THE KOTTLER FORMULATION OF THE LARMOR-LORENTZ PRINCIPLE FOR THE ELECTROMAGNETIC FIELD IN THE UNIAXIAL ANISOTROPIC MEDIUM

By J. Petykiewicz and A. Rynkowski†

Institute of Physics, Warsaw Technical University*

(Received February 3, 1976; revised version received December 3, 1976)

The aim of this paper is to present the transformation of the Larmor–Lorentz principle in an uniaxial anisotropic medium for the Wünsche formulation to the form analogical to Kottler's formulas for the isotropic case. The equivalence of those two formulations for the isotropic case was proved by Rubinowicz. The paper also contains the wave's potential for an electromagnetic field in the examined medium, which is of fundamental importance in the Young–Rubinowicz model of the diffracted phenomena.

1. Introduction

In this paper we consider an electromagnetic field in a medium which is magnetically isotropic and has uniaxial electrical anisotropy. SI units are employed, and ε_0 and μ_0 stand for the dielectric constant and magnetic premeability of vacuum, respectively. We have assumed the coordinate system x_1, x_2, x_3 in which the axes of the system are principal axes of the relative dielectric tensor. Thus, it can be represented as

$$\overset{\rightrightarrows}{\varepsilon} = \overset{\rightrightarrows}{I}\varepsilon_1 + (\varepsilon_3 - \varepsilon_1)\vec{k}\vec{k}, \tag{1.1}$$

where $\overset{\rightrightarrows}{I}$ stands for a unit tensor and $\vec{k}\vec{k}$ for dyadic. The unit vector $\vec{k}$ lies in the direction of the x_3-axis. ε_1 and ε_3 stand for the relative principal dielectric constants. Magnetic features of the medium under consideration are described by the relative permeability tensor

$$\overset{\rightrightarrows}{\mu} = \overset{\rightrightarrows}{I}\mu, \tag{1.2}$$

where μ stands for relative permeability.

The Larmor–Lorentz principle was take as a basis for this paper, since it enabled us to express the state of the monochromatic field at point P by values of the field and its derivatives at all points Q of the surface surrounding the considered point.

---

* Address: Instytut Fizyki, Politechnika Warszawska, Koszykowa 75, 00-662 Warszawa, Poland.
† Deceased.

In the case of the anisotropic medium, the tensor Green's functions were found by Wünsche [1]. There are two tensors for the uniaxiall anisotropic medium: electric tensor Green's function and magnetic tensor Green's function. The equations, which satisfy these tensor functions have the following forms:

$$\nabla \times \vec{\vec{\mu}}^{-1} \cdot (\nabla \times \vec{G}_l^{(E)}) - k_0^2 \vec{\vec{\varepsilon}} \cdot \vec{G}_l^{(E)} = \vec{j}_l^{\text{el}} \delta(P, Q), \tag{1.3}$$

$$\nabla \times \vec{\vec{\varepsilon}}^{-1} \cdot (\nabla \times \vec{G}_l^{(H)}) - k_0^2 \vec{\vec{\mu}} \cdot \vec{G}_l^{(H)} = \vec{j}_l^{\text{mag}} \delta(P, Q),$$

where $\vec{\vec{\varepsilon}}^{-1}$ and $\vec{\vec{\mu}}^{-1}$ are inverse tensors for the dielectric tensor and the magnetic permeability tensor, respectively. The subscript l denotes the column number of the tensor. The physical sense of the quantities $\vec{j}_l^{\text{el}}$ or $\vec{j}_l^{\text{mag}}$ appearing in (1.3) is as follows: $\vec{j}_l^{\text{el}} = j^{\text{el}} \vec{l}$ is the electric current density, $\vec{l}$ is a unit vector of the appropiate axis of the chosen system of coordinates. $\delta(P, Q)$ is the delta function. $\vec{j}_l^{\text{mag}} = j_l^{\text{mag}} \vec{l}$ is magnetic current density. The first column of $\vec{\vec{G}}^{(E)}$ tensor corresponds to the unit vector $\vec{i}$, the second one — to the unit vector $\vec{j}$, and the third one — to the unit vector $\vec{k}$. $k_0^2 = \omega^2 \varepsilon_0 \mu_0$, where ω is the frequency of the field.

The fundamental tensor Green's functions $\vec{\vec{G}}^{(E)}$ and $\vec{\vec{G}}^{(H)}$ are functions of two points — that of integration Q and that of observation P.

Knowledge of the $\vec{\vec{G}}^{(E)}$ and $\vec{\vec{G}}^{(H)}$ permits one to write the Larmor–Lorentz principle for the uniaxial anisotropic medium in the form

$$E_l(P) = \oiint_S df [(\vec{E}(Q) \times \vec{n}) \cdot \vec{\vec{\mu}}^{-1} \cdot (\nabla \times \vec{G}_l^{(E)}) - i\omega\mu_0 (\vec{n} \times \vec{H}(Q)) \cdot \vec{G}_l^{(E)}],$$

$$H_l(P) = \oiint_S df [i\omega\varepsilon_0 (\vec{E}(Q) \times \vec{n}) \cdot G_l^{(H)} - (\vec{n} \times \vec{H}(Q)) \cdot \vec{\vec{\varepsilon}}^{-1} \cdot (\nabla \times \vec{G}_l^{(H)})], \tag{1.4}$$

where E_l and H_l are the components of the electric or magnetic vector field at point P. They are related to the tangential components of the electric and magnetic fields on the surface S, which surrounds the examined area. This form (see (1.4)) of the Larmor–Lorentz principle was formulated for the anisotropic medium by Wünsche [1].

The explicit form of the tensors $\vec{\vec{G}}^{(E)}$ and $\vec{\vec{G}}^{(H)}$ was found for uniaxial anisotropy by Wünsche [1] and may be written in the form

$$\vec{\vec{G}}^{(E)} = -\frac{1}{4\pi} \left\{ \left[\frac{\sqrt{\mu}}{k_0^2 \sqrt{\varepsilon_1}} \nabla\nabla + \mu\varepsilon_3 \sqrt{\mu\varepsilon_1}\, \vec{\vec{\varepsilon}}^{-1} \right] \frac{\exp(-ik_0 R_\varepsilon)}{R_\varepsilon} \right.$$

$$\left. + \frac{\sqrt{\varepsilon_1 \mu}}{ik_0} \vec{\vec{\varepsilon}}^{-1} \left[\nabla \times \vec{k} \frac{\vec{\varrho} \times \vec{k}}{\varrho^2} (\exp(-ik_0 R_\varepsilon) - \exp(-ik_0 R)) \right] \right\}, \tag{1.5}$$

$$\vec{\vec{G}}^{(H)} = -\frac{1}{4\pi} \left\{ \left[\frac{\sqrt{\varepsilon_1}}{k_0^2 \sqrt{\mu}} \nabla\nabla + \varepsilon_1 \sqrt{\mu\varepsilon_1}\, \vec{\vec{I}} \right] \frac{\exp(-ik_0 R)}{R} \right.$$

$$\left. + \frac{1}{ik_0} \sqrt{\frac{\varepsilon_1}{\mu}} \left[\nabla \times \vec{k} \frac{\vec{\varrho} \times \vec{k}}{\varrho^2} (\exp(-ik_0 R) - \exp(-ik_0 R_\varepsilon)) \right] \right\},$$

where

$$R = \sqrt{\varepsilon_1\mu[(x_{1Q}-x_{1P})^2+(x_{2Q}-x_{2P})^2+(x_{3Q}-x_{3P})^2]},$$

$$R_\varepsilon = \sqrt{\mu\varepsilon_3[(x_{1Q}-x_{1P})^2+(x_{2Q}-x_{2P})^2]+\mu\varepsilon_1(x_{3Q}-x_{3P})^2},$$

$$\vec{\varrho} = [x_{1P}-x_{1Q}, x_{2P}-x_{2Q}], \tag{1.6}$$

$$\varrho = |\vec{\varrho}| = \sqrt{(x_{1P}-x_{1Q})^2+(x_{2P}-x_{2Q})^2}.$$

We regard x_{1P}, x_{2P}, x_{3P} and x_{1Q}, x_{2Q}, x_{3Q} as coordinates of points P and Q, respectively. $\varrho^{-2}\vec{k}(\vec{\varrho}\times\vec{k})$ represents the dyadic of two vectors: $\vec{k}$ and $\left[(\vec{\varrho}\times\vec{k})\frac{1}{\varrho^{-2}}\right]$. ∇ is an operator of Nabel and $\nabla\nabla$ is a dyadic product of two Nabel's operators. The unit vector $\vec{k}$ denotes the direction of an axis x_3, which corresponds to the principal axis of the dielectric tensor $\vec{\vec{\varepsilon}}$.

2. *The Kottler's formulation of the Larmor–Lorentz principle in the isotropic medium*

In an isotropic medium $\varepsilon_1 = \varepsilon_3 = \varepsilon$ and thus

$$R_\varepsilon = R = \sqrt{\varepsilon\mu[(x_{1Q}-x_{1P})^2+(x_{2Q}-x_{2P})^2+(x_{3Q}-x_{3P})^2]}. \tag{2.1}$$

We obtain a relation for the tensors $\vec{\vec{G}}^{(E)}$ and $\vec{\vec{G}}^{(H)}$ in the form

$$\vec{\vec{G}}^{(E)} = \frac{\mu}{\varepsilon}\vec{\vec{G}}^{(H)}, \tag{2.2}$$

and the tensor $\vec{\vec{G}}^{(E)}$ may be written as

$$\vec{\vec{G}}^{(E)} = -\frac{1}{4\pi}\left\{\left[\frac{\sqrt{\mu}}{k_0^2\sqrt{\varepsilon}}\nabla\nabla+\mu\sqrt{\varepsilon\mu}\,\vec{\vec{I}}\right]\frac{\exp(-ik_0R)}{R}\right\}. \tag{2.3}$$

This closed form of the electric Green's functions is found in many textbooks [2]. The Larmor–Lorentz principle for the isotropic medium can be transformed into

$$\vec{E}(P) = \frac{\sqrt{\varepsilon\mu}}{4\pi}\oint\!\!\oint_S df\left\{\vec{E}(Q)\frac{\partial}{\partial n_Q}\frac{\exp(-ik_0R)}{R} - \frac{\exp(-ik_0R)}{R}\frac{\partial}{\partial n_Q}\vec{E}(Q)\right\}$$

$$+\frac{\sqrt{\varepsilon\mu}}{4\pi}\oint_{(K)}\frac{\exp(-ik_0R)}{R}(\overrightarrow{\delta E}\times\overrightarrow{dl})$$

$$+\frac{1}{4\pi ik_0}\sqrt{\frac{\mu_0\mu}{\varepsilon_0\varepsilon}}\nabla_P\oint_{(K)}\frac{\exp(-ik_0R)}{R}\overrightarrow{\delta H}\cdot\overrightarrow{dl}, \tag{2.4}$$

for the electric field, and

$$\vec{H}(P) = \frac{\sqrt{\varepsilon\mu}}{4\pi} \oiint_S df \left\{ \vec{H}(Q) \frac{\partial}{\partial n_Q} \frac{\exp(-ik_0R)}{R} - \frac{\exp(-ik_0R)}{R} \frac{\partial}{\partial n_Q} \vec{H}(Q) \right\}$$

$$+ \frac{\sqrt{\varepsilon\mu}}{4\pi} \oint_{(K)} \frac{\exp(-ik_0R)}{R} (\overrightarrow{\delta H} \times \overrightarrow{dl}) + \frac{1}{4\pi} \sqrt{\frac{\varepsilon_0\varepsilon}{\mu_0\mu}} \nabla_P \oint_{(K)} \frac{\exp(-ik_0R)}{R} \overrightarrow{\delta E} \cdot \overrightarrow{dl} \tag{2.5}$$

for the magnetic field.

As we have seen, the field at the point P is defined by the contributions from the surface integral, which are take from the closed surface. Those from the curvilinear integrals are taken from the curve line of discontinuity in the fields $\vec{E}$ and $\vec{H}$ [3]. The latter two are described by $\overrightarrow{\delta E}$ and $\overrightarrow{\delta H}$. The curve line of discontinuity is on the closed surface of integration. The subscript P or Q in formulas (2.4) and (2.5) at the differential operator denotes differentiation with respect to the point P or Q, respectively.

The expressions (2.4) and (2.5) are connected with Kottler's formulae [4], which have been formulated in Kirchhoff's theory of diffraction of the electromagnetic waves. In this case the line of discontinuity of the field is a screen's edge.

The question is whether a similar form of the Larmor–Lorentz principle exists for the uniaxial anisotropy medium. As it is shown below, such a possibility exists for the medium defined by (1.1) and (1.2).

These expressions are similar to (2.4) and (2.5). Some other ones were presented at the International Symposium of Electomagnetic Waves [5]. Both of the formulations are extensions of the formulae (2.4) and (2.5) for the case of an anisotropic medium.

The form found below is very useful for the application of the Young–Rubinowicz diffraction model.

3. Kottler's formulation of the Larmor–Lorentz principle for the uniaxial anisotropic medium

To get Kottler's formulation of the Larmor–Lotentz principle (1.4) for the uniaxial anisotropic medium, we resolve any arbitrarily polarized incident wave into two waves, one of which is a wave of TE-type and the other a wave of TM-type. This division is taken with regard to the distinguished axis of anisotropy, which is the x_3 axis,

The incident waves are written in the form

$$\vec{E} = \vec{E}^{TE} + \vec{E}^{TM}$$

$$\vec{H} = \vec{H}^{TE} + \vec{H}^{TM} \tag{3.1}$$

and every case of the field TE or TM is to be treated separately.

3.1. The case of the TE field

When the field in the medium is of the TE-type, the Larmor–Lorentz principle gives us the magnetic field $\vec{H}$ at the considered point P in the form

$$H_l(P) = \oiint_S d f \vec{n} \cdot \{-\vec{H}^{\mathrm{TE}} \times \vec{\vec{\varepsilon}}^{-1} \cdot (\nabla \times \vec{G}_l^{(H)}) + \vec{G}_l^{(H)} \times \vec{\vec{\varepsilon}}^{-1} \cdot (\nabla \times \vec{H}^{\mathrm{TE}})\}, \tag{3.1.1}$$

where the subscript $l(= 1, 2, 3)$ — defines the different components of the field. The evident feature of the vector $\vec{G}_l^{(H)}$ is the l's column of tensor $\vec{\vec{G}}^{(H)}$ and is as follows:

$$\vec{G}_l^{(H)} = \frac{1}{4\pi} \left\{ \left[\frac{\sqrt{\varepsilon_1}}{k_0^2 \sqrt{\mu}} \nabla \frac{\partial}{\partial x_l} + \varepsilon_1 \sqrt{\varepsilon_1 \mu}\, \vec{l} \right] \frac{\exp(-ik_0 R)}{R} \right.$$
$$\left. + \frac{1}{ik_0} \sqrt{\frac{\varepsilon_1}{\mu}} \nabla \times \left[\vec{k} \frac{(\vec{\varrho} \times \vec{k})\vec{l}}{\varrho^2} (\exp(-ik_0 R) - \exp(-ik_0 R_\varepsilon)) \right] \right. . \tag{3.1.2}$$

One should notice, that in the case of $l = 3$ (i. e. where $\vec{l} = \vec{k}$) the rotational component in (3.1.2) disappears and vector $\vec{G}_3^{(H)}$ takes a simpler form.

When we introduce (3.1.2) to (3.1.1) and use well-known vector indentities, we may expresse the component of the field in the following form:

$$H_l(P) = \frac{\sqrt{\varepsilon_1 \mu}}{4\pi} \oiint_S df \left\{ H_l^{\mathrm{TE}} \frac{\partial}{\partial n_Q} \frac{\exp(-ik_0 R)}{R} - \frac{\exp(-ik_0 R)}{R} \frac{\partial H_l^{\mathrm{TE}}}{\partial n_Q} \right\}$$
$$+ \frac{\sqrt{\varepsilon_1 \mu}}{4\pi} \oiint_S df \vec{n} \cdot \left[\nabla_Q \times \frac{\exp(-ik_0 R)}{R} (\vec{l} \times \vec{H}^{\mathrm{TE}}) \right]$$
$$+ \frac{i \sqrt{\varepsilon_1 \varepsilon_0}}{4\pi k_0 \sqrt{\mu \mu_0}} \oiint_S df \vec{n} \cdot \left[\nabla_Q \times \left(\vec{E}^{\mathrm{TE}} \frac{\partial}{\partial x_l} \frac{\exp(-ik_0 R)}{R} \right) \right]$$
$$+ \frac{\sqrt{\varepsilon_1}}{4\pi i k_0 \sqrt{\mu}} \oiint_S df \vec{n} \cdot \left\{ \nabla_Q \times \left[\vec{H}^{\mathrm{TE}} \frac{\vec{\varrho} \cdot (\vec{k} \times \vec{l})}{\varrho^2} \frac{\partial}{\partial x_3} (\exp(-ik_0 R) - \exp(-ik_0 R_\varepsilon)) \right] \right\}$$
$$- \frac{\sqrt{\varepsilon_1}}{4\pi i k_0 \sqrt{\mu}} \oiint_S df \vec{n} \cdot \left\{ \nabla_Q \times \left[(\vec{k} \cdot \vec{H}^{\mathrm{TE}}) \nabla_Q \frac{\vec{\varrho} \cdot (\vec{k} \times \vec{l})}{\varrho^2} (\exp(-ik_0 R) - \exp(-ik_0 R_\varepsilon)) \right] \right\}$$
$$- \frac{\sqrt{\varepsilon_1}}{4\pi i k_0 \sqrt{\mu}} \oiint_S df \vec{n} \cdot \left\{ \nabla_Q \times \left[\frac{\partial \vec{H}^{\mathrm{TE}}}{\partial x_3} \frac{\vec{\varrho} \cdot (\vec{k} \times \vec{l})}{\varrho^2} (\exp(-ik_0 R) - \exp(-ik_0 R_\varepsilon)) \right] \right\} \tag{3.1.3}$$

If the functions $\vec{E}^{\mathrm{TE}}$, $\vec{H}^{\mathrm{TE}}$ satisfied the regularity conditions on all points of the surface S, then only the first of the integrals does not vanish and for the value of the field at the point P we obtain

$$H_l(P) = \frac{\sqrt{\varepsilon_1 \mu}}{4\pi} \oiint_S df \left\{ H_l^{\mathrm{TE}}(Q) \frac{\partial}{\partial n_Q} \frac{\exp(-ik_0 R)}{R} - \frac{\exp(-ik_0 R)}{R} \frac{\partial}{\partial n_Q} H_l^{\mathrm{TE}}(Q) \right\}. \tag{3.1.4}$$

When the field is regular in all of the points of the surface, then the field is of the same type as on the surface. Thus, the Larmor–Lorentz principle simply reconstructs the primary wave. The situation, as discussed in the work [6], exists in the case of free propagation of the field in the anisotropic medium when the diffracted bodies are absent. In the case of diffraction on the objects with a sharp edge, the lines of discontinuity of the field can appear on the surface of integration. This is the situation in the case of Kirchhoff's diffraction theory for electromagnetic waves and then the field at P-points equals

$$H_l(P) = \frac{\sqrt{\varepsilon_1 \mu}}{4\pi} \oiint_S df \left\{ H_l^{\mathrm{TE}} \frac{\partial}{\partial n_Q} \frac{\exp(-ik_0 R)}{R} - \frac{\exp(-ik_0 R)}{R} \frac{\partial H_l^{\mathrm{TE}}}{\partial n_Q} \right\}$$

$$+ \frac{\sqrt{\varepsilon_1 \mu}}{4\pi} \oint_{(K)} \frac{\exp(-ik_0 R)}{R} (\vec{l} \times \delta \vec{H}^{\mathrm{TE}}) \cdot \vec{dl} + \frac{i}{4\pi k_0} \sqrt{\frac{\varepsilon_1 \varepsilon_0}{\mu \mu_0}} \oint_{(K)} \left(\frac{\partial}{\partial x_l} \frac{\exp(-ik_0 R)}{R} \right) \delta \vec{E}^{\mathrm{TE}} \cdot \vec{dl}$$

$$+ \frac{\sqrt{\varepsilon_1}}{4\pi i k_0 \sqrt{\mu}} \oint_{(K)} \frac{\partial}{\partial x_3} \left[\frac{\vec{\varrho} \cdot (\vec{k} \times \vec{l})}{\varrho^2} (\exp(-ik_0 R) - \exp(-ik_0 R_\varepsilon)) \right] \delta \vec{H}^{\mathrm{TE}} \cdot \vec{dl}$$

$$- \frac{\sqrt{\varepsilon_1}}{4\pi i k_0 \sqrt{\mu}} \oint_{(K)} \delta H_3^{\mathrm{TE}} \nabla \left[\frac{\vec{\varrho} \cdot (\vec{k} \times \vec{l})}{\varrho^2} (\exp(-ik_0 R) - \exp(-ik_0 R_\varepsilon)) \right] \cdot \vec{dl}$$

$$- \frac{\sqrt{\varepsilon_1}}{4\pi i k_0 \sqrt{\mu}} \oint_{(K)} \left[\frac{\vec{\varrho} \cdot (\vec{k} \times \vec{l})}{\varrho^2} (\exp(-ik_0 R) - \exp(-ik_0 R_\varepsilon)) \delta \frac{\overrightarrow{\partial H^{\mathrm{TE}}}}{\partial x_3} \cdot \vec{dl}, \right. \tag{3.1.5}$$

where

$$\delta \vec{H}^{\mathrm{TE}}, \delta \vec{E}^{\mathrm{TE}}, \delta H_3^{\mathrm{TE}} \quad \text{and} \quad \delta \frac{\overrightarrow{\partial H}^{\mathrm{TE}}}{\partial x_3}$$

are the discontinuities of the respective quantities which appear on the lines of discontinuity of the field, found on the surface of integration S around the considered point P.

3.2. The case of the TM-field

When the propagated field in a given medium is a field of the TM-type i. e., when $H_3^{\mathrm{TM}} = 0$, the electric vector of a field at the P-point can be expressed with the help of the Larmor-Lorentz principle, in the form

$$E_l(P) = -\oiint_S df \vec{n} \cdot [\vec{E}^{\mathrm{TM}} \times (\nabla_Q \times \vec{G}_l^{(E)}) - \vec{G}_l^{(E)} \times (\nabla_Q \times \vec{E}^{\mathrm{TM}})], \tag{3.2.1}$$

where the index l correspond to the different components of the vector $\vec{E}(p)$. The vector $\vec{G}_l^{(E)}$ is defined by the relation

$$\vec{G}_l^{(E)} = \frac{1}{4\pi} \left\{ \left[\frac{\sqrt{\mu}}{k_0^2 \sqrt{\varepsilon_1}} \nabla_Q \frac{\partial}{\partial x_l} + \mu \varepsilon_2 \frac{1}{\varepsilon_l} \sqrt{\mu \varepsilon_1} \vec{l} \right] \frac{\exp(-ik_0 R_\varepsilon)}{R_\varepsilon} \right.$$

$$\left. + \frac{\sqrt{\mu \varepsilon_1}}{ik_0} \overleftrightarrow{\varepsilon}^{-1} \cdot \left[\nabla_Q \times \vec{k} \frac{\varrho \cdot (k \times) \vec{l}}{\varrho^2} (\exp(-ik_0 R) - \exp(-ik_0 R_\varepsilon)) \right] \right\}. \tag{3.2.2}$$

In the above expression, the second component is zero when $l = 3$ (it is when $\vec{l} = \vec{k}$).

When we introduce the vector $\vec{G}_l^{(E)}$ in the form (3.2.2) to the expression (3.2.1), we can express the vector $\vec{E}$ at the point of observation P with the help of this relation

$$E_l(P) = \frac{1}{4\pi}\frac{\varepsilon_3}{\varepsilon_l}\sqrt{\frac{\mu}{\varepsilon_1}}\oint\!\!\!\oint_S d\vec{fn}\cdot\left\{E_l^{\mathrm{TM}}\overset{\leftrightarrow}{\varepsilon}\cdot\nabla_Q\frac{\exp(-ik_0R_\varepsilon)}{R_\varepsilon}-\frac{\exp(-ik_0R_\varepsilon)}{R_\varepsilon}\overset{\leftrightarrow}{\varepsilon}\cdot\nabla_Q E_l^{\mathrm{TM}}\right\}$$

$$+\frac{i}{4\pi k_0}\sqrt{\frac{\mu\mu_0}{\varepsilon\varepsilon_0}}\oint\!\!\!\oint_S d\vec{fn}\cdot\left\{\nabla_Q\times\left[\vec{H}^{\mathrm{TM}}\frac{\partial}{\partial x_l}\frac{\exp(-ik_0R_\varepsilon)}{R_\varepsilon}\right]\right\}$$

$$+\frac{\varepsilon_3\sqrt{\mu\varepsilon_1}}{4\pi\varepsilon_l}\oint\!\!\!\oint_S d\vec{fn}\cdot\left\{\nabla_Q\times\left[\vec{E}^{\mathrm{TM}}\times\vec{l}\,\frac{\exp(-ik_0R_\varepsilon)}{R_\varepsilon}\right]\right\}$$

$$-\frac{\varepsilon_3\sqrt{\mu}}{4\pi\varepsilon_l\sqrt{\varepsilon_1}}(\varepsilon_3-\varepsilon_1)\oint\!\!\!\oint_S d\vec{fn}\cdot\left\{\nabla_Q\times\left[\frac{\exp(-ik_0R_\varepsilon)}{R_\varepsilon}\vec{k}(\vec{l}\times\vec{k})\cdot\vec{E}^{\mathrm{TM}}\right]\right\}$$

$$-\frac{i\sqrt{\mu}}{4\pi\sqrt{\varepsilon_1}}\oint\!\!\!\oint_S d\vec{fn}\cdot\left\{\nabla_Q\times\left[E_3^{\mathrm{TM}}\nabla\frac{\vec{\varrho}\cdot(\vec{k}\times\vec{l})}{\varrho^2}(\exp(-ik_0R_\varepsilon)-\exp(-ik_0R))\right]\right\}$$

$$-\frac{i\sqrt{\mu}}{4\pi k_0\sqrt{\varepsilon_1}}\oint\!\!\!\oint_S d\vec{fn}\cdot\left\{\nabla_Q\times\left[\frac{\partial\vec{E}^{\mathrm{TM}}}{\partial x_3}\frac{\vec{\varrho}\cdot(\vec{k}\times\vec{l})}{\varrho^2}(\exp(-ik_0R_\varepsilon)-\exp(-ik_0R))\right]\right\}. \qquad (3.2.3)$$

Relation (3.2.3) was obtained from (3.21) with the help of common vector identities. In the case of lack of lines of discontinuity on the integration surface, all integrals in the expresssion (3.2.3), which have the integrand of the form: $\vec{n}\cdot[\nabla\times\vec{A}]$, equal zero and the field at the point of observation P can be expressed by the formula

$$E_l(P) = \frac{1}{4\pi}\frac{\varepsilon_3}{\varepsilon_l}\sqrt{\frac{\mu}{\varepsilon_1}}\oint\!\!\!\oint_S d\vec{fn}\cdot\left\{E_l^{\mathrm{TM}}\overset{\leftrightarrow}{\varepsilon}\cdot\nabla_Q\frac{\exp(-ik_0R_\varepsilon)}{R_\varepsilon}-\frac{\exp(-ik_0R_\varepsilon)}{R_\varepsilon}\overset{\leftrightarrow}{\varepsilon}\cdot\nabla_Q E_l^{\mathrm{TM}}\right\}. \qquad (3.2.4)$$

When the lines of discontinuity appear on the integration surface, as in the case of diffraction on the object with the edge (see the Kirchhoff theory of diffraction), then the field at the point of observation P will be

$$E_l(P) = \frac{\varepsilon_3}{4\pi\varepsilon_l}\sqrt{\frac{\mu}{\varepsilon_1}}\oint\!\!\!\oint_S d\vec{fn}\cdot\left\{E_l^{\mathrm{TM}}\overset{\leftrightarrow}{\varepsilon}\cdot\nabla\frac{\exp(-ik_0R_\varepsilon)}{R_\varepsilon}-\frac{\exp(-ik_0R_\varepsilon)}{R_\varepsilon}\overset{\leftrightarrow}{\varepsilon}\cdot\nabla E_l^{\mathrm{TM}}\right\}$$

$$+\frac{i}{4\pi k_0}\sqrt{\frac{\mu\mu_0}{\varepsilon\varepsilon_0}}\oint_{(K)}\left[\frac{\partial}{\partial x_l}\frac{\exp(-ik_0R_\varepsilon)}{R_\varepsilon}\right]\overrightarrow{\delta H}^{\mathrm{TM}}\cdot d\vec{s}$$

$$+\frac{\varepsilon_3\sqrt{\mu\varepsilon_1}}{4\pi\varepsilon_l}\oint_{(K)}\frac{\exp(-ik_0R_\varepsilon)}{R_\varepsilon}(\overrightarrow{\delta E}^{\mathrm{TM}}\times\vec{l})\cdot d\vec{s}$$

$$-\frac{\varepsilon_3(\varepsilon_3-\varepsilon_1)}{4\pi\varepsilon_l}\sqrt{\frac{\mu}{\varepsilon_1}}\oint_{(K)}\frac{\exp(-ik_0R_\varepsilon)}{R_\varepsilon}(\vec{l}\times\vec{k})\cdot\overrightarrow{\delta E}^{\mathrm{TM}}\vec{k}\cdot d\vec{s}$$

$$+\frac{i\sqrt{\mu}}{4\pi k_0\sqrt{\varepsilon_1}}\oint_{(K)}\frac{\partial}{\partial x_3}\left\{\frac{\vec{\varrho}\cdot(\vec{k}\times\vec{l})}{\varrho^2}[\exp(-ik_0R_\varepsilon)-\exp(-ik_0R)]\right\}\overrightarrow{\delta E}^{\mathrm{TM}}\cdot d\vec{s}$$

$$-\frac{i\sqrt{\mu}}{4\pi k_0\sqrt{\varepsilon_1}}\oint_{(K)}\delta E_3^{\mathrm{TM}}\nabla\left\{\frac{\vec{\varrho}\cdot(\vec{k}\times\vec{l})}{\varrho^2}[\exp(-ik_0R_\varepsilon)-\exp(-ik_0R)]\right\}\cdot d\vec{s}$$

$$-\frac{i\sqrt{\mu}}{4\pi k_0\sqrt{\varepsilon_1}}\oint_{(K)}\left\{\frac{\vec{\varrho}\cdot(\vec{k}\times\vec{l})}{\varrho^2}[\exp(-ik_0R_\varepsilon)-\exp(-ik_0R)]\right\}\delta\frac{\partial\vec{E}^{\mathrm{TM}}}{\partial x_3}\cdot d\vec{s}. \qquad (3.2.5)$$

Where

$$\overrightarrow{\delta H}^{\mathrm{TM}},\ \overrightarrow{\delta E}^{\mathrm{TM}},\ \delta E_3^{\mathrm{TM}} \quad \text{and} \quad \delta\left(\frac{\partial\vec{E}^{\mathrm{TM}}}{\partial x_3}\right)$$

are the discontinuities of the respective quantities which appear on the lines of discontinuity of the field, found on the surface of integration S around the considered point P.

3.3. The case of the arbitrary field

If the arbitrary field propagates in a given uniaxially anisotropic medium, then in accordance with the results [7], we can always split it into two components, which we have considered above and find the final result in the form of the sum of the two above results. Therefore, the result for the l-component of the field $\vec{E}$ at the point of observation P be can written in the form

$$E_l(P) = E_l^{\mathrm{II}}(P)+\frac{\vec{l}\cdot\overleftrightarrow{\varepsilon}^{-1}}{i\omega\varepsilon_0}[\nabla_P\times\vec{H}^{\mathrm{I}}(P)], \qquad (3.3.1)$$

where $\vec{E}^{\mathrm{II}}$ and $\vec{H}^{\mathrm{I}}$ are defined by the expressions (3.1.3) and (3.2.3) and are connected respectively with one defined type of the incident field by the Larmor–Lorentz principle.

When we pass from the anisotropic medium to the isotropic one, (i.e. when we have $\overleftrightarrow{\varepsilon} = \overleftrightarrow{I}\varepsilon$), formula (3.3.1) is equivalent to (2.4).

Kottler's form of the Larmor–Lorentz principle is very useful in Kirchhoff's theory of diffraction, since it provides a quite simple interpretation of its results according to the Young–Rubinowicz model of diffraction phenomena. This is mainly due to the fact that Kottler's form of the Larmor–Lorentz principle permits one to obtain in a very simple way the wave potential of Miyamoto and Wolf, which in turn is the basis for this model.

4. Kirchhoff's theory of diffraction of the electromagnetic field in uniaxial anisotropic medium

The formulas (3.1.5) and (3.2.5) are the starting points for Kirchhoff's theory of diffraction which employes the Larmor-Lorentz principle for definition of the electromagnetic field at a given point P. To find the distrurbance at P we take the integral over a surface S formed by the opening A, a portion S_1 of the non-illuminated side of the screen, and a portion Ω_R of a large sphere of radius R, center at P which, together with A and S_1, form a closed surface.

If the field is radiating at infinity, the integrals over Ω_R tend to zero as the radius of the hemisphere tends to infinity and the surface of integration can be replaced throughout by S_1+A.

S comprises a perfectly conducting screen S_1 and an aperture A. If the screen is perfectly conducting, then we have at all its points

$$1^\circ \qquad \vec{n}\times\vec{E}|_{S_1} = 0 \quad \text{and} \quad \vec{n}\times\vec{H}|_{S_1} = 0,$$

where $\vec{n}$ is the unit vector outward normal to a closed surface of integration. It is the first of Kirchhoff's assumptions.

The second of Kirchhoff's assumptions is that at the aperture points $\vec{E}$ and $\vec{H}$ have the same value as in the incident field

$$2^\circ \qquad \vec{n}\times\vec{E}|_A = \vec{n}\times\vec{E}_0 \quad \text{and} \quad \vec{n}\times\vec{H}|_A = \vec{n}\times\vec{H}_0,$$

where $\vec{E}_0$ and $\vec{H}_0$ are the vectors of the incident field.

For these assumptions there is a line of discontinuity on the surface of integration, which is an edge of the screen.

If we take the conditions 1° and 2°, we have on the edge the next discontinuities of the value for the vectors of the field,

$$\overrightarrow{\delta H} = \vec{H}_0, \quad \overrightarrow{\delta E} = \vec{E}_0, \quad \overrightarrow{\delta\left(\frac{\partial H}{\partial x_3}\right)} = \frac{\partial \vec{H}_0}{\partial x_3}, \quad \overrightarrow{\delta\left(\frac{\partial E}{\partial x_3}\right)} = \frac{\partial \vec{E}_0}{\partial x_3}. \tag{4.1}$$

For the uniaxial anisotropic case the conditions of radiation (the Sommerfeld conditions) have been formulated by Wabia [8]. If we repeat the consideration of Kirchhoff for this case, we can define the field at the P point by the field at the points of the aperture.

And so we can express l-component of the $\vec{H}$ for the case of the field of the TE-type, in the form (3.1.5), where the discontinuities of the values of the field are suitably defined by (4.1). The integrals on the closed curve (K) are taken now on the edge of the screen. We may deal in the same way with the case of the field of TM-type. The l-components of $\vec{E}$ are defined by (3.2.5), where the discontinuities of the values of the field are given by (4.1).

The analysis of the results, which are given by formulae (3.1.5) and (3.2.5), show that the diffraction field, in the case of the incident field of TM-type or TE-type always consists of two components. The first component is the field of TM-type (or TE-type), and second is the field of TE-type (or TM-type). This second kind of field, absent in the incident field is connected with the sum curvilinear integrals. We can affirm, in agreement with the results of works [8], [9], that in the case of Kirchhoff's diffraction theory for electromagnetic waves in an anisotropic medium, the edge generates the new type of the field.

5. The wave potential of the Miyamoto and Wolf's type and the Young–Rubinowicz interpretation of the diffraction at the uniaxiall anisotropic medium

The results of Kirchhoff's diffraction theory in the uniaxially electric anisotropic medium may be interpreted in agreement with the Young–Rubinowicz model of the diffracted phenomena [10].

This model gives a posibility of representing the diffraction field as a superposition of the so-called diffracted waves arising as a result of reflection of the incidence field from the edge and of so-called geometrical waves.

To give this interpretation of the results of Kirchhoff's diffraction theory, it suffices to show the possibility of representing the expression (3.1.3) and (3.2.3) in the form

$$H_l = \oint\!\!\oint_S \vec{n} \cdot (\nabla \times \vec{W}_l^{(H)}) df,$$

$$E_l + \oint\!\!\oint_S \vec{n} \cdot (\nabla \times \vec{W}_l^{(E)}) df, \tag{5.1}$$

where $\vec{W}_l^{(H)}$ and $\vec{W}_l^{(E)}$ are the wave potentials. Once this is done then Young–Rubinowicz's interpretation [11] can be emploeyd.

If the incident field is of the TE-type, then we can use the expression (3.1.3). The integrands in (3.1.3) are already partially in the required form. One exception is the integral

$$\oint\!\!\oint_S \vec{n} \cdot \left\{ H_l^{\mathrm{TE}} \nabla \frac{\exp(-ik_0 R)}{R} - \frac{\exp(-ik_0 R)}{R} \nabla H_l^{\mathrm{TE}} \right\} df. \tag{5.2}$$

But in this case we can use the results of the work [12]. This integral can be represented in the form

$$\oint\!\!\oint_S \vec{n} \cdot \left\{ \nabla \times \frac{\sqrt{\varepsilon_1 \mu}}{4\pi} \left[\nabla R \times \frac{1}{R} \int_R^\infty \exp(-ik_0 R') \nabla' H_l^{\mathrm{TE}} dR' \right] \right\} df. \tag{5.3}$$

In (5.3) R is the length of vector $\overrightarrow{PQ}$, where P is the point of the observation and Q — the point of the integration on the closed surface S. The integration in (5.3) must be carried out with respect to R over a halfray beginning at the point Q, the extension of which passes through P.

Thus, we can give an explicit form of the wave's potential for l-component of the $\vec{H}$-field in the uniaxial anisotropic medium by the following expressions:

$$\vec{W}_l^{(H)}(P, Q) = -\frac{\sqrt{\varepsilon_1 \mu}}{4\pi}\left\{\left(\frac{\vec{R}}{R} \times \frac{1}{R}\int_R^\infty \exp(-ik_0 R')\nabla' H_l^{\mathrm{TE}} dR'\right)\right.$$

$$+\frac{\exp(-ik_0 R)}{R}(\vec{H}^{\mathrm{TE}} \times \vec{l}) - \frac{i\sqrt{\varepsilon_0}}{k_0 \mu \sqrt{\mu_0}} \vec{E}^{\mathrm{TE}} \frac{\partial}{\partial x_l} \frac{\exp(-ik_0 R_\varepsilon)}{R_\varepsilon}$$

$$+\frac{i}{k_0 \mu \varepsilon_1} \vec{k} \times \left[\vec{H}^{\mathrm{TE}} \times \nabla \frac{\vec{\varrho} \cdot (\vec{k} \times \vec{l})}{\varrho^2} (\exp(-ik_0 R) - \exp(-ik_0 R_\varepsilon))\right]$$

$$\left. -\frac{i}{k_0 \mu \varepsilon_1} \frac{\vec{\varrho} \cdot (\vec{k} \times \vec{l})}{\varrho^2} [\exp(-ik_0 R) - \exp(-ik_0 R_\varepsilon)] \frac{\partial \vec{H}^{\mathrm{TE}}}{\partial x_3}\right\}. \tag{5.4}$$

In the case, when the incident field is of the TM-type, the situation is similar to the above. This is due to the fact that in expression (3.2.3) one of the integrals has the integrand in the form

$$\frac{\varepsilon_3 \sqrt{\mu}}{4\pi \varepsilon_l \sqrt{\varepsilon_1}} \vec{n} \cdot \left\{E_l^{\mathrm{TM}} \overset{\rightleftarrows}{\varepsilon} \cdot \nabla \frac{\exp(-ik_0 R_\varepsilon)}{R_\varepsilon} - \frac{\exp(-ik_0 R_\varepsilon)}{R_\varepsilon} \overset{\rightleftarrows}{\varepsilon} \cdot \nabla E_l^{\mathrm{TM}}\right\} \tag{5.5}$$

which can be written down by the following expression (see [6]):

$$\vec{n} \cdot \frac{\varepsilon_3 \sqrt{\mu}}{4\pi \varepsilon_l} \left\{\nabla \times \overset{\rightleftarrows}{\varepsilon}^{-1} \cdot \left(\nabla R_\varepsilon \times \frac{1}{R_\varepsilon} \int_{R_\varepsilon}^\infty \exp(-ik_0 R_\varepsilon') \nabla' E_l^{\mathrm{TM}} dR_\varepsilon'\right)\right\}. \tag{5.6}$$

Thus, when the electromagnetic field is of the TM-type, then the explicit expression of the potential for l-component of the field can be written in the form

$$\vec{W}_l^{(E)}(P, Q) = \frac{\sqrt{\varepsilon_1 \mu}}{4\pi}\left\{\frac{1}{\varepsilon_l}\sqrt{\frac{\varepsilon_3}{\varepsilon_1}}\, \overset{\rightleftarrows}{\varepsilon}^{-1} \cdot \left(\nabla R_\varepsilon \times \frac{1}{R_\varepsilon}\int_{R_\varepsilon}^\infty \exp(-ik_0 R_\varepsilon')\nabla' E_l^{\mathrm{TM}} dR_\varepsilon'\right)\right.$$

$$+\frac{\varepsilon_3}{\varepsilon_1 \varepsilon_l}(\vec{E}^{\mathrm{TM}} \times \vec{l}) \frac{\exp(-ik_0 R_\varepsilon)}{R_\varepsilon} + \frac{i}{k_0 \varepsilon_1}\sqrt{\frac{\mu_0}{\varepsilon_0}}\, \vec{H}^{\mathrm{TM}} \frac{\partial}{\partial x_l} \frac{\exp(-ik_0 R_\varepsilon)}{R_\varepsilon}$$

$$\left. -\frac{\varepsilon_3(\varepsilon_1 - \varepsilon_3)}{\varepsilon_1 \varepsilon_l} \vec{k} \vec{E}^{\mathrm{TM}} \cdot (\vec{l} \times \vec{k}) \frac{\exp(-ik_0 R_\varepsilon)}{R_\varepsilon}\right\}$$

$$+\frac{i\sqrt{\mu}}{4\pi k_0 \sqrt{\varepsilon_1}} \vec{k} \times \left\{\vec{E}^{\mathrm{TM}} \times \nabla \left[\frac{\vec{\varrho} \cdot (\vec{k} \times \vec{l})}{\varrho^2} (\exp(-ik_0 R_\varepsilon) - \exp(-ik_0 R))\right]\right\}$$

$$-\frac{i\sqrt{\mu}}{4\pi k_0 \sqrt{\varepsilon_1}} \frac{\vec{\varrho} \cdot (\vec{k} \times \vec{l})}{\varrho^2} [\exp(-ik_0 R_\varepsilon) - \exp(-ik_0 R)] \frac{\partial \vec{E}^{\mathrm{TM}}}{\partial x_3}. \tag{5.7}$$

The expressions (5.4) and (5.7) for the vectors potentials, when the propagated field is of the TE-type or TM-type, are related by the subscript l with differential components of the incident fields H_l^{TE} or E_l^{TM}.

If the field's vectors are of interest, then one must use the tensor potentials. The l-columns of these tensors are the respective vector-potentials (5.4) for the TM-type or (5.7) — for the TE-type.

REFERENCES

[1] A. Wünsche, *Ann. Phys.* (Germany) **25**, 179 (1970).
[2] H. Hönl, A. Maue, K. Westpfahl, *Theorie der Beugung*, Handbuch der Physik Bd-XXV 1, Springer--Verlag, Berlin 1961.
[3] A. Rubinowicz, *Acta Phys. Pol.* **27**, 435 (1965).
[4] F. Kottler, *Ann. Phys.* (Germany) **71**, 457 (1923).
[5] A. Kujawski, J. Petykiewicz, International Symposium Electromagnetic Wave, Tbilisi 1971.
[6] A. Kujawski, J. Petykiewicz, Prace IPPT **8** (1972), (in Polish).
[7] A. Kujawski, S. Przeździecki, *Bull. Acad. Pol. Sci., Ser. Sci. Math. Astron. Phys.* **25**, 955 (1975).
[8] M. Wabia, *Acta Phys. Pol.* **A51**, 431 (1977).
[9] A. Rynkowski, *Acta Phys. Pol.* **A48**, 231 (1975).
[10] A. Rubinowicz, *Progress in Optics*, Vol. 4, North-Holland Publishing Company, Amsterdam 1965, p. 201.
[11] A. Rubinowicz, *Acta Phys. Pol.* **21**, 61 (1962).
[12] K. Miyamoto, E. Wolf, *J. Opt. Soc. Am.* **52**, 615 (1962).