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COMPLETELY INTEGRABLE SYSTEMS OF N PARTICLES IN ONE DIMENSION

By S. Wojciechowski

Institute of Theoretical Physics, Warsaw University*

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A general prescription for constructing completely integrable one-dimensional systems of N interacting particles is found. It appears that within this approach the only integrable potential for systems of particles interacting with first or second neighbours is given by the exponential function. The integrability conditions for systems of particles interacting with each other lead to a functional-differential equation for the potential, with Weierstrass function as the most general solutions. The complete integrability of all considered systems is proved.

1. Introduction

The integrability of Hamiltonian systems is an exceptional property. As is well known, the problem of N bodies is non-integrable for $N \geqslant 3$. Even in one dimension a system of N particles interacting on a line or circle can be integrated completely only for some special interacting potentials. Therefore, it is not surprising that the efforts of mathematicians of the nineteenth century succeded in integrating explicitly only some classical cases, and then progress stopped. Poincare was the first to show that most Hamiltonian systems are nonintegrable. This conclusion turned investigations in classical mechanics into another direction, i. e. approximate methods and stability problems.

But the knowledge of integrable cases is still very useful for many purposes. Approximation techniques can be tested by examining exact solutions. Moreover, every solvable N-body model improves our understanding of classical statistical mechanics.

New possibilities for finding integrable systems have arisen quite recently. They happen to be closely related to the discovery by Kruskal and other authors [1] of the strongly stable wave solutions, the so-called solitons, for the Korteweg-de Vries equation and other special nonlinear wave equations (Sine-Gordon, modified Korteweg-de Vries and nonlinear Schrodinger equation). It has been shown by Zakharov and Faddeev [2] that the existence of infinitely many conservation laws for the K-dV equation is closely related to

* Address: Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, Hoża 69, 00-681 Warszawa, Poland.

its Hamiltonian structure. As was further explained, all the exceptional properties of these nonlinear equations, particularly the existence of exact analytical solutions, are due to the fact that these equations describe completely integrable Hamiltonian systems with an infinite (continuum) number of degrees of freedom. Similarly, in [3, 4] the complete integrability of the Toda lattice as a Hamiltonian system with infinite (countable) number of degrees of freedom was proved by Flaschka and independently by Manakov. These results made it possible to find some examples of completely integrable systems of N(N-arbitrary) interacting particles on the line. The best known is the finite Toda lattice and other ones are given by Moser [5] and Calogero [6].

Below, only Hamiltonian systems with a finite number of degree of freedom will be investigated. Starting from the particular results in [3, 5, 6], some general integrability conditions are formulated and some new completely integrable systems are found. The possibility of transition to infinite lattices and to problems of statistical mechanics is open.

2. *The notion of integrability*

The notion of integrability is intuitive but nevertheless it is convenient to adopt some rigorous definition. Let $\{A, B\}$ denote the Poisson bracket of quantities A and B. A quantity I is an integral of a system with a given Hamiltonian H if $\{I, H\} = 0$. Furthermore, N functions $I_1; \ldots, I_N$ are said to be in involution if $\{I_k, I_l\} = 0$ for every pair (k, l) where $k, l = 1, 2, \ldots, N$. An adequate definition of integrability is based on the following theorem of Liouville.

Theorem

If for a Hamiltonian dynamical system with N degrees of freedom $\dot{q}_i = \dfrac{\partial H}{\partial p_i}$, $\dot{p}_i = -\dfrac{\partial H}{\partial q_i}$, $i = 1, \ldots, N$, there exist N univalent, functionally independent integrals $I_1, \ldots, I_N$ in involution, then it is possible to construct N new integrals and the system is integrable by quadratures. Therefore the appropriate definition of the complete integrability we set as below.

Definition

The system of Hamiltonian equations $\dot{q}_i = \dfrac{\partial H}{\partial p_i}$, $\dot{p}_i = -\dfrac{\partial H}{\partial q_i}$, $i = 1, \ldots, N$ is called completely integrable if there exist N time independent univalent, functionally independent integrals in involution. Note that the assumptions of the Liouville theorem are strengthen by time independence of the integrals. For a better understanding of this definition, we outline here the proof of Liouville's theorem (see [7] or [8]). The main task is to construct a canonical transformation from (p, q) to new generalized coordinates (β, α) such that the dynamical equations trivialize to $\dot{\alpha}_r = 0$, $\dot{\beta}_r = 0$. Let the quantities $I_k(p, q, t) = \alpha_k$ be the independent integrals in involution, $\{\alpha_i, \alpha_j\}_{p,q} = 0$. Therefore they can be taken as new generalized "position variables". To determine new generalized momenta, it is convenient

to construct a generating function $W(q, \alpha, t)$. By the independence of I_k, the $I_k(p, q, t) = \alpha_k$ can be resolved, leading to $p_r = f_r(q, \alpha, t)$. The generating function can now be introduced according to

$$dW(q, \alpha, t) = \sum_{r=1}^{N} p_r dq_r - H dt = \sum_{r=1}^{N} f_r(q, \alpha, t) dq_r - H(q, f(q, \alpha, t), t) dt.$$

The identities $\frac{\partial f_r}{\partial q_s} = \frac{\partial f_s}{\partial q_r}$ and $\frac{\partial f_r}{\partial t} = -\frac{\partial H}{\partial q_r}$ can be proved by confirming that $\sum_r p_r dq_r - H dt$ is in fact a complete differential (the computations are omitted). The canonical transformation corresponding to $W(q, \alpha, t)$ is given by

$$p_r = \frac{\partial W}{\partial q_r}, \quad \beta_r = -\frac{\partial W}{\partial \alpha_r}; \quad H' = H + \frac{\partial W}{\partial t} = H - H = 0, \quad r = 1, \ldots, N.$$

Here the $\beta_r = -\frac{\partial W}{\partial \alpha_r}$ are generalized momenta. As is obvious, the dynamical equations in the new coordinates (β, α) are

$$\dot{\alpha}_r = 0, \quad \dot{\beta}_r = 0, \quad r = 1, \ldots, N$$

with solutions

$$\alpha_r = \text{const} = f_r(p, q, t), \quad \beta_r = -\frac{\partial W(q, \alpha, t)}{\partial \alpha_r} = \text{const.} \tag{2.1}$$

The quantities $-\frac{\partial W(q, \alpha, t)}{\partial \alpha_r} = \beta_r$ are thus the new N independent integrals. Their independency results from $\det \left\| \frac{\partial^2 W}{\partial q_r \partial \alpha_s} \right\| \neq 0$. It is essential that the $\beta_1, \ldots, \beta_N$ are obtained by quadratures. A phase trajectory of motion could be obtained by determining $p(\alpha, \beta, t)$ and $q(\alpha, \beta, t)$ from (2.1). The existence of N integrals in involution is a very strong property because it gives information about the global structure of the trajectories. Note that locally, by the theorem asserting the existence and uniqueness of solutions of ordinary differential equations, there exist $2N$ integrals p_0, q_0 having the sense of the initial conditions for $p(t) = p(p_0, q_0, t)$ and $q(t) = q(p_0, q_0, t)$. But in general, they may not be given by quadratures or they may not exist globally, and usually depend explicitly upon time.

3. Dynamical equations of completely integrable Hamiltonian systems

It is usually very difficult to decide whether some given equations are completely integrable. It is, however, possible to give a constructive prescription which allows one to obtain a wide class of integrable systems. The right idea may be derived from the work of Lax [9].

As is known from linear algebra, all the eigenvalues of the matrix are invariants of the similarity transformations. This suggests that we limit consideration to the subclass

of cases when the states (p, q) of the system in phase space can be adequately described by some matrices L, and time evolution of the system in given by the similarity transformation

$$L(t) = K(t)L(0)K^{-1}(t), \tag{3.1}$$

where

$$K(t)K^{-1}(t) = I. \tag{3.2}$$

Then all invariants of the matrix $L(t)$ are automatically integrals of the motion. To have a sufficient number of independent integrals it is reasonable to assume that the matrix L is in general diagonalizable for the assumption of other Jordan types would imply some degeneration of invariants. Below, the diagonalizability of L will be achieved by assuming its hermicity. The equations for $L(t)$ can be obtained by differentiating (3.1) and (3.2) with respect to t. One has

$$(K^{-1})_t LK + K^{-1}L_t K + K^{-1}LK_t = 0, \quad K_t K^{-1} + K(K^{-1})_t = 0.$$

Hence:

$$L_t = -K(K^{-1})_t L - LK_t K^{-1} = K_t K^{-1} L - LK_t K^{-1}.$$

Writing $K_t K^{-1} = -A$, one obtains the evolution equation in the Lax form

$$\frac{d}{dt} L = LA - AL = [L, A]. \tag{3.3}$$

Here the one-parameter family of operators $K(t)$ satisfies the differential equation $\frac{d}{dt} K = -AK$. In particular, for $L(t)$ hermitian and $K(t)$ unitary, the matrix $A(t)$ is skew-hermitian, i. e.

$$A^* = -(K_t K^{-1})^* = -KK_t^* = K_t K^* = -A.$$

These considerations suggest that Eq. (3.3) is adequate as a starting point for the construction of completely integrable systems. Let us now assume that Eq. (3.3) is simultaneously of Hamiltonian type. To find the hypothetical Hamiltonian, let us restrict our attention to conservative systems. Then the Hamiltonian is an integral of motion, and so it can be constructed from the invariants of the matrix L. What invariants can be used? Here the method of embodying the Hamiltonian system into the Lax system still admits some ambiguity. The most natural possibility arises by assuming that L is constructed linearly from the momenta p and some functions of the coordinates. In the natural coordinates the Hamiltonian is a bilinear form in the momenta. This implies that only the two invariants of first and second order can be used, namely $J_1 = \operatorname{Tr} L$ and $J_2 = \operatorname{Tr} L^2$, and that the Hamiltonian should be

$$H = \alpha J_1 + \beta J_2 + \gamma (J_2)^2 \text{ with } \alpha, \beta, \gamma \text{ real.} \tag{3.4}$$

The basic structural condition is that the Hamiltonian equations obtained from H

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad i = 1, \ldots, N$$

should agree with (3.3). The above method of constructing dynamical integrable systems is useful not only in embodying the subcases of Hamiltonian mechanics, but it can also be associated with other interesting evolution processes. For instance, in the work of Moser [5] and Kac and van Moerbeke [10], non-Hamiltonian dynamics has been investigated for the so-called discrete Korteweg-de Vries equations (though it can be embedded in the Hamiltonian system of the Toda lattice). Below we apply this method in order to study an N-point system in one space dimension. The concrete shape of the matrix L used below is partly justified by general arguments, but the specific work of Flashka [3] and Moser [5] is also very helpful.

4. Systems of particles interacting with nearest and next neighbours

The result is that in this class the only integrable potential is the exponential function. Among such systems the best known is the Toda lattice with periodic boundary condition ([11–13]). In the possible modifications, the interactions with the next, the third and even further neighbours could be taken into account. But it seems that in all cases the potential should be exponential.

Let us examine N particles interacting on a circle with both its nearest neighbours by the same potential. The positions of the particles are described by the x_n variables measuring the displacement of the n-th particle from equilibrium. Because none of the points is distinguished from the others, the numeration can start from an arbitrary point. This means that the matrix L describing a state of the system should be invariant under any displacement of the numeration of points. This condition is satisfied by the matrix L in the form

$$L = \begin{Vmatrix} p_1 & ia_{12} & 0 & \ldots & 0 & -ia_{N1} \\ -ia_{12} & p_2 & ia_{23} & \ldots & 0 & 0 \\ 0 & -ia_{23} & p_3 & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & ia_{N-1,N} \\ ia_{N1} & 0 & & \ldots & -ia_{N-1,N} & p_N \end{Vmatrix}, \tag{4.1}$$

where p_n are the momenta and $a_{n,n+1} = a(x_n - x_{n+1})$ with $a(x)$ a certain given real function. Moreover it is clear why all the other matrix elements are equal to zero, for there is no interaction between distant points. For the above L matrix, now choose the Hamiltonian

to be $H = \frac{1}{2}\,\mathrm{Tr}\,L^2$. This choice is general enough as the term $\mathrm{Tr}\,L = \sum_{i=1}^{N} p_i$ commutes with the Hamiltonian (3.4) and vanishes in the mass center coordinates. Henceforth we take

$$H = \tfrac{1}{2}\,\mathrm{Tr}\,L^2 = \tfrac{1}{2}\sum_{k=1}^{N} p_k^2 + \sum_{k=1}^{N} a_{k,k+1}^2, \quad \text{with } N+1 = 1(\mathrm{mod}\,N) \tag{4.2}$$

and the equations of motion are given by

$$\dot{x}_k = \frac{\partial H}{\partial p_k} = p_k, \tag{4.3a}$$

$$\dot{p}_k = -\frac{\partial H}{\partial x_k} = 2(-a'_{k,k+1}a_{k,k+1} + a'_{k-1,k}a_{k-1,k}), \quad k = 1, \ldots, N, \tag{4.3b}$$

where $a'(x)$ means differentiation with respect to the argument. Furthermore, we note that

$$\dot{a}_{k,k+1} = a'_{k,k+1}(p_k - p_{k+1}). \tag{4.3c}$$

To identify the equations (4.3a, b, c) with those obtained from (3.3), it is convenient to assume the skew-hermitian matrix A to be

$$A = i\begin{Vmatrix} G_1 & f_{12} & 0 & \ldots & 0 & f_{N1} \\ f_{12} & G_2 & f_{23} & & & 0 \\ 0 & f_{23} & G_3 & & & \\ . & & & & & \\ . & & & & & \\ 0 & & & & & f_{N-1,N} \\ f_{N1} & 0 & . & \ldots & f_{N-1,N} & G_N \end{Vmatrix}, \tag{4.4}$$

where $f_{k,k+1} = f(x_k - x_{k+1})$. Here $f(x)$ and $G_k(\cdot)$ are given real functions. Then

$$\dot{p}_k = 2(a_{k-1,k}f_{k-1,k} - a_{k,k+1}f_{k,k+1}), \tag{4.5a}$$

$$\dot{a}_{k,k+1} = f_{k,k+1}(p_k - p_{k+1}) + ia_{k,k+1}(G_{k+1} - G_k)\,(\mathrm{mod}\,N) \tag{4.5b}$$

$$0 = a_{k,k+1}f_{k,k+2} - f_{k,k+1}a_{k,k+2}. \tag{4.5c}$$

The choice $G_k = \mathrm{const} = 0$ and

$$f_{k,k+1} = a'_{k,k+1}, \tag{4.6a}$$

$$0 = a_{k,k+1}f_{k,k+2} - f_{k,k+1}a_{k,k+2}, \tag{4.6b}$$

ensures the consistency of Eqs (4.5a, b, c) and (4.3a, b, c). By resolving (4.6a, b) one gets $a'(x_k - x_{k+1})/a(x_k - x_{k+1}) = c$ with the unique solution $a(x_k - x_{k+1}) = \varepsilon \exp[c(x_k - x_{k+1})]$ where ε, c are real constants. This means that within this approach the exponential function is the unique potential that makes the "circular system" of N particles interacting with nearest neighbours completely integrable.

A generalized system having interactions with nearest and next neighbours can be considered in a similar manner. Here symmetry arguments suggest that we take the matrix L in the form

$$L = \begin{Vmatrix} p_1 & ia_{12} & ia_{13} & 0 & \dots & -ia_{N-1,1} & -ia_{N1} \\ -ia_{12} & p_2 & ia_{23} & & & 0 & -ia_{N2} \\ ia_{13} & -ia_{23} & p_3 & & & & 0 \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ ia_{N-1,1} & & & & & & ia_{N-1,N} \\ ia_{N1} & ia_{N2} & \cdot & \cdot & \dots & -ia_{N-1,N} & p_N \end{Vmatrix}, \tag{4.7}$$

where $a_{k,k+1} = a(x_k - x_{k+1})$, $a_{k,k+2} = b(x_k - x_{k+2})$ and $a(x)$, $b(x)$ are some given real functions (it is assumed that the interaction with nearest neighbours might be different from the interaction between the next ones). The Hamiltonian is

$$H = \tfrac{1}{2}\,\mathrm{Tr}\,L^2 = \tfrac{1}{2}\sum_{k=1}^{N} p_k^2 + \sum_{k=1}^{N}(a_{k,k+1}^2 + a_{k,k+2}^2)\ (\mathrm{mod}\ N) \tag{4.8}$$

and accordingly the Hamiltonian equations are

$$\dot{x}_k = \frac{\partial H}{\partial p_k} = p_k, \tag{4.9a}$$

$$\dot{p}_k = 2(a'_{k-1,k}a_{k-1,k} - a'_{k,k+1}a_{k,k+1}) + 2(a'_{k-2,k}a_{k-2,k} - a'_{k,k+2}a_{k,k+2}), \tag{4.9b}$$

implying

$$\dot{a}_{k,k+1} = a'_{k,k+1}(p_k - p_{k+1}); \quad \dot{a}_{k,k+2} = a'_{k,k+2}(p_k - p_{k+2}). \tag{4.9c}$$

The matrix A is assumed to have the same form as previously but to be similarly enlarged by an additional skew row above and under the diagonal, whose elements $f_{k,k+2}, f_{k+2,k}$ satisfy $f_{k,k+2} = f_{k+2,k}$. The resulting equations (3.3) are

$$\dot{p}_k = 2(a_{k-1,k}f_{k-1,k} - a_{k,k+1}f_{k,k+1}) + 2(a_{k-2,k}f_{k-2,k} - a_{k,k+2}f_{k,k+2}), \tag{4.10a}$$

$$\dot{a}_{k,k+1} = f_{k,k+1}(p_k - p_{k+1}) + ia_{k,k+1}(G_{k+1} - G_k)$$
$$+ i[(a_{k,k-1}f_{k-1,k+1} - f_{k,k-1}a_{k-1,k+1}) + (a_{k,k+2}f_{k+2,k+1} - f_{k,k+2}a_{k+2,k+1})], \tag{4.10b}$$

$$\dot{a}_{k,k+2} = f_{k,k+2}(p_k - p_{k+2}) + i(a_{k,k+1}f_{k+1,k+2} - f_{k,k+1}a_{k+1,k+2}) + ia_{k,k+2}(G_{k+2} - G_k), \tag{4.10c}$$

$$0 = ia_{k,k+3} = -a_{k,k+1}f_{k+1,k+3} - a_{k,k+2}f_{k+2,k+3} + f_{k,k+1}a_{k+1,k+3} + f_{k,k+2}a_{k+2,k+3}, \tag{4.10d}$$

$$0 = ia_{k,k+4} = -a_{k,k+2}f_{k+2,k+4} + f_{k,k+2}a_{k+2,k+4}. \tag{4.10e}$$

The compatibility conditions for (4.10a, b, c, d, e) and (4.9a, b, c) give us $f_{km} = a'_{km}$ motivating the assumed form of A. Furthermore (4.10) yields $f_{k,k+2}/a_{k,k+2} = a'_{k,k+2}/a_{k,k+2} = d = \text{const}$ and so $a_{k,k+2} = \varepsilon \exp[d(x_k - x_{k+2})]$. This reduces (4.10d) to:

$$a_{k+1,k+3}(da_{k,k+1} - f_{k,k+1}) = a_{k,k+2}(da_{k+2,k+3} - f_{k+2,k+3}).$$

The solution is given by

$$a_{k,k+1} = a(x_k - x_{k+1}) = [-d(x_k - x_{k+1}) + h] \exp[c(x_k - x_{k+1})],$$

with ε, c, d, h real constants. For non-trivial constants c, d the inspection of purely imaginary terms in Eqs (4.10b, c) leads now to a contradiction. This means that assumed system of points interacting non-trivially with nearest and next neighbours is not integrable by means of the variant of the Lax trick applied here. Nevertheless, the equations expressing the interaction with the next neighbours only are solvable, and the potential is the exponential function as found previously. In that case the structure of the system essentially depends upon the parity of the number N of points on the circle. For N even it splits into two independently interacting Toda lattices. But for N odd all particles constitute one joined system in which any particle interacts with its nearest neighbours through $(N\text{-}3)/2$ other intermediary particles. These systems have not been investigated yet.

5. Systems of particles interacting with each other

This class includes some potentials already investigated by Moser [4] and Calogero [6] namely $V(x_i - x_j) = 1/(x_i - x_j)^2$, $V(x_i - x_j) = 1/\sin^2(x_i - x_j)$, $V(x_i - x_j) = 1/\mathrm{sh}^2(x_i - x_j)$, where x_i denotes variable on the line or circle respectively. Their integrability has been proved on the basis of concrete properties of the functions considered. The formulation presented below states the problem more generally and provides a functional differential equation for the potential.

Suppose N particles interact with each other through an identical potential depending upon mutual distances. Material points are numbered from 1 to N and the numeration does not change in time. The description of the system should be insensitive to any permutation of numbers. This suggests the choice of the following matrix L as it possesses the required symmetry:

$$L = \begin{Vmatrix} p_1 & ia_{12} & ia_{13} & \cdots & ia_{1N} \\ ia_{21} & p_2 & & & \cdot \\ ia_{31} & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ ia_{N1} & & & & p_N \end{Vmatrix}, \tag{5.1}$$

where $a_{ij} = a(x_i - x_j)$ with a certain real function $a(x)$. The hermicity of L, i.e. $a_{km} + a_{mk} = a(x_k - x_m) + a(x_m - x_k) = 0$ implies that the function $a(x)$ is odd.

One could think about a different Lax algebraization of the system with the more general L. Here there would be no immediate conclusion about an odd parity of $a(x)$. The resulting Hamiltonian is

$$H = \tfrac{1}{2}\,\mathrm{Tr}\,L^2 = \tfrac{1}{2}\sum_{k=1}^{N} p_k^2 + \sum_{k,r}{}' a_{kr}^2, \tag{5.2}$$

where $\sum_{r,k}'$ denotes an independent summation of k and r excluding $k = r$. The Hamiltonian equations are of the form

$$\dot{x}_k = \frac{\partial H}{\partial p_k} = p_k, \tag{5.3a}$$

$$\dot{p}_k = -\frac{\partial H}{\partial x_k} = -2\sum_{\substack{r=1\\ r\neq k}}^{N} a_{kr}a'_{kr}, \tag{5.3b}$$

and consistently

$$\dot{a}_{kr} = a'_{kr}(p_k - p_r). \tag{5.3c}$$

For these equations the matrix A should be assumed to be

$$A = i\begin{Vmatrix} G_1 & f_{12} & f_{13} & \cdots & f_{1N} \\ f_{21} & G_1 & & & \cdot \\ f_{31} & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ f_{N1} & & & & G_N \end{Vmatrix}, \tag{5.4}$$

where $f_{ij} = f(x_i - x_j)$ and $f(x)$ is some real function. From the antihermicity of A, the function $f(x)$ should satisfy the condition $f(x_i - x_j) = f_{ij} = f_{ji} = f(x_j - x_i)$ which means that $f(x)$ should be even. Eq. (3.3) leads to

$$\dot{p}_k = -2\sum_{\substack{r=1\\ r\neq k}}^{N} a_{kr}f_{kr}, \tag{5.5a}$$

$$\dot{a}_{kl} = f_{kl}(p_k - p_l) + i\Big[\sum_{\substack{r=1\\ r\neq k,l}}^{N} (a_{kr}f_{rl} - f_{kr}a_{rl}) + a_{kl}(G_l - G_k)\Big]. \tag{5.5b}$$

The system (5.3a, b, c) is consistent with (5.5a, b) if

$$f_{kr} = a'_{kr}, \tag{5.6a}$$

$$\sum_{\substack{r=1\\ r\neq k,l}}^{N} (a_{kr}f_{rl} - f_{kr}a_{rl}) + a_{kl}(G_l - G_k) = 0. \tag{5.6b}$$

So we arrive at the following functional-differential system of equations involving both $a(x)$ and $G(\cdot)$ functions

$$\sum_{\substack{r=1 \\ r\neq k,l}}^{N} (a_{kr}a'_{rl}-a'_{kr}a_{rl})+a_{kl}(G_l-G_k) = 0. \tag{5.7a}$$

If $a(x) \neq 0$ this is equivalent to

$$\sum_{r=1}^{N} [a(x_k-x_r)a'(x_r-x_l)-a'(x_k-x_r)a(x_r-x_l)]/a(x_k-x_l)+(G_l-G_k) = 0 \tag{5.7b}$$

for all pairs $k, l = 1, \ldots, N$; $k \neq l$. The functions $G_k = G_k(x_1 \ldots, x_N)$ can be arbitrarily chosen to satisfy (5.7b). It may seem that the number $(N(N-1)/2)$ of equations for a_{kr} is somewhat abundant. In fact, (5.7) can be resolved with respect to G_k only for certain special functions $a(x)$. Here the odd parity of $a(x)$ is essential. The form of Eqs. (5.7) suggests that the function $a(x)$, which makes the whole structure solvable, should fulfill some law similar to the addition theorem.

An important class of solutions can be obtained under the asumption that every term of (5.7b) splits into the difference of two identical functions depending upon different arguments; that is,

$$\frac{a(x_k-x_r)a'(x_r-x_l)-a'(x_k-x_r)a(x_r-x_l)}{a(x_k-x_l)} = f(x_k-x_r)-f(x_r-x_l). \tag{5.8}$$

Then G_k, in the form

$$G_k = \sum_{\substack{r=1 \\ r\neq k}}^{N} f(x_k-x_r) \tag{5.9}$$

satisfies the system of Eqs. (5.7b). If, moreover, $a(x)$ is assumed to be of odd parity, then (5.8) reduces to the functional-differential equation

$$\frac{a(x)a'(y)-a'(x)a(y)}{a(x+y)} = f(x)-f(y), \tag{5.10}$$

which was also investigated (independently but a bit earlier) by Calogero [14]. The function $f(x)$ in Eq. (5.10) should be appropriately chosen for every solution $a(x)$.

There are known several solutions $a(x)$ of Eq. (5.10) namely

(*i*) $1/x$ — in the class of rational functions,

(*ii*) $1/\sin(x)$, $\operatorname{ctg}(x)$, $1/\operatorname{sh}(x)$, $\operatorname{cth}(x)$ — in the class of simply periodic functions,

(*iii*) $1/\operatorname{sn}(x)$, $\operatorname{cn}(x)/\operatorname{sn}(x)$, $\operatorname{dn}(x)/\operatorname{sn}(x)$, $\operatorname{cn}(x)/\operatorname{sn}(x)\operatorname{dn}(x)$, $\operatorname{cn}(x)\operatorname{dn}(x)/\operatorname{sn}(x)$, $\operatorname{dn}(x)/\operatorname{sn}(x)\operatorname{cn}(x)$, — in the class of doubly periodic functions.

All these solutions are equivalent in the sense that they yield the same potential, the Weierstrass function $\mathscr{P}(ax|\omega, \omega')$ [15]. So, instead of asking for the most general solution of (5.8), it is reasonable, at least from a physical point of view, to look for the most general

even potential generated by the equality $V(x) = a^2(x)$. Adding all Eqs. (5.8) corresponding to all permutations of (k, r, l), we have

$$\frac{a(x_r-x_l)a'(x_l-x_k)-a'(x_r-x_l)a(x_l-x_k)}{a(x_k-x_r)}+\frac{a(x_l-x_k)a'(x_k-x_r)}{a(x_r-x_l)}$$

$$-\frac{a'(x_l-x_k)a(x_k-x_r)}{a(x_r-x_l)}+\frac{a(x_k-x_r)a'(x_r-x_l)-a'(x_k-x_r)a(x_r-x_l)}{a(x_l-x_k)}=0 \qquad (5.11)$$

and therefore the equation for potential is

$$[V(x)V'(y)-V'(x)V(y)]+[V(y)V'(z)-V'(y)V(z)]+[V(z)V'(x)-V'(z)V(x)]=0 \qquad (5.12)$$

with $x+y+z = (x_k-x_r)+(x_r-x_l)+(x_l-x_k) = 0$. But this equation is nothing other than the addition formula

$$\det\begin{Vmatrix} 1 & V(x) & V'(x) \\ 1 & V(y) & V'(y) \\ 1 & V(x+y) & -V'(x+y) \end{Vmatrix} = 0$$

for the Weierstrass function $\mathcal{P}(ax|\omega, \omega')$. It is known that this formula determines the even function V up to a multiplying constant. Notify that for $N = 3$ the equation (5.11) is the straightforward result of the (5.7b) without the restricting assumption made in (5.8). Therefore for the system of particles interacting with each other the most general potential integrable within this approach and independent of number of particles is given by the Weierstrass function.

6. *Integrals of motion, involutivity*

The basic assumption for the whole class of problems investigated here was that the eigenvalues (generally invariants) of L are the integrals of motion. Now it is necessary to answer the question whether they are functionally independent and in involution; that is whether they constitute a fundamental set of integrals for the completely integrable systems. The independency of the integrals results in a simple manner from the sets of integrals other than eigenvalues, but the proof of involutivity is not so trivial.

Sets of integrals usually considered are $J_n = \operatorname{Tr} L^n$ or the coefficients K_n in the characteristic polynomial

$$\det[-L_{ij}+\lambda\delta_{ij}] = (+\lambda)^N - \sum_{n=1}^{N} K_n(+\lambda)^{N-n}.$$

It is known that K_n are sums of all principal minors of the n-th order, and that they are connected with J_n by the Newton formula

$$nK_n = J_n - K_1J_{n-1} - \ldots - K_{n-1}J_1.$$

The functional independence of the K_n results from the fact that in every principal minor of n-th order there exists one term of the form $p_{i_1} \dots p_{i_n}$, where $i_1 \dots i_n$ are the numbers of the rows included in that minor. Then every invariant can be written in the form

$$K_n = \sum_{1 \leqslant i_1 < i_2 < \dots < i_n \leqslant N} p_{i_1} \times \dots \times p_{i_n} + (\text{terms depending upon } \varepsilon a(\cdot)),$$

where every matrix element is given as $a_{ij} = \varepsilon a(x_i - x_j)$. Note that substitution of ε does not influence any properties of the matrices considered in the previous paragraphs. Therefore, if some general relation existed between all the K_n it would exist in particular for $\varepsilon = 0$. But in that case K_n reduces to the symmetric polynomials of order n of the momenta, which are independent.

Now, in a way similar to Sawada and Kotera [16], we define here another, third set of integrals I_n. For this set of integrals we prove involutivity.

Let the integrals which are highest in p_k be given by the expressions

$$I_N = \exp\left[-\tfrac{1}{2}\varepsilon \sum_{i,j}{}' V(x_i - x_j) \frac{\partial}{\partial p_i} \frac{\partial}{\partial p_j}\right] \Pi p,$$

$$I_N = \exp\left[-\tfrac{1}{2}\varepsilon \sum_{i=1}^{N} V(x_i - x_{i+1}) \frac{\partial}{\partial p_i} \frac{\partial}{\partial p_{i+1}}\right] \Pi p,$$

$$I_N = \exp\left[-\tfrac{1}{2}\varepsilon \sum_{i=1}^{N} V(x_i - x_{i+2}) \frac{\partial}{\partial p_i} \frac{\partial}{\partial p_{i+2}}\right] \Pi p, \tag{6.1}$$

for systems (5.2), (4.2) and (4.8) respectively, where $\Pi p = p_1 \times p_2 \dots p_N$. These quantities are well defined (because the number of terms is finite), completely symmetric N-th order functions of p_k. Moreover, it is easily seen that I_N are translation invariant, i.e. $\{\sum_k p_k, I_N\} = 0$. But the proof that I_N are the first integrals i.e. that $\{H, I_N\} = 0$, involves more computations and will be given at the end. The usual Poisson bracket is denoted here by $\{S; T\} = \sum_{r=1}^{N} \frac{\partial S}{\partial p_r} \frac{\partial T}{\partial x_r} - \frac{\partial T}{\partial p_r} \frac{\partial S}{\partial x_r}$. All the other integrals I_n are defined inductively in the following way

$$I_{n-1} = \{I_n, \sum_{k=1}^{N} x_k\}.$$

Then, by virtue of the Jacobi identities,

$$\{H, \{I_N, \sum_{k=1}^{N} x_k\}\} + \{I_N, \{\sum_{k=1}^{N} x_k, H\}\} + \{\sum_{k=1}^{N} x_k, \{H, I_N\}\} = 0,$$

$$\{\sum_{j=1}^{N} p_j, \{I_N, \sum_{k=1}^{N} x_k\}\} + \{I_N, \{\sum_{k=1}^{N} x_k, \sum_{j=1}^{N} p_j\}\} + \{\sum_{k=1}^{N} x_k, \{\sum_{j=1}^{N} p_j, I_N\}\} = 0,$$

and the equality

$$\{\sum_{k=1}^{N} x_k, H\} = -\sum_{j=1}^{N} p_j$$

finite induction arguments indicate that the I_n are in fact translationally invariant integrals. Furthermore, they are completely symmetric n-th order functions of p_k, which guarantees functional independence of I_n. It can be observed that the operators $\partial = \sum_{k=1}^{N} \frac{\partial}{\partial p_k}$ and

$$D = \sum_{i,j}{}' V(x_i - x_j) \frac{\partial}{\partial p_j} \frac{\partial}{\partial p_j} \Bigg(D = \sum_{i=1}^{N} V(x_i - x_{i+1}) \frac{\partial}{\partial p_i} \frac{\partial}{\partial p_{i+1}} \text{ or}$$

$$D = \sum_{i=1}^{N} V(x_i - x_{i+2}) \frac{\partial}{\partial p_i} \frac{\partial}{\partial p_{i+2}} \text{ respectively} \Bigg)$$

commute, and as a consequence,

$$I_{n-1} = \left\{ I_n, \sum_{k=1}^{N} x_k \right\} = \left(\sum_{k=1}^{N} \frac{\partial}{\partial p_k} \right) I_n = (\partial) I_n = (\partial)^{N-n+1} I_N.$$

Thus we can now examine the quantity $\{I_m, I_n\}$. This is certainly the integral of motion that might be independent of I_k. But, by the functional independency of I_k, the equalities $I_k(p, x) = \alpha_k = \text{const}$ (α_k stands for the concrete value of each integral I_k) can be resolved, leading to $p_k = g_k(x, \alpha)$. So the quantities $\{I_m, I_n\}$ can be expressed as

$$\{I_m, I_n\} = G_{mn}(p, x) = F_{mn}(I_1, \ldots, I_N, x). \tag{6.2}$$

It is important here that in such a new integral F_{mn} the momenta are included through the old integrals I_k. Therefore the application of ∂ to both sides of (6.2) yields

$$\{I_{m-1}, I_n\} + \{I_m, I_{n-1}\} = \sum_{k=1}^{N} \frac{\partial F_{mn}}{\partial I_k} (\partial I_k) = \sum_{k=1}^{N} \frac{\partial F_{mn}}{\partial I_k} I_{k-1}. \tag{6.3}$$

Now, by finite induction, it is easy to justify that $F_{mn}(I_1, \ldots, I_N, x) = 0$. Let us assume that all the Poisson brackets $\{I_r, I_s\}$ of order $r+s = m+n-1$ vanish. Then the left-hand side of (6.3) vanishes, and the independency of I_k implies that $\frac{\partial F_{mn}}{\partial I_k} = 0$. This means that the integral $F_{mn}(I_1, \ldots, I_N, x) = F_{mn}(x)$ could only depend upon the position coordinates. But, from the independency of p_k and the form of the Hamiltonians (5.2), (4.8) and (4.2), it immediately follows that $\frac{\partial F_{mn}}{\partial x_k} = 0$. An appropriate choice of (p, x) easily shows that

the constant F_{mn} has to be zero. Lastly, it is obvious that $\{I_1, I_2\} = 0$ what proves the involutivity of I_n.

Now it is easy to notify that this proof depends not on special choice of the integrals I_n but on its recurrent structure. Integrals J_n also exhibit the same recurrent property

$$\partial J_N = N \operatorname{Tr} L^{N-1}(\partial L) = N \operatorname{Tr} L^{N-1} = N J_{N-1}$$

because ∂L = identity matrix for all already considered systems. Therefore by virtue of preceding arguments integrals J_n (and K_n as well) are in involution too.

It would seem that the procedure introduced here proves additionally that I_n are integrals, and that there is no need to verify the identity $\{H, I_N\} = 0$. If this were true, it would mean that the involutivity of I_n is only a property of matrix L and not a property of a particular potential $V(x)$, so that all even potentials are integrable. This is obviously false because in fact we have earlier used the property that $F_{mn}(I_1, \ldots, I_N)$ is the integral of H.

To complete this proof it is necessary to show that $\{H, I_N\} = 0$. We will do it for the systems (5.2). Computation for systems (4.2) and (4.8) as very similar are ommited. Let us now examine the first term of

$$\{H, I_N\} = \sum_{k=1}^{N} \frac{\partial H}{\partial p_k} \frac{\partial I_N}{\partial x_k} - \sum_{k=1}^{N} \frac{\partial I_N}{\partial p_k} \frac{\partial H}{\partial x_k} = \sum_{k=1}^{N} p_k \frac{\partial I_N}{\partial x_k} - \sum_{k=1}^{N} \frac{\partial I_N}{\partial p_k} \frac{\partial H}{\partial x_k}. \tag{6.4}$$

Then, for a single n-th component of the exponential I_N, we have

$$\left(\sum_{k=1}^{N} p_k \frac{\partial}{\partial x_k}\right)\left\{\frac{1}{n!}(-\tfrac{1}{2}\varepsilon)^n D^n\right\}\Pi p = -\varepsilon \sum_{k,l}{}' V'(x_k - x_l) p_k \frac{\partial}{\partial p_k}\frac{\partial}{\partial p_l}$$

$$\left\{\frac{1}{(n-1)!}(-\tfrac{1}{2}\varepsilon)^{n-1} D^{n-1}\right\}\Pi p = \varepsilon \sum_{k,l}{}' V'(x_k - x_l) \frac{\partial}{\partial p_k} p_l \frac{\partial}{\partial p_l}\left\{\frac{1}{(n-1)!}(-\tfrac{1}{2}\varepsilon)^{n-1} D^{n-1}\right\}\Pi p,$$

where the prime denotes differentiation with respect to the argument. In the second term of (6.4) the $(n-1)$-th component of I_N gives

$$\varepsilon \sum_{k,l}{}' V'(x_k - x_l) \frac{\partial}{\partial p_k}\left\{\frac{1}{(n-1)!}(-\tfrac{1}{2}\varepsilon)^{n-1} D^{n-1}\right\}\Pi p.$$

Now, adding in (6.4) the terms corresponing to the same power of ε, we obtain

$$-\varepsilon \frac{1}{(n-1)!}(-\tfrac{1}{2}\varepsilon)^{n-1} \sum_{k,l}{}' V(x_k - x_l) \frac{\partial}{\partial p_k}\left(1 - p_l \frac{\partial}{\partial p_l}\right)\{D^{n-1}\}\Pi p. \tag{6.5}$$

The only non-vanishing terms in (6.5) are those in $D^{n-1}\Pi p$ for which p_k is present and p_l is absent. Let us assume (without loss of generality) that differentiation with respect to p_l

appears in the left factor of $D\times(D^{n-2}\Pi p)$. Then this factor can be rewritten in the form

$$\sum_{\substack{m=1\\ m\neq l}}^{N}[V(x_l-x_m)+V(x_m-x_l)]\frac{\partial}{\partial p_l}\frac{\partial}{\partial p_m}+(\text{nonessential terms}),$$

and, with accuracy to a constant factor, expression (6.5) yields

$$\sum_{k,l,m}{}' V'(x_k-x_l)\,[V(x_l-x_m)+V(x_m-x_l)]\frac{\partial}{\partial p_k}\frac{\partial}{\partial p_l}\frac{\partial}{\partial p_m}D^{n-1}\Pi p \tag{6.6}$$

where $\sum_{k,l,m}'$ means independent summation of all k, l, m excluding triples with $k = l$ or $k = m$ or $l = m$. By permuting k, l, adding and multiplying by $\frac{1}{2}$, (6.6) becomes

$$\sum_{k,l,m}{}' V'(x_k-x_l)\,[V(x_l-x_m)-V(x_m-x_k)]\frac{\partial}{\partial p_k}\frac{\partial}{\partial p_l}\frac{\partial}{\partial p_m}D^{n-1}\Pi p. \tag{6.7}$$

Note here that for fixed (k, l, m) the facor $\dfrac{\partial}{\partial p_k}\dfrac{\partial}{\partial p_l}\dfrac{\partial}{\partial p_m}$ is multiplied by

$$V'(x_k-x_l)\,[V(x_l-x_m)-V(x_m-x_k)]+V'(x_l-x_m)\,[V(x_m-x_k)$$
$$-V(x_k-x_l)]+V'(x_m-x_k)\,[V(x_k-x_l)-V(x_l-x_m)]$$

which vanishes by virtue of Eq. (5.11) and the identity $(x_k-x_l)+(x_l-x_m)+(x_m-x_k)=0$. Thus every term labelled by ε^n (n arbitrary) is equal to zero, and consequently $\{H, I_N\}=0$.

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