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INFORMATION-THEORETICAL METHOD IN INVESTIGATION OF FIRST-ORDER STATISTICS OF A POLARIZED SPECKLE PATTERN

By B. Grzegorzewski

Institute of Mathematics and Physics, Technical and Agricultural Academy, Bydgoszcz*

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The first-order statistics of a polarized speckle pattern is obtained by using the information-theoretical method. The results are in agreement with those recently obtained on the basis of "random walk in the complex plane" and are their generalization.

1. Introduction

The scattered radiation from an optically rough surface illuminated by coherent light forms a speckle pattern due to random differences in optical path length. The statistical properties of a speckle pattern have been investigated by many authors. The recent book edited by Dainty [1] contains an extensive bibliography on this subject. Many investigators have performed a theoretical study on the statistics of a speckle pattern. Goodman [2, 3] suggested the model of the scattering medium, in which one was treated as a collection of a large number of independent scatterers. The amplitude of the electric field at a given point of speckle consists of multimode contributions from different scattering regions. The statistics of the complex field of speckle was performed on the basis of a "random walk in the plane" [2]. It follows from the central limit theorem (CLT) that, as the number of scattering regions is large, the statistics of the complex field is asymptotically Gaussian. However, if the number of scatterers is small, so that the CLT cannot be applied, the statistics of speckle will deviate from the Gaussian.

The statistical properties of speckle may be used in the study of the statistical properties of the scattered object if they are investigated with relation to the point response function of an optical system applied for producing speckles, and the coherence of incident light. Asakura and his collaborators [4, 5, 9] have studied the probability density function of intensity and the average contrast of a speckle pattern as a function of surface roughness

* Address: Instytut Matematyki i Fizyki, Akademia Techniczno-Rolnicza, Hanki Sawickiej 28, 85-084 Bydgoszcz, Poland.

and coherence of light. They have also paid attention to the entropy of speckle patterns [10], which is more independent parameter of speckle than the average contrast. These investigations were carried out for the conditions under which the statistics of the complex field was Gaussian. We introduce the method to study the statistical properties of speckle, which obey the general case of the statistics of a complex field.

In the theory of statistical inference there are many methods of estimating unknown statistical parameters and testing statistical hypothesis. One of these methods, called information-theoretical, is based on maximizing a certain functional (information) with respect to the prescribed mean values of some quantities. Information-theoretical methods were introduced to theoretical physics by Jaynes [6] and Ingarden [7, 8]. Application of these methods gave rise to the formulation of the so-called information-thermodynamics being a far reaching generalization of common thermodynamics. This paper is an introduction to the study of statistical properties of a speckle pattern on the basis of the information theory. The principle of maximum information is applied to obtain the most probable intensity distribution of polarized laser speckle pattern.

2. Statistics of the complex amplitude

Let us consider a polarized speckle pattern obtained from laser light. The electric field at the point $\vec{r}$ and time instant t has the form:

$$E(\vec{r}, t) = A(\vec{r}) \exp(i2\pi\nu t). \tag{1}$$

$A(\vec{r})$ is a complex phasor amplitude and ν is the optical frequency. Function $A(\vec{r})$ represents the statistical properties of speckle. This is a complex-valued function of space.

$$A = A_r + iA_i, \tag{2}$$

A_r and A_i are the random variables of an unknown probability density function. Let us assume that these random variables A_r and A_i are statistically independent. In this situation

$$p(A_r, A_i) = p(A_r)p(A_i). \tag{3}$$

In order to find $p(A_r)$ and $p(A_i)$ we assume that the values

$$\langle A_r^k \rangle \text{ and } \langle A_i^k \rangle, \quad k = 1, 2, \ldots, n, \tag{4}$$

of n statistical moments are known, and also

$$\langle A_r^0 \rangle = \langle A_i^0 \rangle = 1. \tag{5}$$

It follows from the maximum information principle that the most probable distribution corresponding to the given knowledge (4), (5) realizes the maximum of information S, where

$$S[p(A_r)] = -\int_{-\infty}^{+\infty} p(A_r) \ln p(A_r) dA_r. \tag{6}$$

For $p(A_i)$, S will have the same form. In order to solve this problem one should introduce the function

$$L[p(A_r)] = S[p(A_r)] - \sum_{k=0}^{n} \lambda_{r,k} \int_{-\infty}^{+\infty} A_r^k p(A_r) dA_r, \tag{7}$$

where λ_k are Lagrange undetermined multipliers. The most likely probability density function $p(A_r)$ satisfies the condition

$$\frac{\delta L[p(A_r)]}{\delta p(A_r)} = 0|_{p(A_r)=p^0(A_r)}. \tag{8}$$

The maximization yields

$$p^0(A_r) = Z_r^{-1}(\lambda_{r,1}, \ldots, \lambda_{r,n}) \exp\left(-\sum_{k=1}^{n} \lambda_{r,k} A_r^k\right), \tag{9}$$

where

$$Z_r(\lambda_{r,1}, \ldots, \lambda_{r,n}) = \int_{-\infty}^{+\infty} \exp\left(-\sum_{k=1}^{n} \lambda_{r,k} A_r^k\right) dA_r. \tag{10}$$

The Lagrange multipliers can be calculated from the equations

$$\langle A_r^k \rangle = -\frac{\partial \ln Z_r(\lambda_{r,1}, \ldots, \lambda_{r,n})}{\partial \lambda_{r,k}}. \tag{11}$$

For the $p(A_i)$ we obtain

$$p^0(A_i) = Z_i^{-1}(\lambda_{i,1}, \ldots, \lambda_{i,n}) \exp\left(-\sum_{k=1}^{n} \lambda_{i,k} A_i^k\right). \tag{12}$$

Thus, on the basis (2) we may write

$$p^0(A_r, A_i) = \mathscr{Z}^{-1} \exp\left[-\sum_{k=1}^{n} (\lambda_{r,k} A_r^k + \lambda_{i,k} A_i^k)\right], \tag{13}$$

where

$$\mathscr{Z}^{-1} = Z_r^{-1} Z_i^{-1}. \tag{14}$$

The formula (13) is the probability density function of real and imaginary parts of the field.

We must now pay attention to the fact that the maximization is possible if integral (10) — for the real, as well as for the imaginary field — has finite values. This will be obeyed if the number n of known statistical moments of A_r and A_i is even, and

$$\lambda_{r,k} > 0, \quad \lambda_{i,k} > 0 \quad \text{for } k = n. \tag{15}$$

3. *Statistics of intensity*

The intensity I and phase θ of the resultant field are related to the real and imaginary parts of the complex amplitude by the transformation

$$A_r = \sqrt{I}\cos\theta, \quad A_i = \sqrt{I}\sin\theta. \tag{16}$$

Applying this transformation to equation (13) we have

$$p^0(I, \theta) = \tfrac{1}{2}\mathscr{Z}^{-1}\exp\{-\sum_{k=1}^{n}[\lambda_{r,k}(\sqrt{I}\cos\theta)^k + \lambda_{i,k}(\sqrt{I}\sin\theta)^k]\}. \tag{17}$$

The probability distribution of intensity alone is obtained by integrating (17) over the phase.

$$p^0(I) = \tfrac{1}{2}\mathscr{Z}^{-1}\int_{-\pi}^{+\pi}\exp\{-\sum_{k=1}^{n}[\lambda_{r,k}(\sqrt{I}\cos\theta)^k + \lambda_{i,k}(\sqrt{I}\sin\theta)^k]\}d\theta. \tag{18}$$

This is a general form for the probability density function of the intensity of a polarized laser speckle pattern.

From the experimental point of view it is interesting to find the function $p^0(A_r, A_i)$ and moments (4) when we have the intensity distribution. This problem can be solved numerically. Having a probability density function of intensity from the experiment we can find values of Lagrange multipliers using our probability density function (18). The Lagrange multipliers give immediately the probability density function of real and imaginary parts of the field (13), and from (10), (11) the values of moments (4).

Ohtsubo and Assakura [9] have studied the statistical properties of a laser speckle pattern by investigating their probability density function of intensity and the contrast on the basis of Goodman's theory. It is easy to show that their results are related to our investigations when Lagrange multipliers take the values

$$\lambda_{r,1} = \lambda_1, \quad \lambda_{i,1} = 0, \quad \lambda_{r,2} = \lambda_{i,2} = \lambda_2, \quad \lambda_{i,k} = \lambda_{r,k} = 0 \quad \text{for } k \geqslant 3. \tag{19}$$

From (19) and (18) we see that the probability density function of intensity has the form

$$p^0(I) = \lambda_2 \exp[-\lambda_2(I+\beta_1^2)]I_0(2\lambda_2\sqrt{I}\,\beta_1), \tag{20}$$

where

$$\beta_1 = -\frac{\lambda_1}{2\lambda_2}.$$

$I_0(\ldots)$ is a modified Bessel function. From (13) we see that the probability density function of real and imaginary parts of field is Gaussian, and from (11)

$$\frac{1}{2\lambda_2} = \langle A_r^2\rangle - \langle A_r\rangle^2, \tag{21}$$

$$\lambda_1 = \langle A_r\rangle. \tag{22}$$

From (19) we obtain

$$\langle A_r^2 \rangle = \langle A_i^2 \rangle \tag{23}$$

and

$$\langle A_i \rangle = 0. \tag{24}$$

The probability density function of intensity (20) has the same form as the one previously derived by Goodman.

4. *Conclusions*

The general formula giving the probability density function of intensity of a polarized laser speckle pattern has been derived by using the information-theoretical method. It was shown that the Gaussian distributed field is a special case in our theoretical results. Investigations based on the information-theoretical method may be especially useful for non-Gaussian distributed fields.

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REFERENCES

[1] *Laser Speckle and Related Phenomena*, edited by J. C. Dainty, Springer-Verlag, New York 1975.
[2] J. W. Goodman, p. 9 in Ref. [1].
[3] J. W. Goodman, *J. Opt. Soc. Am.* **66**, 1145 (1976).
[4] H. Fujii, T. Asakura, *Opt. Commun.* **11**, 35 (1974).
[5] H. Fujii, T. Asakura, *Opt. Commun.* **12**, 32 (1974).
[6] E. T. Jaynes, *Phys. Rev.* **106**, 620 (1957).
[7] R. S. Ingarden, *Fortschr. Phys.* **13**, 755 (1965).
[8] R. S. Ingarden, A. Kossakowski, *Ann. Phys.* (USA) **89**, 451 (1975).
[9] J. Ohtsubo, T. Asakura, *Optik* **45**, 65 (1976).
[10] T. Asakura, J. Ohtsubo, *Optik* **46**, 19 (1976).