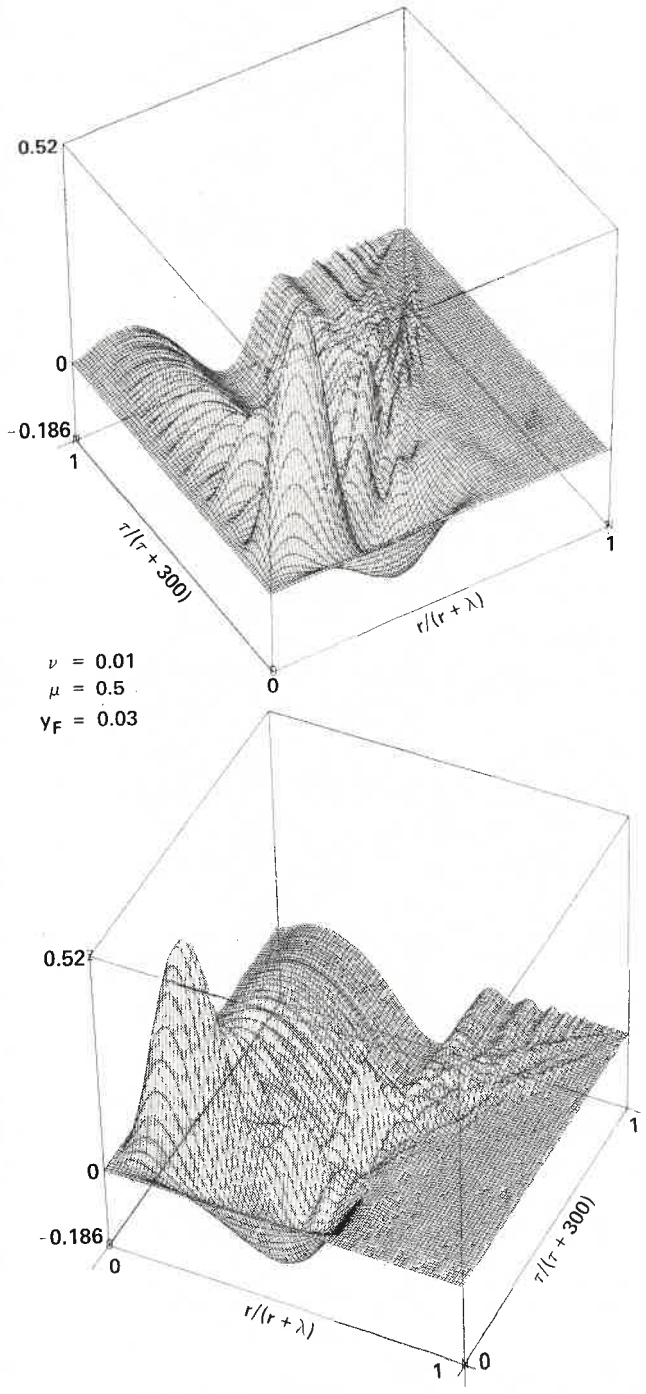

Fig. 3. See Fig. 1

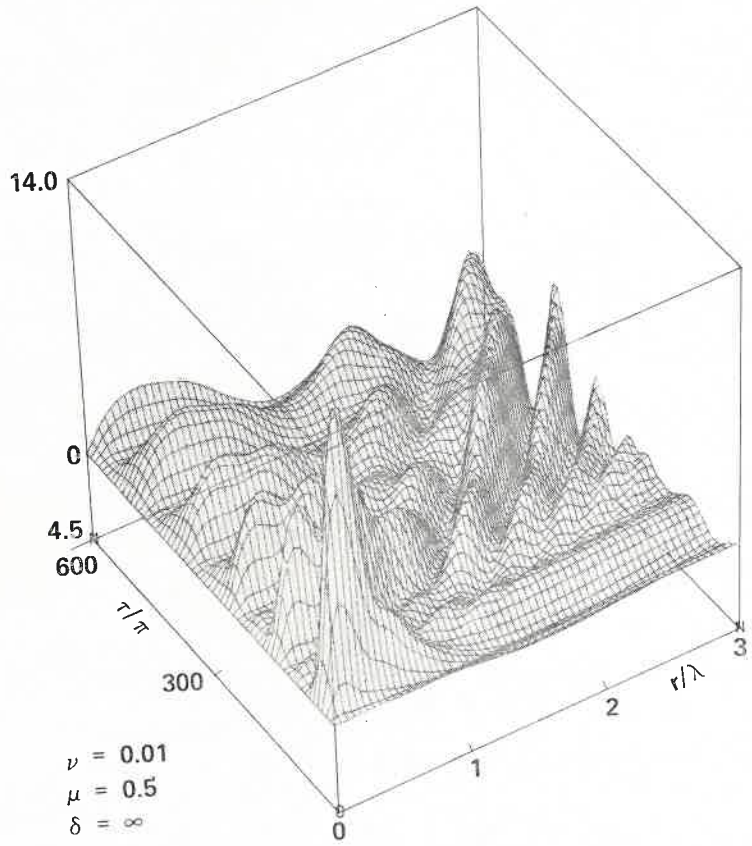

Fig. 4. See Fig. 2

temperature. At time $t = 0$ an external interaction is switched on in the system, described by a static potential (an infinitely heavy localized impurity) $V_{\vec{k}\vec{k}'} = g v_k v_{k'}$ which can transform a into b particles in a (momentum mixing) scattering event. The system will respond not only with a creation of b particles but also with a time and space dependent redistribution of a particles which we want to follow through the local a particle density $\varrho_a(\vec{r}, t)$, Eq. (11).

In this section we will outline briefly the main calculational steps involved in the computation of $\varrho_a(\vec{r}, t)$. With the help of Eq. (12) we can rewrite Eq. (11) as

$$\varrho_a(\vec{r}, t) = \Omega^{-1} \sum_{\vec{k},\vec{k}'} e^{-i(\vec{k}-\vec{k}')\cdot\vec{r}} \operatorname{Tr}\left(a_{\vec{k}}^{+}(t) a_{\vec{k}'}(t) \hat{\varrho}_0\right) / \operatorname{Tr} \hat{\varrho}_0 \tag{13}$$

where the particle operators are subject to the equations-of-motion (choosing $\hbar = 1$)

$$i\dot{a}_{\vec{k}} = [a_{\vec{k}}, H] = \varepsilon_{\vec{k}}^{(a)} a_{\vec{k}} + \theta(t) \Omega^{-1} \sum_{\vec{k}'} V_{\vec{k}\vec{k}'} b_{\vec{k}'} \tag{14a}$$

$$i\dot{b}_{\vec{k}} = [b_{\vec{k}}, H] = \varepsilon_{\vec{k}}^{(b)} b_{\vec{k}} + \theta(t) \Omega^{-1} \sum_{\vec{k}'} V_{\vec{k}\vec{k}'} a_{\vec{k}'} \tag{14b}$$

and H is given by Eqs (1) and (10). We can solve Eq. (14b) formally with the help of the free retarded Green's function to get

$$b_{\vec{k}'}(t) = e^{-i\varepsilon_{k'}^{(b)} t} b_{\vec{k}'}(0) - i \sum_{\vec{k}''} V_{\vec{k}'\vec{k}''} \int_0^t dt' e^{-i\varepsilon_{k'}^{(b)}(t-t')} a_{\vec{k}''}(t'). \tag{15}$$

Inserting this solution into Eq. (14a), we can again solve formally and get after exchanging the two time integrations

$$a_{\vec{k}}(t) = e^{-i\varepsilon_k^{(a)} t} a_{\vec{k}}(0) - i \int_0^t dt' e^{-i\varepsilon_k^{(a)}(t-t')} \Omega^{-1} \sum_{\vec{k}'} V_{\vec{k}\vec{k}'} e^{-i\varepsilon_{k'}^{(a)} t'} b_{\vec{k}'}(0)$$

$$+ i\Omega^{-2} \sum_{\vec{k}',\vec{k}''} \frac{V_{\vec{k}\vec{k}'} V_{\vec{k}'\vec{k}''}}{\varepsilon_k^{(a)} - \varepsilon_{k'}^{(b)}} \int_0^t dt' a_{\vec{k}''}(t') \left[e^{-i\varepsilon_{k'}^{(b)}(t-t')} - e^{-i\varepsilon_k^{(a)}(t-t')}\right]. \tag{16}$$

To solve this integral equation we next take Laplace transforms with respect to the variable t and find

$$A_{\vec{k}}(z) = \int_0^\infty dt e^{-zt} a_{\vec{k}}(t) = A_{\vec{k}}^{\text{in}}(z)$$

$$- \frac{i}{z + i\varepsilon_k^{(a)}} \Omega^{-1} \sum_{\vec{k}'} V_{\vec{k}\vec{k}'} \left[B_{\vec{k}'}^{\text{in}}(z) - i\Omega^{-1} \sum_{\vec{k}''} V_{\vec{k}'\vec{k}''} \frac{1}{z + i\varepsilon_{\vec{k}'}^{(b)}} A_{\vec{k}''}(z) \right] \tag{17}$$

where

$$A_{\vec{k}}^{\text{in}}(z) = \frac{1}{z + i\varepsilon_k^{(a)}} a_{\vec{k}}(0)$$

$$B_{\vec{k}}^{\text{in}}(z) = \frac{1}{z + i\varepsilon_k^{(b)}} b_{\vec{k}}(0). \tag{18}$$

If $V_{\vec{k}\vec{k}'} = g v_k v_{k'}$ is assumed separable, we can solve Eq. (17) easily. By multiplying it with v_k and summing over $\vec{k}$, we can determine the auxiliary operator

$$A(z) = \Omega^{-1} \sum_{\vec{k}} v_k A_{\vec{k}}(z)$$

$$= (A^{\text{in}}(z) - i I_a(z) B^{\text{in}}(z))/(1 + I_a(z) I_b(z)) \tag{19}$$

where

$$A^{\text{in}}(z) = \Omega^{-1} \sum_{\vec{k}} v_k A^{\text{in}}_{\vec{k}}(z)$$

$$B^{\text{in}}(z) = \Omega^{-1} \sum_{\vec{k}} v_k B^{\text{in}}_{\vec{k}}(z) \tag{20}$$

and

$$I_a(z) = g\Omega^{-1} \sum_{\vec{k}} \frac{v_k^2}{z + i\varepsilon_k^{(a)}}$$

$$I_b(z) = g\Omega^{-1} \sum_{\vec{k}} \frac{v_k^2}{z + i\varepsilon_k^{(b)}} . \tag{21}$$

With the help of Eq. (19) we find the solution to Eq. (17)

$$A_{\vec{k}}(z) = A^{\text{in}}_{\vec{k}}(z) - i \frac{v_k}{z + i\varepsilon_k^{(a)}} B^{\text{in}}(z) - \frac{v_k^k}{z + i\varepsilon_k^{(a)}} I_b(z) A(z). \tag{22}$$

To carry the calculation further, we assume that the single-particle energies are $\varepsilon_k^{(a)} = k^2/2m_a$ and $\varepsilon_k^{(b)} = k^2/2m_b$ (no threshold energy) and specify the potential form factors to be of Yamaguchi [8] type [10]

$$v_k = (k^2 + \gamma^2)^{-1}. \tag{23}$$

Replacing in Eq. (22) the summation over momenta by an integration in the large volume limit, we get

$$I_a(z) = \frac{g}{8\pi\gamma} \left(\sqrt{z} + e^{i\pi/4} \frac{\gamma}{\sqrt{2m_a}} \right)^{-2} \tag{24}$$

and a similar expression for $I_b(z)$ which we insert into Eqs. (19) and (22). After factorization of the denominator of Eq. (19) and partial fraction decomposition, we get

$$A_k(z) = A^{\text{in}}_k(z) - g \sum_{\vec{p}} \frac{v_k}{z + i\varepsilon_k^{(a)}} \frac{v_p}{z + i\varepsilon_p^{(b)}} \sum_{j=1}^{4} \frac{C_j}{\sqrt{z} - e^{i\pi/4}\alpha_j} p_p(0) \tag{25}$$

where

$$C_j = e^{-i\pi/4}\frac{g}{8\pi\gamma}\left(\alpha_j+\frac{\gamma}{\sqrt{2m_a}}\right)^2\Bigg/\prod_{\substack{l=1\\ l\neq j}}^{4}(\alpha_j-\alpha_l) \tag{26}$$

with the four roots of

$$1+I_a(z)I_b(z) = 0 \tag{27}$$

given by

$$\alpha_j = -\frac{\gamma}{2}\left(\frac{1}{\sqrt{2m_a}}+\frac{1}{\sqrt{2m_b}}\right)\pm\left[\frac{\gamma^2}{4}\left(\frac{1}{\sqrt{2m_a}}-\frac{1}{\sqrt{2m_b}}\right)^2\pm\frac{g}{8\pi\gamma}\right]^{1/2}. \tag{28}$$

We have dropped the in-field of the b particles in Eq. (25), because it will not contribute due to our initial conditions, that no b particles are present at $t = 0$. We are now in a position of taking inverse Laplace transforms on Eq. (25) by noting that [11]

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dz e^{zt}\frac{1}{z+i\varepsilon}\frac{1}{\sqrt{z}-e^{i\pi/4}\alpha}$$

$$= e^{-i\pi/4}\frac{1}{\varepsilon+\alpha^2}\{(i\sqrt{\varepsilon}-\alpha)e^{-i\varepsilon t}+\alpha e^{i\alpha^2 t}\,\mathrm{Erfc}\,(-\alpha\sqrt{it})-i\sqrt{\varepsilon}\,e^{-i\varepsilon t}\,\mathrm{Erfc}\,(\sqrt{-i\varepsilon t})\} \tag{29}$$

where

$$\mathrm{Erfc}\,(x) = 1-\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \tag{30}$$

is an error function. The resulting exact time evolution of the a particle annihilation operators is then given by

$$a_k^\dagger(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dz e^{zt}A_k^\dagger(z)$$

$$= e^{-i\varepsilon_k t}a_k^\dagger(0)+\Omega^{-1}\sum_{\vec{p}} F_{\vec{k}\vec{p}}(t)a_p^\dagger(0), \tag{31}$$

where $F_{\vec{k}\vec{p}}(t)$ is a rather complicated function which we will give presently after we have introduced a set of dimensionless variables, namely $V_0 = g\lambda$, $\lambda_0^2 = \hbar^2/(4m_aV_0)$, $\nu = \lambda_0/\lambda$, $\mu = m_a/m_b$, $a_i = \alpha_i/\sqrt{2V_0}$, $x = k\lambda_0$, $y = p\lambda_0$ and $\tau = 2V_0t/\hbar$. We then find

$$F_{\vec{k}\vec{p}}(t) = \lambda_0^3\nu/(32\pi)F(x, y, \tau)$$

$$= \frac{1}{x^2+\nu^2}\frac{1}{y^2+\nu^2}\frac{1}{y^2-x^2}\sum_{j=1}^{4}(a_j+\nu)^2\Bigg/\prod_{\substack{l=1\\ l\neq j}}^{4}(a_j-a_l)$$

$$\left\{\frac{1}{x^2+a_j^2}\left[(ix-a_j)e^{-ix^2\tau}+a_je^{ia_j^2\tau}\,\mathrm{Erfc}\,(-a_je^{i\pi/4}\sqrt{\tau})-ixe^{-ix^2\tau}\,\mathrm{Erfc}\,(e^{-i\pi/4}x\sqrt{\tau})\right]\right.$$

$$-\frac{1}{y^2+a_j^2}\left[(iy-a_j)e^{-iy^2\tau}+a_je^{ia_j^2\tau}\,\mathrm{Erfc}\,(-a_je^{i\pi/4}\sqrt{\tau})\right.$$

$$\left.\left.-iye^{-iy^2\tau}\,\mathrm{Erfc}\,(e^{-i\pi/4}y\sqrt{\tau})\right]\right\}. \tag{32}$$

We are now in a position to calculate $\varrho_a(\vec{r}, t)$. Inserting Eq. (12) into (11) and using Eq. (31) for $a_{\vec{k}}^+(t)$ we get

$$\varrho_a(\vec{r}, t) = \varrho_a(t=0)+\Omega^{-2}\sum_{\vec{k},\vec{k}'}(e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}n_{\vec{k}'}^{(a)}(0)e^{i\varepsilon_k t}F_{\vec{k}\vec{k}'}(t)+\text{c.c.})$$

$$+\Omega^{-3}\sum_{\vec{p}}n_{\vec{p}}^{(a)}(0)\sum_{\vec{k},\vec{k}'}e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}F_{\vec{k}'\vec{p}}^{*}(t)F_{\vec{k}\vec{p}}(t) \tag{33}$$

where the initial conditions imposed in the system enter in the final traces in the initial ideal gas of a particles

$$\mathrm{Tr}\,(a_{\vec{k}}^+(0)a_{\vec{k}'}(0)\hat{\varrho}_0)/\mathrm{Tr}\,\hat{\varrho}_0 = \delta_{\vec{k}\vec{k}'}n_{\vec{k}}^{(a)}(0) \tag{34}$$

where

$$n_{\vec{k}}^{(a)}(0) = (e^{\beta(\varepsilon_k^{(a)}-\mu)}\pm 1)^{-1} \tag{35}$$

for Bose-Einstein (negative sign) or Fermi-Dirac statistics (positive sign) or

$$n_{\vec{k}}^{(a)}(0) = e^{-\beta(\varepsilon_k^{(a)}-\mu)} \tag{36}$$

in the classical limit of Maxwell-Boltzmann statistics. Here μ is the chemical potential (per particle) in an ideal gas. To procede with the evaluation of $\varrho_a(\vec{r}, t)$ we next take the large volume limit by replacing sums $\Omega^{-1}\sum_{\vec{k}}$ by integrals $(2\pi)^{-3}\int d^3k$ in Eq. (33) and observe that for a spherical potential $\varrho_a(\vec{r}, t)$ will only depend on the radial distance $r = |\vec{r}|$ from the potential center which makes the angular integrations trivial. If the gas of a particles was initially described by Maxwell-Boltzmann statistics, Eq. (36), we find in the dimensionless variables introduced above

$$\varrho_a(r, \tau)/\varrho_a(\tau=0) = 1+\frac{1}{16\pi^2}v\left(\frac{\lambda_0}{r}\right)^2\left(\frac{\delta}{\pi}\right)^{3/2}$$

$$\times\int_0^\infty ydy\,\sin(yr/\lambda_0)e^{-\delta y^2}[e^{iy^2\tau}\int_0^\infty xdx\,\sin(xr/\lambda_0)F(x, y, \tau)+\text{complex conjugate}]$$

$$+\frac{v^2}{(4\pi)^5}\left(\frac{\lambda_0}{r}\right)^2\left(\frac{\delta}{\pi}\right)^{3/2}\int y^2dye^{-\delta y^2}$$

$$\times\left|\int_0^\infty xdx\,\sin(xr/\lambda_0)F(x, y, \tau)\right|^2. \tag{37}$$

Here $\delta = 2V_0\beta = 2V_0/k_BT$ is a measure for the temperature in the gas. Expressions quite similar to Eq. (37) can be written down for gases with quantum statistics. As a final technical remark let us note that the x-integration in Eq. (37) can actually be carried out explicitly yielding a very lengthy but also very interesting expression which we have given in Ref. [4] for a one-component system. The final, but rather simple y-integration in Eq. (37) has to be done numerically.

3. Examples

Our final result, Eq. (37), in the last section entails the following physics: we have prepared a times $t < 0$ an ideal gas of a particles uniformly distributed throughout space at a constant temperature. At time $t = 0$ we switch on a separable external potential that can in a scattering event within its range transform a into b particles. The local density of a particles $\varrho_a(r, t)$, Eq. (37), describes exactly the space- and time-dependent response of the system to such an external action for any strength of the potential.

The analytic discussion of $\varrho_a(r, t)$ for a slightly different potential form-factor was given in Ref. [3] and will not be repeated here. Rather we proceed to some numerical examples for $\varrho_a(r, t)$ concentrating on the new features emerging in a two-component system over and above those fascinating structures that already appear in a one-component system.

To illustrate the complex dynamics of the two-component system studied here, let us write down the single particle Schrödinger equations for the stationary states (in momentum representation)

$$\frac{k^2}{2m_a}\Psi_a(\vec{k}) - E\Psi_a(\vec{k}) = -gv_k \sum_{\vec{p}} v_p \Psi_b(\vec{p}) \tag{38a}$$

$$\frac{k^2}{2m_b}\Psi_b(\vec{k}) - (E-\varepsilon_b)\Psi_b(\vec{k}) = -gv_k \sum_{\vec{p}} v_p \Psi_a(\vec{p}) \tag{38b}$$

where the Ψ's are single-particle wave-functions. (In nuclear physics and potential scattering, such a system is referred to as a two-channel system [8, 9, 12].) Solving (38b) we get

$$\Psi_b(\vec{k}) = -\frac{1}{\frac{k^2}{2m_b} - E + \varepsilon_b - i\varepsilon} gv_k \sum_{\vec{p}} v_p \Psi_a(\vec{p}) \tag{39}$$

which inserted into Eq. (39a) yields

$$\frac{k^2}{2m_a}\Psi_a(\vec{k}) - E\Psi_a(\vec{k}) = -\sum_{\vec{p}} \langle \vec{k}|V_{\text{opt}}|\vec{p}\rangle \Psi_a(\vec{p}) \tag{40a}$$

where

$$\langle \vec{k}|V_{\text{opt}}|\vec{p}\rangle = G(E)v_k v_p \tag{40b}$$

is usually referred to as the generalized optical potential [12] for the first channel with the effective energy-dependent coupling constant

$$G(E) = -g^2 \sum_{\vec{p}} \frac{v_p^2}{\frac{p^2}{2m_b} - E + \varepsilon_b - i\varepsilon} . \tag{40c}$$

Observe that $G(E) < 0$ for $E < \varepsilon_b$ ($\varepsilon_b = 0$ in our case) and thus can allow a boundstate for sufficiently large g. Also note that Re $G(E) > 0$ somewhere above threshold and this can lead to a simultaneous resonance. The imaginary part of $G(E)$ describes the loss of a particles once the b channel is open for $E > \varepsilon_b$.

For a Yamaguchi [8] potential form factor $v_p = (p^2+\gamma^2)^{-1}$ we have

$$G(E) = \frac{g^2}{8\pi\gamma}\left(\sqrt{E} + i\frac{\gamma}{\sqrt{2m_b}}\right)^{-2} \tag{41a}$$

which is for $E < 0$

$$G(-|E|) = -\frac{g^2}{8\pi\gamma}\left(\sqrt{|E|} + \frac{\gamma}{\sqrt{2m_b}}\right)^{-2} \tag{41b}$$

and for $E > 0$

$$G(E) = \frac{g^2}{8\pi\gamma}\frac{(\sqrt{E} + i\gamma/\sqrt{2m_b})^2}{(E+\gamma^2/(2m_b))^2} . \tag{41c}$$

In a Yamaguchi potential a boundstate can occur (of course only in s-waves) at

$$E_{\text{bs}} = -\left(\frac{\gamma}{\sqrt{2m_b}} - \left(\frac{-G(E_{\text{bs}})}{8\pi\gamma}\right)^{1/2}\right)^2 \tag{42}$$

if

$$\left(\frac{-G(E_{\text{bs}})}{8\pi\gamma}\right)^{1/2} - \frac{\gamma}{\sqrt{2m_b}} > 0. \tag{43}$$

We had seen in our discussion of the one-component system [4] that the occurrence of a boundstate is reflected in $\varrho(r, t)$ as time dependent density oscillations of period $\hbar/|E_{\text{bs}}|$ within the range of the potential. This feature should again show up in the two-component system but this time with a typical dependence on the mass ratio $\mu = m_a/m_b$. This is illustrated in Fig. 1. We plot here, and in subsequent pictures, the quantity $(r/\lambda)^2(\varrho_a(r, t)/\varrho_a(\tau = 0)-1)$, i.e. the a particle mass in a spherical shell at distance r from the potential center, as a function of this radial distance and also as a function of the (dimensionless) time variable $\tau = 2V_0 t/\hbar$. The potential parameters in Fig. 1 are such that a boundstate barely develops. To suppress temperature smearing we have also taken $\delta = \infty$ in Eq. (37). (Some cautionary remarks about this limit can be found in Ref. [4].) Observe first the pronounced density ridges and valleys running more or less diagonally across the (r, τ)-plane

very reminiscent of the density evolution in a one-component system with a weak external potential [4]. These structures are obviously radially outgoing density waves emanating from the potential center when the interaction was suddenly switched on. In addition we can see in Fig. 1, within the range of the potential, i.e. for $r/\lambda \lesssim 1$, time dependent density oscillations that obviously get shorter as the mass ratio $\mu = m_a/m_b$ is changed from 1 to 0.5 to 0.1. These oscillations reflect the attempt of the system to fill a barely existing boundstate. Because temperature smearing effects are suppressed in the $\delta = \infty$ limit, these oscillations will repeat themselves for ever. For $\delta < \infty$ these oscillations would be damped. But they are also damped in a Fermi-Dirac gas at zero temperature, as illustrated in Fig. 2. In this case the integration over the Boltzmann factor in Eq. (37) has to be replaced by an integration over the Fermi sphere according to

$$(\delta/\pi)^{3/2} \int_0^\infty e^{-\delta y^2} \dots dy \to \frac{1}{4\pi} \frac{3}{y_F^3} \int_0^{y_F} \dots dy \tag{44}$$

where $y_F = \lambda_0 k_F = (6\pi^2 \lambda_0^3 \varrho_a(0))^{1/3}$ is the dimensionless Fermi momentum. Let us not overlook that Fig. 2 shows the complete a particle density evolution over the entire (r, τ)-plane which we have rescaled into a finite section by the transformation $r \to r/(r+r_0)$ and $\tau \to \tau/(\tau+\tau_0)$ with appropriately chosen constants r_0 and τ_0. We thus can also see that the outgoing density waves remain of a finite (eventually decreasing) amplitude. The boundstate oscillations within the range λ of the potential are clearly damped. The depression in $\varrho_a(r, \tau)$ for distances r twice or three times λ stem partly from the shape of the boundstate wavefunction in our potential which is of the Hulthén form [13], namely

$$\Psi_{bs}(r) \sim (e^{-r/\lambda} - e^{-r/\lambda_{bs}}) \tag{45}$$

where $\lambda_{bs} = \hbar/(2m_a|E_{bs}|)^{1/2}$. The depression in $\varrho_a(r, t)$, however, also signals the fact that the a particle subsystem can loose mass in the creation of b particles. Note that as $\tau \to \infty$, i.e. $\tau/(\tau+\tau_0) \to 1$, the system settles in its new steady state appropriate for an ideal two-component gas in the external potential.

In Fig. 3 we can investigate the effects in a stronger potential as compared to Fig. 2. We again observe damped boundstate oscillations, but in this potential they are predated by a huge mass enhancement peak which is a consequence of the fact that this potential, in a single particle picture, not only has a bounstate for negative energies but also develops a resonance for positive energies which would e.g. manifest itself as a peak in the single particle scattering cross section. Because a resonance can be pictured as a boundstate with a finite lifetime, this great mass enhancement disappears again as time goes on. It is very interesting to see in the lower picture of Fig. 3 that at some later time the resonating mass enhancement peak can reconstruct itself once more further out in space. (When picturing the true behavior, always bare in mind the spherical symmetry of the system!)

In Fig. 4 we have a look at the initial stage of the time evolution of the same two-component system as Fig. 3 but with the effects of temperature smearing (or momentum mixing due to the integration over the Fermi sphere) removed in the limit $\delta = \infty$. We see very beautifully the towering mass enhancement peak shortly after the potential was

switched on, followed by boundstate oscillations within the range of the potential as time goes on. All this action leads to outgoing spherical density waves represented in such a plot as ridges and valleys running more or less diagonally across the (r, τ)-plane. This figure can be analyzed in detail in comparison with the structures in the density evolution of a one-component system for a pure boundstate case (e.g. Fig. 5 of Ref. [4]) and a resonance case (e.g. Fig. 8 of Ref. [4]).

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