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THREE DIMENSIONAL STABILITY OF KORTEWEG DE VRIES WAVES AND SOLITONS

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Characteristics of the Korteweg de Vries (K-de V) equation are found in three dimensions. As in the one dimensional case, the equation is hyperbolic and both wave and soliton solutions are stable. Previous results are recovered as special cases.

1. Introduction

We will consider the problem of the stability of ion acoustic waves and solitons in a two component plasma in which the electrons are hot and weightless and the ions are cold ($m_e/m_i \to 0$, $T_i/T_e \to 0$). The relevant equations are, in dimensionless form [1]

$$\frac{\partial n}{\partial t}+\frac{\partial}{\partial x}(nu)+\frac{\partial}{\partial y}(nv)+\frac{\partial}{\partial z}(nw)=0,$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{\partial \phi}{\partial x},$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{\partial \phi}{\partial y},$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{\partial \phi}{\partial z},$$

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}+\frac{\partial^2 \phi}{\partial z^2}=e^{\phi}-n. \tag{1.1}$$

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Here n is the density, ϕ the electric potential, and u, v, w are the x, y, and z components of the ion velocity. The z coordinate will be cyclic in our problem.

In this analysis we will consider modifications to stationary solutions (functions of $X = x - Ut$). These modifications will be described by their generalized fourier-laplace components with space and time dependence given by a function of x times $e^{i(kr+\omega t)}$. Without loss of generality we can choose our coordinate system so that

$$k = (k_x, k_y, 0) \tag{1.2}$$

and so z will be cyclic.

The primary wave or soliton propagates in the x direction with consant velocity U. We will not investigate the full system (1.1), but a model equation for it. If we introduce stretched coordinates according to the scheme

$$\xi = \varepsilon^{1/2}(x - Ut), \quad \tau = \varepsilon^{3/2}t, \quad \eta = \varepsilon y, \tag{1.3}$$

assume amplitudes of all waves and solitons to be small and of the form

$$n = 1 + \varepsilon n^{(1)} + \ldots, \quad u = \varepsilon u^{(1)} + \ldots,$$
$$v = \varepsilon^{3/2} v^{(1)} + \ldots, \quad \phi = \varepsilon \phi^{(1)} + \ldots, \tag{1.4}$$

we obtain the three dimensional generalization of the K-de V equation. (It is often called the two dimensional K-de V equation as z is cyclic.) It is, with $n^{(1)} = \phi^{(1)} = u^{(1)}/U = \phi$

$$\frac{\partial \phi}{\partial \tau} + \phi \frac{\partial \phi}{\partial \xi} + \alpha \frac{\partial^3 \phi}{\partial \xi^3} + \tfrac{1}{2} \frac{\partial v}{\partial \eta} = 0$$
$$\frac{\partial v}{\partial \xi} = \frac{\partial \phi}{\partial \eta}. \tag{1.5}$$

For $v = 0$ this equation reduces to the ordinary K-de V equation, obtained by the above method by Washimi and Taniuti for the first time in this context [2]. In its present, two dimensional form it was obtained first phenomenologically [3] and, somewhat later, rigorously [4]. Both [3] and [4] go on to obtain stability when $k_x = 0$ for the soliton limit. The present analysis, which is general in k and amplitude, will agree with these previous results for that very restricted case.

2. *Basic wave and soliton solutions*

We start with a stationary wave or soliton solution to (1.5), depending on $X = \xi - U\tau$ only. Integrating the first equation in (1.5) twice we obtain successively

$$\alpha \frac{d^2 \phi_0}{dX^2} + \tfrac{1}{2} \phi_0^2 = C, \quad \phi_0 = \phi - U, \tag{2.1}$$

$$\frac{\alpha}{2} \left(\frac{d\phi_0}{dX} \right)^2 = C\phi_0 - \tfrac{1}{6} \phi_0^3 + D. \tag{2.2}$$

Here C and D are constants. Physically meaningful solutions exist if the right-hand side of (2.2) is nonnegative (if $\alpha > 0$). When C is fixed this will occur for D varying between a minimum value, corresponding to the linear wave limit, to a maximum value corresponding a soliton. Intermediate D give nonlinear waves, often called cnoidal waves. Their form is so well known that we will not go into them in any more detail, a good reference being [5].

3. *The dispersion relation for modulations*

Now assume the solution (2.2) to be perturbed. The modulations will be $\sim f(x)e^{i(\mathbf{k}\cdot\mathbf{r}+\omega t)}$, $f(x)$ periodic, and

$$\phi = \phi_0+\delta\phi \qquad v = \delta v. \tag{3.1}$$

We wish to find the dispersion relation

$$\Delta(\omega, \mathbf{k}) = 0 \tag{3.2}$$

so defined for a nonlinear problem. There are several methods for doing this and they are outlined in [6]. Here we work in the X, t coordinate system and k expansion will therefore be particularly suited [7]. To obtain the characteristic velocities we then add U to $d\omega/d\mathbf{k}$ found from the dispersion relation. So assume k small and

$$\begin{aligned} \mathbf{k} &= k(\cos\theta, \sin\theta) \\ \delta\phi &= \delta\phi_0+k\delta\phi_1+k^2\delta\phi_2+ \ldots, \\ \delta v &= \delta v_0+k\delta v_1+k^2\delta v_2+ \ldots, \\ \omega &= \omega_1(\theta)k+\omega_2(\theta)k^2+ \ldots, \end{aligned} \tag{3.3}$$

thus only long wavelength modulations will be considered.

Equation (1.5) becomes (we no longer capitalize x)

$$\begin{aligned} \alpha \frac{d^2}{dx^3}\delta\phi+\frac{d}{dx}(\phi_0\delta\phi) &= -i\omega\delta\phi-ik\cos\theta\phi_0\delta\phi \\ &-\alpha[3ik\cos\theta\delta\phi_{xx}-3k^2\cos^2\theta\delta\phi_x-ik^3\delta\phi]-\tfrac{1}{2}k\sin\theta\delta v, \\ \frac{d}{dx}\delta v &= -ik\sin\theta\delta v+ik\sin\theta\delta\phi. \end{aligned} \tag{3.4}$$

The zero order solutions are

$$\delta v_0 = E \qquad \delta\phi_0 = d\phi_0/dx. \tag{3.5}$$

The second equation in (3.4) is, in first order

$$\frac{d}{dx}\delta v_1 = -i\sin\theta E+i\sin\theta d\phi_0/dx. \tag{3.6}$$

Since δv_1 and ϕ_0 are periodic in x, upon integrating over a period we obtain $E = 0$. The first equation in (3.4) gives, in first order

$$\alpha \frac{d^3}{dx^3} \delta\phi_1 + \frac{d}{dx}(\phi_0 \delta\phi_1) = -i\omega_1 \delta\phi_0 - i \cos\theta \frac{d}{dx}\frac{\phi_0^2}{2} - 3i\alpha \cos\theta \frac{d^3}{dx^3}\phi_0. \quad (3.7)$$

Integrating we obtain

$$\alpha \frac{d^2}{dx^2} \delta\phi_1 + \phi_0 \delta\phi_1 = -i\omega_1 \phi_0 - i \cos\theta \frac{\phi_0^2}{2} - 3i\alpha \cos\theta \phi_{0xx} + C_1. \quad (3.8)$$

The homogeneous equation corresponding to (3.8) is solved by $d\phi_0/dx$ and

$$\frac{d\phi_0}{dx} \int \frac{dx}{(d\phi_0/dx)^2} = x\beta \frac{d\phi_0}{dx} + Q_0(x), \qquad Q_0 \text{ periodic} \quad (3.9)$$

(we have decomposed it into its secular and periodic parts). Using (3.9) it is not difficult to construct a nonsecular solution to (3.8)

$$\delta\phi_1 = \delta\hat{\phi}_1 \cos\theta, \quad (3.10)$$

$$\delta\hat{\phi}_1 = -i\hat{\omega} + \frac{1}{\alpha}\overline{C}\overline{Q}(x) + \frac{i}{\beta} Q_0(x), \quad (3.11)$$

where

$$\omega = \hat{\omega}\cos\theta, \qquad C_1 = \hat{C}\cos\theta,$$

$$\phi_x \int \frac{\phi}{\phi_x^2}\, dx = x\gamma\phi_x + Q_1(x),$$

$$\overline{C} = \hat{C} - iC, \qquad \overline{Q} = Q_1 - (\gamma/\beta) Q_0. \quad (3.12)$$

We will also need $\langle \phi_0 \delta\hat{\phi}_1 \rangle$ and $\langle \phi_0^2 \delta\hat{\phi}_1 \rangle$ in further calculations. They are obtained by multiplying (3.8) by ϕ_0 and ϕ_0^2 respectively and integrating. Finally δv_1 is obtained from (3.6). Collecting all these results

$$\delta\hat{\phi}_1 = -i\hat{\omega}_1 + \overline{C}/\alpha\overline{Q} + i/\beta Q_0,$$

$$\delta v_1 = i \sin\theta(\phi_0 + D),$$

$$\langle \phi_0 \delta\hat{\phi}_1 \rangle = -i\hat{\omega}_1 + \overline{C},$$

$$\tfrac{1}{2}\langle \phi_0^2 \delta\hat{\phi}_1 \rangle = -C\langle \delta\hat{\phi}_1 \rangle - i\hat{\omega}\langle \varphi_0^2 \rangle + 2i\alpha\langle \phi_{0x}^2 \rangle + \overline{C}\langle \phi_0 \rangle. \quad (3.13)$$

Second order

The second order δv equation is

$$\frac{d}{dx}\delta v_2 - \sin\theta\cos\theta(\phi_0 + D) = i \sin\theta\cos\theta\,\delta\hat{\phi}_1 \quad (3.14)$$

and so

$$D = -\langle \phi_0 \rangle - i\langle \delta\hat{\phi}_1 \rangle. \quad (3.15)$$

The $\delta\phi$ equation becomes

$$\alpha \frac{d^3}{dx^3}\delta\phi_2 + \frac{d}{dx}(\phi_0\delta\phi_2) = -i\omega_2\delta\phi_0 - i\omega_1\delta\phi_1 - i\cos\theta\phi_0\delta\phi_1$$

$$-\alpha[3i\cos\theta\delta\phi_{1xx} - 3\cos^2\theta\delta\phi_{0x}] + \tfrac{1}{2}\sin^2\theta[\phi_0 - \langle\phi_0\rangle - i\langle\delta\hat{\phi}_1\rangle]. \qquad (3.16)$$

When we integrate over a period and invoke (3.13) we obtain a condition on $\bar{C}$

$$i\bar{C} = \frac{(\hat{\omega} + \frac{1}{2}\,\mathrm{tg}^2\,\theta)\left(-\hat{\omega} + \dfrac{\langle Q_0\rangle}{\beta}\right) - \hat{\omega}\langle\phi_0\rangle}{1 + \dfrac{1}{\alpha}\langle\hat{Q}\rangle(\hat{\omega} + \frac{1}{2}\,\mathrm{tg}^2\,\theta)}. \qquad (3.17)$$

The dispersion relation is now obtained by multiplying (3.16) by ϕ_0 and integrating over a period. The left-hand side is zero by (2.1) and so

$$-i\hat{\omega}_1\langle\phi_0\delta\hat{\phi}_1\rangle - i\langle\phi_0^2\delta\hat{\phi}_1\rangle - 3i\alpha\langle\phi_{0xx}\delta\hat{\phi}_1\rangle - 3\alpha\langle\phi_{0x}^2\rangle$$

$$+\tfrac{1}{2}\,\mathrm{tg}^2\,\theta[\langle\phi_0^2\rangle - \langle\phi_0\rangle^2 - i\langle\phi_0\rangle\langle\delta\hat{\phi}_1\rangle] = 0, \qquad (3.18)$$

yielding

$$-\hat{\omega}\langle\phi_0\rangle + (4C_0 + \tfrac{1}{2}\langle\phi_0\rangle\,\mathrm{tg}^2\,\theta)\left(-\hat{\omega}_1 + \frac{1}{\beta}\langle Q_0\rangle\right) + \hat{\omega}\langle\phi_0^2\rangle$$

$$-5\alpha\langle\phi_{0x}^2\rangle + \tfrac{1}{2}\,\mathrm{tg}^2\,\theta[\langle\phi_0^2\rangle - \langle\phi_0\rangle^2]$$

$$+\frac{[\langle\phi_0\rangle - \hat{\omega} - \hat{Q}/\alpha(4C_0 + \frac{1}{2}\langle\phi_0\rangle\,\mathrm{tg}^2\,\theta)]\left[(\hat{\omega}_1 + \frac{1}{2}\,\mathrm{tg}^2\,\theta)\left(-\hat{\omega} + \dfrac{1}{\beta}\langle Q_0\rangle\right) - \hat{\omega}\langle\phi_0\rangle\right]}{1 + \dfrac{1}{\alpha}\langle\hat{Q}\rangle(\hat{\omega} + \frac{1}{2}\,\mathrm{tg}^2\,\theta)} = 0.$$

This is our dispersion relation involving, in principle, only ω, k, θ and the constants that determine the nonlinear wave.

4. *The dispersion relation expressed as a cubic in ω/k*

To simplify (3.19) we first note that the transformation

$$x = \sqrt{3\alpha}\,z, \quad \phi_0 = \sqrt{C}\,\phi, \quad D = \xi/6 \qquad (4.1)$$

reduces the number of constants in (2.2), which becomes

$$\left(\frac{d\phi_0}{dz}\right)^2 = \phi_0 - \phi_0^3 + \xi. \qquad (4.2)$$

To work in the new notation we take $\alpha = 1/3$, $C = 1/6$, $D = \xi/6$. It will also prove convenient to introduce

$$Y = 5\langle\phi_{0z}^2\rangle - 3\xi = \langle 5\phi_0 - 5\phi_0^3 + 2\xi\rangle. \qquad (4.3)$$

We will be able to express this quantity in terms of complete elliptic integrals.

Now write (4.2) as (using 3.1)

$$\left(\frac{d\phi_0}{dz}\right)^2 = \tfrac{2}{3}\,\phi_0+\tfrac{2}{3}\,\phi_0\phi_{zz}+\xi. \tag{4.4}$$

Dividing by ϕ_{0z}^2 and integrating over a period we obtain

$$\tfrac{1}{3} = \tfrac{2}{3}\,\gamma+\xi\beta. \tag{4.5}$$

Similar manipulations yield

$$\begin{aligned}\langle Q_0\rangle &= -\gamma+\beta\langle\phi_0\rangle,\\ \xi\langle Q_0\rangle &= -\tfrac{4}{3}\,\langle Q_1\rangle+\tfrac{3}{2}\,\langle\phi_0\rangle,\\ \tfrac{5}{3}\,\langle\phi_{0z}^2\rangle &= \xi+\tfrac{2}{3}\,\langle\phi_0\rangle,\\ -\tfrac{1}{3}\,\langle\phi_0\rangle &= \xi\gamma+\tfrac{2}{9}\,\beta.\end{aligned} \tag{4.6}$$

If we now add the identity (3.12)

$$\langle\bar{Q}\rangle = \langle Q_1\rangle-(\gamma/\beta)\,\langle Q_0\rangle \tag{4.7}$$

we will total enough equations to determine $\langle\bar{Q}\rangle$, $\langle Q_0\rangle$, β, γ in terms of $\langle\phi_{0z}^2\rangle$. However, notation is simpler if we use Y in place of $\langle\phi_{0z}^2\rangle$. Simple algebra gives us the following

$$\langle Q_0\rangle = \tfrac{1}{2}\,Y,\qquad \langle\bar{Q}\rangle = \frac{Y^2-\tfrac{4}{3}}{4(Y+3\xi)}$$

$$\frac{\langle Q_0\rangle}{\beta} = \frac{Y^2+6Y\xi+\tfrac{4}{3}}{2(Y+3\xi)} \tag{4.8}$$

So Y is the only function to be evaluated. After some simple calculations we obtain

$$Y = 2(\phi_{-1}+[\phi_1-\phi_{-1}]E(s^2)/K(s^2)),\qquad s^2 = \frac{\phi_1-\phi_0}{\phi_1-\phi_{-1}} \tag{4.9}$$

where $\phi_1, \phi_0, \phi_{-1}$ are the roots of $\phi-\phi^3+\xi$ in decreasing order, and E and K are complete elliptic integrals.

It is now a matter of straightforward if lengthy calculations to obtain a cubic in $\omega_1 = \omega/k$

$$\begin{aligned}(\omega/k)^2(\omega/k\cos\theta+\tfrac{1}{2}\sin^2\theta)+\frac{2(\omega/k)}{3}\,\frac{3Y(Y+6\xi)+4}{Y^3-4Y+8\xi}\left(\frac{\omega}{k}\cos^2\theta+\tfrac{2}{3}\sin^2\theta\cos\theta\right)\\ +\frac{(27\xi^2-4)8/27}{Y^3-4Y-8\xi}\cos^4\theta-\tfrac{2}{9}\,\frac{8Y+9\xi Y^2+12\xi}{Y^3-4Y-8\xi}\sin^2\theta\cos^2\theta\\ +\frac{2(\tfrac{1}{3}-\tfrac{1}{4}\,Y^2)^2}{Y^3-4Y-8\xi}\sin^4\theta = 0\end{aligned}$$

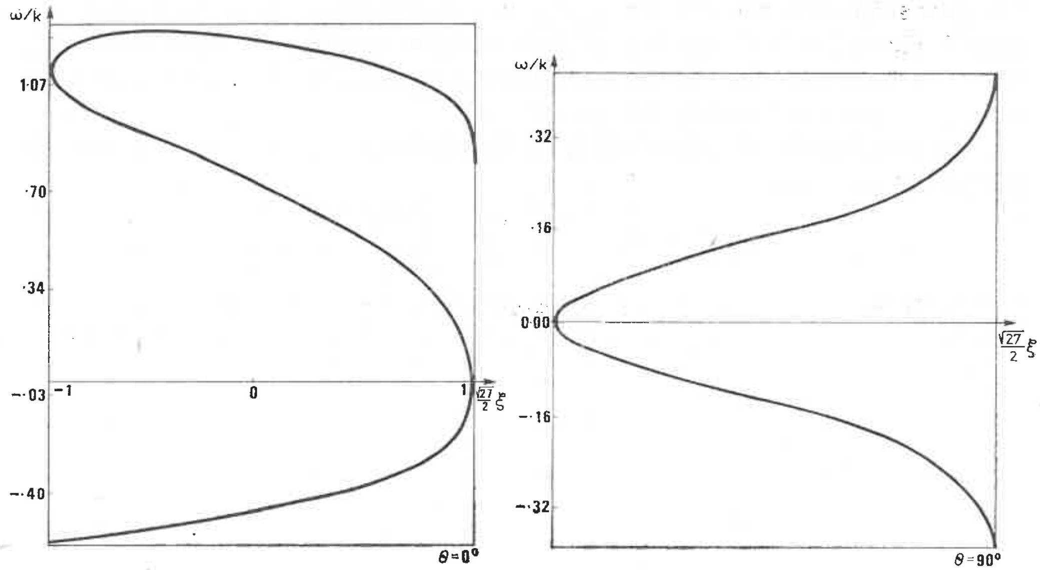

Fig. 1. Phase velocities ω/k as a function of the nonlinear amplitude variable ξ for chosen values of θ. The ξ value at the extreme right corresponds to a soliton

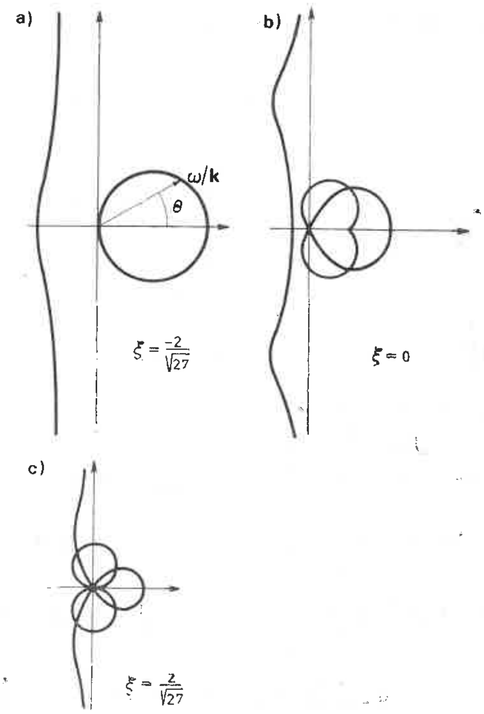

Fig. 2. Polar plots of ω/k for three ξ values. Each plot pictures one physical situation

The three solutions to the cubic are real and are given in figure 1 for chosen values of the angle θ. Figure 2 shows polar plots of phase velocities for the wave amplitude variable ξ fixed. The three plots are for: the linear limit; a nonlinear wave; and the soliton limit, which is of particular interest now-a-days.

The characteristic velocities, which are identical to group velocities as defined here, will be obtained from

$$\boldsymbol{v}_g = U\hat{x}+\boldsymbol{v}_g', \qquad \boldsymbol{v}_g' = \frac{d\omega}{dk} = \left(\frac{\partial\omega}{\partial k}, \frac{1}{k}\frac{\partial\omega}{\partial\theta}\right)$$

in cylindrical coordinates. Thus ω/k found from the cubic is the radial component of $\boldsymbol{v}_g$. The Riemann invariants on our three characteristics are $\phi_{-1}+\phi_0, \phi_{-1}+\phi_1, \phi_0+\phi_1$ [5].

5. *Various limits*

1. In the linear limit $\xi \to -2/\sqrt{27}$ (4.10) yields the roots

$$\omega/k = \frac{2}{\sqrt{3}}\cos\theta, \quad \frac{2}{\sqrt{3}}\cos\theta, \quad -\frac{1}{\sqrt{3}}\cos\theta-\tfrac{1}{2}\sin^2\theta/\cos\theta.$$

They agree with the values obtained from a linear calculation, though one root appears twice. The duality is a linear token of the fact that this root splits in the general, non-linear case (figures 1 and 2).

2. In the soliton limit $\xi \to 2/\sqrt{27}$ the cubic becomes

$$(\omega/k)^2([\omega/k]\cos\theta+\tfrac{1}{2}\sin^2\theta)-\frac{1}{\sqrt{3}}\frac{\omega}{k}\left(\frac{\omega}{k}\cos^2\theta+\tfrac{2}{3}\cos\theta\sin^2\theta\right)$$

$$+\tfrac{2}{9}\sin^2\theta\cos^2\theta-\frac{1}{\sqrt{27}}\sin^4\theta = 0.$$

solved by

$$-\sqrt{\frac{2}{\sqrt{27}}}\sin\theta, \quad \sqrt{\frac{2}{\sqrt{27}}}\sin\theta, \quad \frac{1}{\sqrt{3}}\cos\theta-\tfrac{1}{2}\sin^2\theta/\cos\theta.$$

At $\theta = \operatorname{arc\,tg}\sqrt{2/\sqrt{3}}$ zero is a root. For $\theta = \pi/2$ we regain the result of [3] and [4] ($\omega/k = \pm\sqrt{2/\sqrt{27}}$). See also point 4.

3. The one dimensional case is surprisingly simple

$$\omega/k = -\tfrac{2}{3}(\phi_1-\phi_0)(\phi_1-\phi_{-1})(Y-2\phi_1)^{-1} = \tfrac{1}{3}(\phi_1-\phi_0)(1-E/K)^{-1}$$

$$\omega/k = -\tfrac{2}{3}(\phi_0-\phi_{-1})(\phi_0-\phi_1)(Y-2\phi_0)^{-1} = \tfrac{1}{3}(\phi_1-\phi_0)(1-s^2)\left(\frac{E}{K}-[1-s^2]\right)^{-1}$$

$$\omega/k = -\tfrac{2}{3}(\phi_1-\phi_{-1})(\phi_0-\phi_{-1})(Y-2\phi_{-1})^{-1} = \tfrac{1}{3}(\phi_1-\phi_0)E/K.$$

These values were first obtained by Whitham [8] by a different method.

4. Finally for $\theta = \pi/2$ we have

$$\omega/k = \pm(\tfrac{1}{3}-\tfrac{1}{4}Y^2)(Y^3-4Y-8\xi)^{-1/2}.$$

One root has disappeared. It corresponds to a sound like mode perpendicular to the wave and for it $\omega/k = \infty$. This is consistent with the expansion, since $\partial/\partial y \sim \varepsilon^{-1/2}$ and $\partial/\partial t \sim \varepsilon^{-3/2}$. Therefore the velocity of sound, which was one (and therefore of order zero) before we introduced stretched coordinates, will now appear to be infinite (of order $1/\varepsilon$).

6. *Summary*

Nonlinear ion-acoustic waves and solitons satisfying the Korteweg-de Vries equation are three dimensionally stable. There are in general three distinct values of the phase velocity ω/k. In the vicinity of $\pm\pi/2$ one value of ω/k tends to infinity, whereas the other two become symmetric.

The question of relevance to the full problem as described by (1.1) still remains open. In one dimension some stability properties of K-de V as compared to the full problem have been demonstrated (in the sense that the discarded terms do not alter the equations too drastically). This has been done by general arguments [9]. In [6] we will solve the problem of finding three dimensional characteristics for (1.1). It will then be possible to compare the results of this paper with the solution to the more physical problem, thus obtaining a check on the validity of K-de V.

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