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PROPAGATION OF WEAK MHD DISCONTINUITIES IN RADIATIVE MAGNETOGASDYNAMICS

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Propagation of a weak MHD discontinuity in an optically thick medium of an electrically conducting gas has been studied. The thermal radiation effects on the growth and decay of the wave during propagation have been investigated. A fundamental differential equation governing the growth and decay of the discontinuity is obtained and solved for a plane wave front. It is concluded that if the discontinuity is a compressive wave of order 1, it terminates into a shock wave after a finite critical time t_c, provided that the initial amplitude exceeds a critical value. If the initial discontinuity is an expansive wave, it will decay monotonically and will be damped out ultimately. The effects of thermal radiation and magnetic field accelerate the decaying process. A critical state is also discussed when the compressive wave will either grow or decay.

1. *Introduction*

Thomas [1] studied the growth and decay of sonic discontinuities in ordinary gases. Bürger [2] studied relaxation effects on acceleration waves in non-equilibrium flows of chemically reacting gases. Varley and Rogers [3] studied the high frequency finite acceleration pulses in visco-elastic materials. The effects of diffusion on the growth and decay of acceleration waves were studied by Bowen and Chen [4]. The object of the present paper is to study the propagation of acceleration waves in radiative magnetogasdynamics.

Let the wave surface $\Sigma(t)$ be given by

$$f(x^1, x^2, x^3, t) = 0 \tag{1.1}$$

where x^i are the cartesian coordinates of a point of the surface $\Sigma(t)$ and t is time. Since f represents a wave front, we have

$$\frac{\delta f}{\delta t} \equiv \frac{\partial f}{\partial t} + f_{,i} G n_i = 0, \tag{1.2}$$

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where $\delta/\delta t$ denotes the material derivative as apparent to an observer moving with the wave front. G is the speed of propagation of the wave along the normal direction and n_i are the components of the unit normal vector to the wave surface $\Sigma(t)$. In order to study the propagation of the wave front we use a theory of ray optics which is analogous to that of geometrical optics (Luneberg [5]). If V_i denote components of the ray velocity, we have

$$V_i = \frac{d_r x_i}{dt} = Gn_i + (\delta_{ij} - n_i n_j)\frac{\partial G}{\partial n_j}, \tag{1.3}$$

$$\frac{d_r n_i}{dt} = (n_i n_j - \delta_{ij})\frac{\partial G}{\partial n_j}, \tag{1.4}$$

where dr/dt is the operator of the material derivative along a ray and δ_{ij} is the Kronecker delta.

The geometrical and kinematical compatibility conditions for a singular surface of order one are (Thomas [6])

$$[Z_{,i}] = Bn_i; \quad \frac{\partial Z}{\partial t} = -GB, \tag{1.5}$$

where Z stands for any of the flow variables and B is a scalar function defined over $\Sigma(t)$ by $B = [Z_{,i}]n_i$.

2. *Basic equations*

The system of fundamental differential equations of radiative magnetogasdynamics are

$$\frac{\partial \varrho}{\partial t} + u_i \varrho_{,i} + \varrho u_{i,i} = 0, \tag{2.1}$$

$$\varrho\frac{\partial u_i}{\partial t} + \varrho u_j u_{i,j} + p_{,i} + p^R_{,i} + \mu_0 H_j H_{j,i} - \mu_0 H_j H_{i,j} = 0, \tag{2.2}$$

$$J_i = \mu_0 e_{ijk} H_{k,j}, \quad J^2 = J_i J_i, \tag{2.3}$$

$$\frac{\partial H_i}{\partial t} + u_j H_{i,j} + H_i u_{j,j} - H_j u_{i,j} - \left(\frac{1}{\bar{\sigma}\mu_0}\right) H_{i,jj} = 0, \tag{2.4}$$

$$\left(\frac{l}{v-l}\varrho R + 4a_R T^3\right)\frac{dT}{dt} + (p+4p^R)u_{i,i}$$

$$+\mu_0 u_i H_j (H_{i,j} - H_{j,i}) = \frac{J^2}{\bar{\sigma}} + (K_{\text{eff}} T_{,i})_{,i}, \tag{2.5}$$

where p, T, H_i, J_i, u_i, ϱ and $\bar{\sigma}$ respectively denote the pressure, the temperature, the magnetic field components, the current density components, the gas velocity components, the density and the electrical conductivity. A comma followed by an index (say i) denotes the partial differentiation with respect to a coordinate x_i. E^R and p^R are the radiant ion energy density and radiation pressure, respectively.

We assume that the discontinuity surface $\Sigma(t)$ is such that the magnetic field with its first derivatives and all the other flow parameters themselves are continuous across $\Sigma(t)$, but the second derivatives of the magnetic field and the gradients of the flow variables are discontinuous across it. This implies that

$$[p] = [\varrho] = [u_i] = [T] = [H_i] = [H_{i,j}] = 0, \tag{2.6}$$

where $[Z]$ denotes the jump in the quantity enclosed.

3. Growth equation

The law of conservation of energy across a discontinuity surface implies that

$$\varrho_1(u_{1n}-G)\left[\frac{\nu}{\nu-l}\frac{p}{\varrho}+\mu_0\frac{H^2}{\varrho}+\tfrac{1}{2}u^2+4\frac{p^R}{\varrho}\right]-[K_{\text{eff}}T_{,i}]n_i$$

$$+\frac{c^2\mu_0}{\bar{\sigma}}[H_j(H_{i,j}-H_{j,i})]n_i = 0$$

which in view of (2.6) reduces to

$$[T_{,i}]n_i = 0. \tag{3.1}$$

Now taking jumps in equations (2.1), (2.3), (2.4) and making use of equations (1.5) and (3.1), we obtain

$$(u_n-G)\zeta+\varrho\lambda_i n_i = 0, \tag{3.2}$$

$$\varrho(u_n-G)\lambda_i+\mu n_i = 0, \tag{3.3}$$

$$\varepsilon_i = \bar{\sigma}\mu_0(H_i\lambda_n-H_n\lambda_i), \tag{3.4}$$

where

$$\lambda_i = [u_{i,j}]n_j, \quad \zeta = [\varrho_{,j}]n_j, \quad \mu = [p_{,j}]n_j, \quad \varepsilon_i = [H_{i,jk}]n_j n_k.$$

The equation of state for a polytropic gas model provides us with the relation

$$\mu = a_0^2\zeta, \tag{3.5}$$

where a_0 is the isothermal speed of sound.

Multiplying (3.3) by n_i, summing for i and substituting for μ from (3.5), we get

$$\varrho(u_n-G)\lambda_n+a_0^2\zeta = 0. \tag{3.6}$$

From (3.2), (3.3) and (3.6) we obtain

$$\{(u_n-G)^2-a_0^2\}\zeta = 0. \tag{3.7}$$

The assumption that $\Sigma(t)$ is a discontinuity surface of order 1 implies that $\zeta \neq 0$. Hence we get

$$G = u_n + a_0. \tag{3.8}$$

From (3.2), (3.3) and (3.4) we get

$$\lambda_i = \psi n_i, \quad \zeta = (\varrho | a_0)\psi, \quad \varepsilon_i = \bar{\sigma}\mu_0 H(l_i - l_i n_i)\psi,$$

where ψ is the amplitude of the wave and l_i are components of the unit vector in the direction of the magnetic field.

Differentiating (2.1) and (2.2) with respect to x_k, multiplying by n_k and taking jumps with the help of the second order compatibility conditions of Thomas [6] we obtain

$$(\varrho | a_0)\left(\frac{\delta \psi}{\delta t} + u^\alpha \psi_{,\alpha}\right) = -2\frac{\varrho}{a_0}\psi^2 + 2\varrho\psi\Omega n_i + G\bar{\zeta} - \varrho\bar{\lambda}_i n_i, \tag{3.9}$$

$$\varrho n_i\left(\frac{\delta \psi}{\delta t} + u^\alpha \psi_{,\alpha}\right) = \varrho a_0 \bar{\lambda}_i n_i - \tfrac{4}{3} a_R T^3 \bar{\phi} - \varrho\bar{\sigma} b_0^2 (1 - l_n^2)\psi n_i$$

$$+ \varrho\bar{\sigma} b_0^2 l_n (l_i - l_n n_i), \tag{3.10}$$

where

$$\bar{\lambda}_i = [u_{i,jk}]n_j n_k, \quad \bar{\mu} = [p_{,jk}]n_j n_k, \quad \bar{\zeta} = [\varrho_{,jk}]n_j n_k,$$

and Ω is the mean curvature of $\Sigma(t)$and $b_0^2 = \mu H^2/\varrho$. Here Greek indices denote components of the surface tensor and the latin indices denote components of space tensors.

Taking jump in (2.5) and using compatibility conditions we get

$$[T_{,ii}] = \frac{p + 4p^R}{K_{\text{eff}}}\psi, \tag{3.11}$$

where K_{eff} is the coefficient of effective conductivity due to heat convection and thermal radiation. Differentiating the equation of state successively with respect to x_i and x_j and taking jump we obtain

$$\bar{\mu} = a_0^2\bar{\zeta} + \frac{\varrho R(p + 4p^R)}{K_{\text{eff}}}\psi. \tag{3.12}$$

Using (3.12) in (3.10) we get

$$\varrho G\bar{\lambda}_i - a_0^2\bar{\zeta} = \varrho_0\left(\frac{\delta\psi}{\delta t} + u^\alpha\psi_{,\alpha}\right) - \varrho\bar{\sigma} b_0^2(1 - l_n^2)\psi$$

$$+ \varrho b_0^2 \bar{\sigma} l_n (l_i n_i - l_n) + \frac{\varrho R(p + 4p^R)}{K_{\text{eff}}}\psi n_i + \frac{4a_R T^3 (p + 4p^R)}{3K_{\text{eff}}}\psi n_i. \tag{3.13}$$

Eliminating ζ and $\bar{\lambda}_i$ from (3.9) and (3.13) we get

$$\frac{\delta\psi}{\delta t}+u^{\alpha}\psi_{,\alpha}+\beta\psi+\psi^2 = 0, \tag{3.14}$$

where

$$\beta = \frac{\varrho R(p+4p^R)}{2K_{\text{eff}}}+\frac{2a_R T^3(p+4p^R)}{3K_{\text{eff}}}+\tfrac{1}{2}\,\bar{\sigma}b_0^2\sin^2\theta-a_0\Omega.$$

Here θ is the angle between the magnetic field and the direction of wave propagation. If F is any quantity defined over $\Sigma(t)$, its time derivative along the ray is given by

$$\frac{d_r F}{dt} = \frac{\partial F}{\partial t}+V_i F_{,i}. \tag{3.15}$$

Using (3.8) and (1.3) in (3.15) we get

$$\frac{d_r F}{dt} = \frac{\partial F}{\partial t}+u^{\alpha}F_{,\alpha}. \tag{3.16}$$

Substituting from (3.16) in (3.14) we get

$$\frac{d_r\psi}{dt}+\beta\psi+\psi^2 = 0. \tag{3.17}$$

Now we consider the case of a plane wave front $\Sigma(t)$ for which $\Omega = 0$. Due to planar symmetry the ray direction coincides with the direction of propagation of the wave and hence we have

$$\frac{d_r\psi}{dt} = G\frac{d\psi}{d\sigma}, \tag{3.18}$$

where σ is the distance traversed by the wave front such that $\sigma = 0$ at $t = 0$. Using (3.18) in (3.17) and integrating we get

$$\frac{\psi}{\psi_0} = \frac{1}{\left[\exp(\beta\sigma|G)+\frac{\psi_0}{\beta}(\exp(\beta\sigma|G)-1)\right]}, \tag{3.19}$$

where ψ_0 is the initial amplitude of the wave. When ψ_0 is positive, $\Sigma(t)$ is an expansion wave front which goes on decaying and ultimately damps out. On the other hand when ψ_0 is negative the wave front $\Sigma(t)$ represents a compressive wave which terminates into a shock wave after a finite critical time t_c given by

$$t_c = t_0+\frac{1}{\beta G}\log\left\{\frac{|\psi_0|}{|\psi_0|+\beta}\right\}.$$

4. Conclusions

When ψ_0 is negative, i.e. the initial discontinuity with strength ψ_0 is a compressive wave and has the same numerical value as that of β then the discontinuity will perpetuate. If ψ_0 is negative but less than β then a compressive wave will also decay and will vanish ultimately. If ψ_0 is negative but numerically greater than β then the characteristics will pile up at the wave front to form a shock wave at the instant t_c. The thermal radiation and magnetic field effects will accelerate the decaying process of the wave. In the case of termination into a shock wave the thermal radiation effects will increase the critical time t_c for the formation of a shock wave.

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