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SUBQUANTUM THERMODYNAMICS

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It is shown that an unusual classical model for the one dimensional Schrödinger equation can be cleanly fitted to the pattern of classical thermodynamics. The model involves an interaction between two fluid systems, one with a positive mass distribution and the other with a negative mass distribution. Temperatures, pressures, entropies and chemical potentials are all identified. The relations between them are examined and shown to take satisfactory forms.

1. Introduction

There have been many attempts to explain quantum mechanics in classical terms. Much of the history of such attempts can be found in Max Jammer's book (1974), where many references are listed. The work and writings of Bohm (1957) are particularly important in this area of research. Work related to the line to be developed in this paper can be found in papers by Madelung (1926), Wigner (1932), Barlett (1949), Moyal (1949), Vigier (1954), Kershaw (1956), Hornquist (1969). References of some earlier work by the present author are (Gilson, 1965, 1968, 1971). It is also anticipated that this work has some significant relation with superfluid theory such as discussed in the book by London (1954). However, although these studies often look very promising, progress at the technical level does not seem to have been great. This author believes that this is because workers in this field have not hit upon some of the essential ingredients for a complete and satisfactory model structure or classically compatible image for describing the basic one dimensional Schrödinger equation. Here it will be shown that there is a complete solution to this problem which vindicates Bohm's suggestions that under the conventional Schrödinger structure there are deeper, possibly classical like strata of processes taking place. The essential ingredients referred to above are the incorporation of negative mass density movements and the recognition that the one space dimensional Schrödinger equation only has a meaningful classical image when related to a two dimensional spatial surface. This

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paper is mostly concerned with the negative mass aspect of the problem. The two dimensional configuration space aspect has been dealt with elsewhere (Gilson, 1977a). It will be shown here that when these two missing features are used in model building, the classical thermodynamics behind the Schrödinger equation shows up with remarkable clarity and even progress at the technical level seems to be straightforward.

2. *Analytic continuation of local energy*

Working from the usual Schrödinger quantum mechanics structure it is possible to define a local energy $E(x, t)$ and a local momentum $p(x, t)$ using the operators $i\hbar\partial/\partial t$ and $-i\hbar\partial/\partial x$ and the wave function $\Psi(x, t)$ by,

$$E(x, t) = i\hbar \frac{\partial}{\partial t} \log_e \Psi, \tag{2.1}$$

and

$$p(x, t) = -i\hbar \frac{\partial}{\partial x} \log_e \Psi. \tag{2.2}$$

In general, and when Ψ is not necessarily an eigenfunction, $E(x, t)$ and $p(x, t)$ are complex functions of a real variable x. If we consider the continuation of x into the complex, $z = x+iy$, plane, then

$$E(z) = i\hbar \frac{\partial}{\partial t} \log_e \Psi(z), \tag{2.3}$$

and

$$p(z) = -i\hbar \frac{\partial}{\partial z} \log_e \Psi(z), \tag{2.4}$$

where the time dependence, although assumed to be present, is now not being shown explicitly in the various functions of the complex variable z. By using an analytically continued generalisation of (2.1) with (2.2), we can obtain,

$$\frac{\partial p}{\partial t} = -\frac{\partial E}{\partial z} = -\frac{\partial}{\partial z}(E_1+iE_2), \tag{2.5}$$

with

$$v = p/m = (p_1+ip_2)/m, \tag{2.6}$$

where m is the rest mass for one electron and (2.6) defines a local complex quantum velocity. Thus, using well known properties of complex function derivatives, we obtain,

$$\frac{\partial v_1}{\partial t} = -\frac{\partial E_1}{\partial x}, \tag{2.7}$$

and

$$\frac{\partial v_2}{\partial t} = -\frac{\partial E_2}{\partial x}. \tag{2.8}$$

However, both E and p are being regarded as functions of the complex variable z and so we can use, in regions where they are applicable, the Cauchy Riemann equations,

$$\frac{\partial E_1}{\partial x} - \frac{\partial E_2}{\partial y} = 0, \qquad \frac{\partial E_1}{\partial y} + \frac{\partial E_2}{\partial x} = 0, \tag{2.9}$$

$$\frac{\partial p_1}{\partial x} - \frac{\partial p_2}{\partial y} = 0, \qquad \frac{\partial p_1}{\partial y} + \frac{\partial p_2}{\partial x} = 0. \tag{2.10}$$

Thus using (2.9) with (2.8), we can write (2.8) in the more illuminating form,

$$\frac{\partial v_2}{\partial t} = +\frac{\partial E_1}{\partial y}. \tag{2.11}$$

From the Schrödinger equation,

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi, \tag{2.12}$$

we can easily deduce that,

$$E_1 = \frac{m}{2}(v_1^2 - v_2^2) + \frac{\hbar}{2}\frac{\partial v_2}{\partial x} + V, \tag{2.13}$$

and if we put,

$$\frac{P_1}{\varrho_1} = \frac{\hbar}{2}\frac{\partial v_2}{\partial x} \quad \text{and} \quad \frac{P_2}{\varrho_2} = \frac{\hbar}{2}\frac{\partial v_1}{\partial y}, \tag{2.14}$$

then, using (2.10), we can put the two equations of motion (2.7) and (2.11) into the forms,

$$\frac{\partial v_1}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{P_1}{\varrho_1} + \frac{v_1^2 - v_2^2}{2} + V\right), \tag{2.15}$$

and

$$\frac{\partial v_2}{\partial t} = -\frac{\partial}{\partial y}\left(\frac{P_2}{\varrho_2} + \frac{v_2^2 - v_1^2}{2} - V\right). \tag{2.16}$$

Various interesting results follow from this collection of equations and relations. For example, if we define separate temperatures T_1 and T_2 for the two fluids by,

$$kT_1 = \frac{P_1}{\varrho_1} \quad \text{and} \quad kT_2 = \frac{P_2}{\varrho_2}, \tag{2.17}$$

we see from (2.10) and (2.14) that not only must ϱ_2 be of opposite sign to ϱ_1 but also that

$$T_1 + T_2 = 0. \tag{2.18}$$

That is, with temperatures defined separately for the fluids by (2.17), these two temperatures must be of opposite sign. Further, if we put,

$$\varrho(x, y) = \Psi^*(x-iy)\Psi(x+iy) \tag{2.19}$$

then it follows from the Cauchy Riemann equations that

$$\nabla^2 \log_e \varrho = 0 \tag{2.20}$$

From which, incidentally, if assumed to hold in the first instance for two dimensional fluid dynamical reasons, the correct form (2.19) with $y = 0$ that the quantum probability density ϱ should have on the real axis can be deduced by integration. It turns out, in fact, that (2.18) and (2.20) are the same equation (Gilson, 1977). Equations (2.15)-(2.18) contain all the clues to a classical fluid reinterpretation of the Schrödinger equation. Study of these equations suggests that the following model be adopted. The quantum process can be pictured as a thermal interaction between two fluids. One of these fluids being taken to have a positive mass density $\varrho_1(x, y) \rightarrow \varrho(x)$ in the limit of orthodox quantum mechanics and the other fluid being taken to have a negative mass density $\varrho_2(x, y) \rightarrow -\varrho(x)$ in the limit of orthodox quantum mechanics. Such an interpretation is clearly consistent with the external potential appearing as $+V$ in (2.15) and as $-V$ in (2.16). By introducing a definition for vorticity into the two dimensional (x, y) flow field, the two equations (2.15) and (2.16) can be made completely classical in form. However, this last point has been discussed elsewhere (Gilson, 1977b). We are now in a position to discuss the thermodynamics of this two fluid model.

3. *The underlying thermal process*

In writing down the thermodynamical equations, we shall assume that the two fluid system is displaced from the special equilibrium state that is quantum mechanics as it is usually known. Thus the displaced condition which is not described by the Schrödinger equation as usually constructed will involve $\varrho_1+\varrho_2 = \sigma$ where $\sigma \neq 0$. However, in the quantum limit $\sigma = 0$ and also (2.18) will hold. The use of negative mass distributions opens up the possibilities for forming averages. There are at least two types of average or combination that can be used with the positive — negative mass mixture. Briefly, they are of the forms $\varrho\bar{A}_\pm = \varrho_1 A_1 \pm \varrho_2 A_2$. It should further be remarked that we cannot claim that the thermodynamical pattern to be introduced here is unique or that it cannot be modified or usefully generalised. Generalisation is likely to be possible in various ways and as far as uniqueness is concerned there may be other or better options, but they have yet to be found. Subscripts 1 and 2 will be used to identify the two fluids and their associated parameters. Unsubscripted parameters $\bar{\varepsilon}$, $\bar{s}$, $\bar{P}$ and $\bar{\mu}$ will here be taken to denote the following selections of possible combinations of the corresponding separate fluid parameters,

$$\varrho\bar{\varepsilon} = \varrho_1\varepsilon_1 - \varrho_2\varepsilon_2, \tag{3.1}$$

$$\varrho\overline{Ts} = \varrho_1 T_1 s_1 - \varrho_2 T_2 s_2, \tag{3.2}$$

$$\varrho\overline{P\tau} = \varrho_1 P_1 \tau_1 + \varrho_2 P_2 \tau_2, \tag{3.3}$$

$$\varrho\bar{\mu} = \varrho_1 \mu_1 - \varrho_2 \mu_2, \tag{3.4}$$

$$2\varrho = \varrho_1 - \varrho_2. \tag{3.5}$$

μ in (3.4) is used to represent chemical potential. Further, we shall use

$$\varrho_1 = \frac{\sigma}{2} + \varrho, \quad \varrho_2 = \frac{\sigma}{2} - \varrho, \tag{3.6}$$

as a parametrisation of (3.5). Thus the quantum limiting case which is approached as $\sigma \to 0$ brings into prominence the following combinations of separate fluid variables, if (3.1) to (3.5) are used,

$$\varepsilon = \varepsilon_1 + \varepsilon_2, \tag{3.7}$$

$$Ts = T_1 s_1 + T_2 s_2, \tag{3.8}$$

$$P\tau = P_1 \tau_1 - P_2 \tau_2, \tag{3.9}$$

$$\mu = \mu_1 + \mu_2. \tag{3.10}$$

If we assume that mechanical equilibrium is given by $P_1 = P_2 = P$, (3.9) is replaced by,

$$\tau = \tau_1 - \tau_2. \tag{3.11}$$

Thus in this case the volume τ, defined in this way, is a measure of the compressional difference of the two fluids locally, τ_1 and τ_2 being specific 'volumes' and naturally being taken positive. Consider now the combination of two single component systems separately described by the two fundamental equations,

$$\varepsilon_1 - T_1 s_1 + \frac{P_1}{\varrho_1} - \mu_1 c = X_1, \tag{3.12}$$

$$\varepsilon_2 - T_2 s_2 + \frac{P_2}{\varrho_2} - \mu_2 c' = X_2. \tag{3.13}$$

When the two systems are considered to be united in some way, the various parameters will naturally be assumed to be restricted. That is, the number of independent variables in (3.12) and (3.13) will be reduced in number by the nature of the interaction which is assumed to exist between the two subsystems. We shall assume that c and c' are concentrations. Consequently one reduction in the number of independent variables is induced by the condition,

$$c + c' = 1. \tag{3.14}$$

However, because we are using a positive — negative mass mixture c and c' are not quite the same as the concentrations which are most useful in the purely positive mass mixture (Landau, 1966) and to emphasise that they are different from the more usual c_1

and c_2 the unprimed and primed c will be used here. Forming the total differentials of X_1 and X_2, we obtain the expressions,

$$d\varepsilon_1 - T_1 ds_1 - \frac{P_1}{\varrho_1^2} d\varrho_1 - \mu_1 dc - s_1 dT_1 + \frac{dP_1}{\varrho_1} - c d\mu_1 = dX_1, \tag{3.15}$$

and

$$d\varepsilon_2 - T_2 ds_2 - \frac{P_2}{\varrho_2^2} d\varrho_2 - \mu_2 dc' - s_2 dT_2 + \frac{dP_2}{\varrho_2} - c' d\mu_2 = dX_2. \tag{3.16}$$

If we now assume energy conservation in the differentials for both systems separately in the forms,

$$d\varepsilon_1 - T_1 ds_1 - \frac{P_1}{\varrho_1^2} d\varrho_1 - \mu_1 dc = 0, \tag{3.17}$$

and

$$d\varepsilon_2 - T_2 ds_2 - \frac{P_2}{\varrho_2^2} d\varrho_2 - \mu_2 dc' = 0, \tag{3.18}$$

then we automatically obtain the two Gibbs-Duhem relations,

$$-s_1 dT_1 + \frac{dP_1}{\varrho_1} - c d\mu_1 = dX_1, \tag{3.19}$$

and

$$-s_2 dT_2 + \frac{dP_2}{\varrho_2} - c' d\mu_2 = dX_2. \tag{3.20}$$

However, imposing (3.17) and (3.18) does not have the effect of making the two systems independent for one thing because of (3.14) and also because of other restrictive relations which will be introduced later. We are not working here without clear motivation and, in fact, the introduction of (3.12) and (3.13) is to enable the formation of a pattern of variables which is conformable in a limiting situation to the relation (2.13). The possibility of relating (3.12) and (3.13) to (2.13) may not seem very obvious but considerable evidence has been given to justify making such a connection in earlier work by the present author (Gilson, 1977). In particular, it turns out to be useful to make the identifications,

$$\varepsilon_{1,0} = -\frac{mv_2^2}{2} \quad \text{and} \quad \varepsilon_{2,0} = -\frac{mv_1^2}{2}. \tag{3.21}$$

Thus the negative of local kinetic energy for fluid 2 is used as an internal energy for fluid 1 and vice-versa. This is a rather unusual form of linkage between two fluids but, nevertheless, it seems to be a representation of the form of interaction which is operative in this quantum fluid area. We also remark that we shall introduce the subscript 0 as in (3.21) to denote quantities obtained in the first instance from quantum theory. Quantities without zero subscripts will have to converge onto the 0 subscript quantities in the quantum limit. Thus a thermodynamical $\varepsilon_1 \to \varepsilon_{1,0}$ in the state of quantum equilibrium. We shall return to this point later. Subtracting (3.13) from (3.12), we obtain,

$$\left(\varepsilon_1 - T_1 s_1 + \frac{P_1}{\varrho_1}\right) - \left(\varepsilon_2 - T_2 s_2 + \frac{P_2}{\varrho_2}\right) - \mu_1 c + \mu_2 c' = X_1 - X_2. \tag{3.22}$$

Thus comparing this with (2.13) and bearing in mind the identifications (3.21) and also (2.14), the further identification,

$$\mu_1 c - \mu_2 c' = E(c) - V, \tag{3.23}$$

is suggested, where

$$E(c_0) = -\mathrm{Re}\left(i\hbar \frac{\partial}{\partial t} \log_e \Psi\right) \tag{3.24}$$

and provided $X_1 = X_2$. If we add (3.12) and (3.13), we get

$$\left(\varepsilon_1 - T_1 s_1 + \frac{P_1}{\varrho_1} - \mu_1 c\right) + \left(\varepsilon_2 - T_2 s_2 + \frac{P_2}{\varrho_2} - \mu_2 c'\right) = X_1 + X_2. \tag{3.25}$$

We now introduce another constraint which plays the role of the law of mass action. This is expressed in the form,

$$\mu_1 (c - c_0) + \mu_2 (c' - c_0') = 0 \tag{3.26}$$

or

$$\mu_1 c + \mu_2 c' = \mu_1 c_0 + \mu_2 c_0' = A, \text{ say.} \tag{3.27}$$

From (3.23) and (3.27), we obtain,

$$\begin{aligned} \mu_1 c &= \tfrac{1}{2}(E(c) - V) + \tfrac{1}{2}A, \qquad (3.28) \\ \mu_2 c' &= -\tfrac{1}{2}(E(c) - V) + \tfrac{1}{2}A. \end{aligned}$$

Substituting these into (3.12) and (3.13), we get,

$$\varepsilon_1 - (X_1 + A/2) - T_1 s_1 + \frac{P_1}{\varrho_1} = \frac{E(c) - V}{2}, \tag{3.29}$$

and

$$\varepsilon_2 - (X_2 + A/2) - T_2 s_2 + \frac{P_2}{\varrho_2} = -\frac{E(c) - V}{2}. \tag{3.30}$$

From (3.29) and (3.30), we now see the possibility of a very subtle and interesting limiting process in which (3.29) and (3.30) can both be made to converge onto (2.13) but from opposite mass directions.

4. *The quantum limit*

The possibility of converging onto Schrödinger quantum mechanics from the directions of positive and negative mass using (3.29) and (3.30) can be seen more clearly if X_1 and X_2 are taken to be equal, as we need for (3.23), and both to have a value such that

$$X_1+A/2 = X_2+A/2 = \tfrac{1}{2}(\varepsilon_{1,0}+\varepsilon_{2,0}). \tag{4.1}$$

When this choice is made (3.29) and (3.30) assume the forms,

$$2\varepsilon_1-(\varepsilon_{1,0}+\varepsilon_{2,0})-2T_1s_1+2\frac{P_1}{\varrho_1} = E-V, \tag{4.2}$$

and

$$2\varepsilon_2-(\varepsilon_{1,0}+\varepsilon_{2,0})-2T_2s_2+2\frac{P_2}{\varrho_2} = -(E-V). \tag{4.3}$$

Both of these are conformable to converge onto (2.13) provided,

$$\varepsilon_1 \to \varepsilon_{1,0} \quad \text{and} \quad \varepsilon_2 \to \varepsilon_{2,0}, \tag{4.4}$$

and

$$-2T_1s_1+2\frac{P_1}{\varrho_1} \rightleftarrows -\left(-2T_2s_2+2\frac{P_2}{\varrho_2}\right). \tag{4.5}$$

Elsewhere (Gilson, 1977), we have obtained a solution to this problem but by a different route. However, we can see contained in (4.5), the possible set of limits,

$$s_1 \to s_2 \to s_0, \tag{4.6}$$

$$P_1 \to P_2 \to P_0, \tag{4.7}$$

$$T_1 \to -T_2 \to T_0, \tag{4.8}$$

$$\varrho_1 \to -\varrho_2 \to \varrho_0. \tag{4.9}$$

Thus the left hand side of (4.5) will be subject to the limiting process,

$$-2T_1s_1+2\frac{P_1}{\varrho_1} \to -2T_{1,0}s_{1,0}+2\frac{P_0}{\varrho_0}. \tag{4.10}$$

We have introduced the subscript 0 notation in order to deal with a specific mathematical difficulty inherent in this work. This difficulty will now be explained. Although we can make plausible identifications of thermodynamical variables using quantum mechanical results such as (3.21), we soon come up against the difficulty that these identified functions are not in the first instance functions of the appropriate thermodynamical variables. For example, $\varepsilon_{1,0}$ is a function of x and t and not, as one would wish, a function of ϱ_1, s_1 and c. However, it is a strange characteristic of some of these quantum variables such as $\varepsilon_{1,0}$ that sometimes it is possible to find derived properties such as $\partial\varepsilon_{1,0}/\partial\varrho_1$ without elaborate mathematical methods (Gilson, 1977). In fact, $\varepsilon_{1,0}$ is a functional of ϱ as derived from the usual quantum analysis. This situation does lead to mathematical difficulties in trying to extract thermodynamical information from such quantities as $\varepsilon_{1,0}$. Here we completely overcome this difficulty by working with a related function ε_1, for example, which can be regarded as a straight function of the usual thermodynamical variables and only has to converge onto $\varepsilon_{1,0}$ in the quantum limit. With this particular mathematical problem in mind, we now return to (3.12) and (3.13), but rewrite them in an approximate form,

$$\varepsilon_1 - T_{1,0}s_1 + \frac{P_{1,0}}{\varrho_1} - \mu_{1,0}c - X_{1,0} = 0, \tag{4.11}$$

$$\varepsilon_2 - T_{2,0}s_2 + \frac{P_{2,0}}{\varrho_2} - \mu_{2,0}c' - X_{2,0} = 0. \tag{4.12}$$

Thus we have now frozen-in the eight variables $T_{1,0}, P_{1,0}, \mu_{1,0}, X_{1,0}, T_{2,0}, P_{2,0}, \mu_{2,0}, X_{2,0}$ at values deduced from quantum theory and analytic continuation. For example,

$$T_{1,0} = -\frac{\hbar^2}{4km}\frac{\partial^2 \log_e \varrho(x, y)}{\partial x^2} \quad \text{and} \quad T_{2,0} = -\frac{\hbar^2}{4km}\frac{\partial^2 \log_e \varrho(x, y)}{\partial y^2}.$$

(4.11) and (4.12) are now approximate equations which can be assumed to approach exactness when the variables, $\varepsilon_1, s_1, \varrho_1, c, \varepsilon_2, s_2, \varrho_2, c'$ which are still free, move towards their quantum limiting values in a way consistent with their various interrelations. Now, we consider only variations in the remaining variables which keep the frozen-in variables constant. This procedure has the consequence that the two Gibbs–Duhem equations (3.19) and (3.20) reduce to trivial identities of the form $0 = 0$ and can therefore be forgotten. We can now define functions of the usual thermodynamical variables. From (4.11) and (4.12), we have the functions,

$$\varepsilon_1(s_1, \varrho_1, c) = T_{1,0}s_1 - \frac{P_{1,0}}{\varrho_1} + \mu_{1,0}c + X_{1,0}, \tag{4.13}$$

and

$$\varepsilon_2(s_2, \varrho_2, c') = T_{2,0}s_2 - \frac{P_{2,0}}{\varrho_2} + \mu_{2,0}c' + X_{2,0}. \tag{4.14}$$

Thus, putting $\varepsilon_{1,0} = \varepsilon_1(s_{1,0}, \varrho_{1,0}, c_0)$, and $\varepsilon_{2,0} = \varepsilon_2(s_{2,0}, \varrho_{2,0}, c_0)$, we can form the two relative internal energies,

$$\begin{aligned}\mathscr{E}_1(s_1, \varrho_1, c) &= \varepsilon_1(s_1, \varrho_1, c) - \varepsilon_{1,0}\\ &= T_{1,0}(s_1 - s_{1,0}) - P_{1,0}\left(\frac{1}{\varrho_1} - \frac{1}{\varrho_{1,0}}\right) + \mu_{1,0}(c - c_0),\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}\mathscr{E}_2(s_2, \varrho_2, c') &= \varepsilon_2(s_2, \varrho_2, c') - \varepsilon_{2,0}\\ &= T_{2,0}(s_2 - s_{2,0}) - P_{2,0}\left(\frac{1}{\varrho_2} - \frac{1}{\varrho_{2,0}}\right) + \mu_{2,0}(c' - c'_0).\end{aligned} \tag{4.16}$$

The sum $\mathscr{E}$ of these two relative internal energies plays an important role.

$$\begin{aligned}\mathscr{E} &= \mathscr{E}_1+\mathscr{E}_2\\ &= T_{1,0}(s_1-s_{1,0})+T_{2,0}(s_2-s_{2,0})-P_{1,0}\left(\frac{1}{\varrho_1}-\frac{1}{\varrho_{1,0}}\right)-P_{2,0}\left(\frac{1}{\varrho_2}-\frac{1}{\varrho_{2,0}}\right)\\ &\quad +\mu_{1,0}(c-c_0)+\mu_{2,0}(c'-c'_0).\end{aligned} \tag{4.17}$$

We again impose the constraint (3.26), the law of mass action, and (4.17) reduces to,

$$\begin{aligned}\mathscr{E} &= T_{1,0}(s_1-s_{1,0})+T_{2,0}(s_2-s_{2,0})\\ &\quad -P_{1,0}\left(\frac{1}{\varrho_2}-\frac{1}{\varrho_{1,0}}\right)-P_{2,0}\left(\frac{1}{\varrho_2}-\frac{1}{\varrho_{2,0}}\right).\end{aligned} \tag{4.18}$$

We shall now make the reasonable, though not necessarily the most general, assumption that the entropies are related to the concentrations by,

$$s_1-s_{1,0} = -k\log_e 2c, \qquad s_2-s_{2,0} = -k\log_e 2c', \tag{4.19}$$

and the concentrations which as remarked earlier are not quite the same as for the pure positive mass type of mixture will be defined relative to the density difference 2ϱ as follows,

$$c = \frac{\varrho_1}{2\varrho} = \frac{\sigma}{4\varrho}+\tfrac{1}{2}, \qquad c' = -\frac{\varrho_2}{2\varrho} = -\frac{\sigma}{4\varrho}+\tfrac{1}{2}, \tag{4.20}$$

using (3.6).

Using (3.6) and (4.20), we note various derivatives,

$$\begin{gathered}\frac{\partial c}{\partial \sigma} = \frac{1}{4\varrho}, \qquad \frac{\partial c'}{\partial \sigma} = -\frac{1}{4\varrho},\\ \frac{\partial \varrho_1}{\partial \sigma} = \tfrac{1}{2}, \qquad \frac{\partial \varrho_2}{\partial \sigma} = \tfrac{1}{2},\\ \frac{\partial c}{\partial \varrho} = -\frac{\sigma}{4\varrho^2}, \qquad \frac{\partial c'}{\partial \varrho} = \frac{\sigma}{4\varrho^2},\\ \frac{\partial \varrho_1}{\partial \varrho} = 1, \qquad \frac{\partial \varrho_2}{\partial \varrho} = -1.\end{gathered} \tag{4.21}$$

We can now use these to find the minimum value of $\mathscr{E}$ in the σ, ϱ plane. This position will then give the equilibrium values for the interacting fluid systems. Thus, equating the two partial derivatives of $\mathscr{E}$ to zero, we get,

$$\frac{\partial \mathscr{E}}{\partial \sigma} = -T_{1,0}\frac{k}{c}\left(\frac{1}{4\varrho}\right)+T_{2,0}\frac{k}{c'}\left(\frac{1}{4\varrho}\right)+\frac{P_{1,0}}{2}\frac{1}{\varrho_1^2}+\frac{P_{2,0}}{2}\frac{1}{\varrho_2^2} = 0, \tag{4.22}$$

and

$$\frac{\partial\mathscr{E}}{\partial\varrho} = -T_{2,0}\,\frac{k}{c}\left(-\frac{\sigma}{4\varrho^2}\right) - T_{2,0}\,\frac{k}{c'}\left(\frac{\sigma}{4\varrho^2}\right) + P_{1,0}\left(\frac{1}{\varrho_1^2}\right) - P_{2,0}\left(\frac{1}{\varrho_2^2}\right) = 0. \tag{4.23}$$

Using (4.20), we see that (4.22) can be expressed in the form,

$$\frac{1}{\varrho_1}\left[\frac{P_{1,0}}{\varrho_1} - T_{1,0}k\right] = -\frac{1}{\varrho_2}\left[\frac{P_{2,0}}{\varrho_2} - T_{2,0}k\right] = \alpha, \text{ say.} \tag{4.24}$$

Again using (4.20), we see that (4.23) can be expressed in the form,

$$\frac{1}{\varrho_1}\left[\frac{T_{1,0}k\sigma}{2\varrho} + \frac{P_{1,0}}{\varrho_1}\right] + \frac{1}{\varrho_2}\left[\frac{T_{2,0}k\sigma}{2\varrho} - \frac{P_{2,0}}{\varrho_2}\right] = 0. \tag{4.25}$$

Substituting for the pressures in (4.25) by means of (4.24), we get,

$$\frac{1}{\varrho_1}\left[\frac{T_{1,0}k\sigma}{2\varrho} + \varrho_1\alpha + T_{1,0}k\right] + \frac{1}{\varrho_2}\left[\frac{T_{2,0}k\sigma}{2\varrho} + \varrho_2\alpha - T_{2,0}k\right] = 0$$

or using (4.20), this reduces to,

$$\alpha = -\frac{k}{\varrho}(T_{1,0} + T_{2,0}). \tag{4.26}$$

Thus we see that if (2.18) holds in the quantum equilibrium position then there also $\alpha = 0$. Returning to (4.24) with α given by (4.26), we obtain the two equations of state for our fluids near quantum equilibrium,

$$\frac{P_{1,0}}{\varrho_1} - T_{1,0}k = -\frac{\varrho_1 k}{2\varrho}(T_{1,0} + T_{2,0}) = \varrho_1\alpha, \tag{4.27}$$

$$\frac{P_{2,0}}{\varrho_2} - T_{2,0}k = \frac{\varrho_2 k}{2\varrho}(T_{1,0} + T_{2,0}) = -\varrho_2\alpha. \tag{4.28}$$

If we use the limiting forms of (4.27) and (4.28) in (4.13) and (4.14), we get,

$$\varepsilon_1 - \frac{\varepsilon_{1,0} + \varepsilon_{2,0}}{2} = T_{1,0}(s_1 - k) + \frac{E(c) - V}{2}, \tag{4.29}$$

$$\varepsilon_2 - \frac{\varepsilon_{1,0} + \varepsilon_{2,0}}{2} = T_{2,0}(s_2 - k) - \frac{E(c) - V}{2}. \tag{4.30}$$

Thus comparing these with (2.13) and bearing in mind (2.14), (2.17), (3.21) and (4.19), we see that when $\alpha \to 0$,

$$c \to c_0 = \tfrac{1}{2}, \quad S_1 \to s_{1,0} = k/2 \tag{4.31}$$

and

$$S_2 \to S_{2,0} = k/2.$$

Clearly (4.27) and (4.28) are ideal gas equations in the quantum limit. Also it follows from (4.27) and (4.28) that when $x = 0$ and $\varrho_2 = -\varrho_1 = -\varrho$, say, then $P_{1,0} = P_{2,0}$. Thus we recover the condition for mechanical equilibrium.

5. Conclusions

This work represents very strong evidence for the view that Schrödinger quantum mechanics has a classical background structure. We have demonstrated a convincing conformability of the Schrödinger equation with a classical two fluid system. As remarked earlier, various alternatives differing in some details from the scheme suggested here could certainly be investigated and generalisations could well be considered. In particular, probably, further contributions to the local entropies beyond the terms (4.19) may well be relevant to a deeper understanding of the background quantum fluids. Equation (3.23) is interesting because it puts a new light on the significance of the external potential in quantum mechanics. Here V is seen to be a direct measure of the energy of local polarisation and a close relative of chemical potential. The extent of this vacuum polarisation locally is measured by the quantum probability $\varrho = \frac{1}{2}(\varrho_1 - \varrho_2)$ and the extent that the two fluid state deviates from the limit quantum state is given by the mass density parameter $\sigma = \varrho_1 + \varrho_2$ which is zero in the quantum limit condition.

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