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ON THE GROWTH AND DECAY OF WEAK MAGNETOGASDYNAMIC DISCONTINUITIES

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The modes of propagation of weak magnetogasdynamic (MGD) discontinuities have been determined by using the compatibility conditions. A set of differential equations governing the growth and decay of weak MGD discontinuities has been obtained in the presence of a transverse magnetic field. The particular case of a planar wave front has been studied in detail. It is shown that if a weak discontinuity is a compressive wave of order l, it will terminate into a shock wave after a critical time t_c. On the other hand if it is a rarefaction wave, it will decay and will damp out ultimately. The effects of the magnetic field will cause more rapid damping effects. The expression for t_c has been obtained and it is shown that the magnetic field effects are to decrease the critical time t_c.

1. Introduction

In recent technological developments several researchers [1, 2, 3, 4, 5, 6, 7] have made significant contributions on the spontaneous formation of shock waves in supersonic flows. Becker [5] provided a new physical approach to this problem of a simple model for the formulation of shock waves caused by the accelerated movements of a piston in a tube filled with a quiet gas. Recently Balaban [8] studied the formation and propagation of acceleration waves in elastic-plastic materials and McCarthy [9] studied the growth of thermal waves. McCarthy [10] also investigated the thermodynamical influences on the propagation of waves in electroelastic materiəls. Ram and Srinivasan [14] studied the effects of thermal radiation on the propagation of sonic waves in gases at very high temperature. The object of the present investigation is to study the propagation of weak

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discontinuities in magnetogasdynamics and to investigate the conditions under which these will either terminate into shock waves or will damp out ultimately.

The fundamental system of equations governing the continuous MGD flow are [11]

$$\frac{\partial \varrho}{\partial t}+U_i\varrho_{,i}+\varrho U_{i,i}=0 \tag{1}$$

$$\varrho\frac{\partial U_i}{\partial t}+\varrho U_jU_{i,j}+p_{,i}+\mu(H_jH_{j,i}-H_jH_{i,j})=0 \tag{2}$$

$$\frac{\partial H_i}{\partial t}+U_jH_{i,j}-H_jU_{i,j}+H_iU_{j,j}=0 \tag{3}$$

$$\frac{\partial p}{\partial t}+U_ip_{,i}+\varrho C^2U_{i,i}=0, \tag{4}$$

where ϱ, p, U_i and H_i, respectively, represent density, pressure, velocity components, and magnetic field components and a comma followed by an index (say i) denotes partial differentiation with respect to the corresponding coordinate (X^i).

2. *Modes of propagation*

Let us consider a moving wave $\Sigma(t)$ which is such that the flow parameters are continuous across the wave front $\Sigma(t)$ but their first and higher derivatives are discontinuous. Such a discontinuity is defined as a weak-discontinuity, or a sonic wave [7]. The wave surface $\Sigma(t)$ is assumed to be regular so that there exist limiting values of flow parameters and their derivatives as the surface is approached from either side. The geometric and kinematic compatibility conditions of first and second order for such a singular surface are [12]:

$$[Z_{,i}]=\left[\frac{\partial Z}{\partial n}\right]n_i;\quad \left[\frac{\partial Z}{\partial t}\right]=-G\left[\frac{\partial Z}{\partial n}\right], \tag{5}$$

$$[Z_{,ij}]=\left[\frac{\partial^2 Z}{\partial n^2}\right]n_in_j+g^{\alpha\beta}\left[\frac{\partial Z}{\partial n}\right]_{,\alpha}(n_jX_{i,\beta}+n_iX_{j,\beta})-\left[\frac{\partial Z}{\partial n}\right]g^{\alpha\beta}g^{\sigma\tau}b_{\alpha\sigma}X_{i,\beta}X_{j,\tau}; \tag{6}$$

$$\left[\frac{\partial Z_{,i}}{\partial t}\right]=\left(-G\left[\frac{\partial^2 Z}{\partial n^2}\right]+\frac{\delta}{\delta t}\left[\frac{\partial Z}{\partial n}\right]\right)n_i-Gg^{\alpha\beta}\left[\frac{\partial Z}{\partial n}\right]_{,\alpha}X_{i,\beta},$$

where $g^{\alpha\beta}$, $b_{\alpha\beta}$ are, respectively, the components of first and second fundamental tensor of the surface, n_i are the components of the outward unit normal vector drawn in the direction of propagation and $\frac{\partial Z}{\partial n}=Z_{,i}n_i$. G represents the velocity of propagation of the wave. The capital bracket denotes the jump in the quantity enclosed across the wave. Using (5) in (1), (2), (3) and (4) we get.

$$(U_n-G)\zeta+\varrho\lambda_i n_i = 0 \tag{7}$$

$$\varrho(U_n-G)\lambda_i+\xi n_i+\mu H_j\varepsilon_j n_i-\mu H_n\varepsilon_i = 0 \tag{8}$$

$$(U_n-G)\varepsilon_i+\lambda_j n_j H_i-H_n\lambda_i = 0 \tag{9}$$

$$(U_n-G)\xi+\varrho C^2\lambda_i n_i = 0, \tag{10}$$

where

$$\zeta = \left[\frac{\partial \varrho}{\partial n}\right], \quad \lambda_i = \left[\frac{\partial U_i}{\partial n}\right], \quad \varepsilon_i = \left[\frac{\partial H_i}{\partial n}\right], \quad \xi = \left[\frac{\partial p}{\partial n}\right]$$

$$U_n = U_i n_i \quad \text{and} \quad H_n = H_i n_i.$$

Solving for λ_i we obtain

$$(a_{ij}-\varphi\delta_{ij})\lambda_i = 0 \tag{11}$$

where

$$a_{ij} = C^2 n_i n_j+\frac{\mu H^2}{\varrho} n_i n_j-\frac{\mu H_n}{\varrho} H_i n_j-\frac{\mu H_n}{\varrho} H_j n_i$$

$$\varphi = (U_n-G)^2-\frac{\mu H_n^2}{\varrho}$$

In order that the equations (11) have a non-trivial solution for λ_i the following conditions should hold:

$$U_A^2 = \frac{\mu}{\varrho} H_n^2,$$

$$2U_+^2 = C^2+\frac{\mu}{\varrho} H^2+\sqrt{\left\{\left(C^2+\frac{\mu}{\varrho} H^2\right)^2-4C^2\frac{\mu}{\varrho} H_n^2\right\}},$$

$$2U_-^2 = C^2+\frac{\mu}{\varrho} H^2-\sqrt{\left\{\left(C^2+\frac{\mu}{\varrho} H^2\right)^2-4C^2\frac{\mu}{\varrho} H_n^2\right\}}$$

where $U = G-U_n$ is the relative velocity of propagation of the sonic wave. This shows that there are three possible modes of propagation of weak MGD discontinuities namely U_A, called Alfvén speed. U_+, the fast magnetoacoustic speed and U_-, the slow magnetoacoustic speed.

3. *Growth and decay of sonic waves*

Let us now consider the case when the wave front is moving in a quiet gas with magnetic field transverse to the direction of propagation. In this case the Alfvén and the slow magnetoacoustic modes of propagation will disappear and the wave will propagate with the effective speed of sound. From the equations (7), (8) and (10) we get the relations

$$\zeta = \frac{\varrho\lambda}{G} = \frac{1}{C^2}\xi, \quad \lambda_i = \lambda n_i, \quad G^2 = C^2 + b^2, \tag{11}$$

where $b^2 = \dfrac{\mu H^2}{\varrho}$.

Differentiating (1), (2) and (4) partially with respect to X^k and taking jumps across the wave front with the help of compatibility conditions, we get

$$\frac{\delta\zeta}{\delta t} = G\bar{\zeta} - \varrho\bar{\lambda}_i n_i + 2\lambda\zeta + 2\varrho\Omega\lambda \tag{12}$$

$$\varrho G\frac{\delta\lambda}{\delta t} = -G(\bar{\xi} - \varrho G\bar{\lambda}_i n_i) - \mu H^2\bar{\lambda}_i n_i - \mu(2\varepsilon_i H_i + H^\alpha H^\beta b_{\alpha\beta} - 2H^2\Omega) - \mu\varepsilon_i\varepsilon_i \tag{13}$$

$$\frac{\delta\xi}{\delta t} = G(\bar{\xi} - \varrho G\bar{\lambda}_i n_i) + \mu H^2\bar{\lambda}_i n_i + (\nu+1)\lambda\xi + 2\nu p\Omega\lambda, \tag{14}$$

where Ω is the mean curvature of the wave and $\dfrac{\delta}{\delta t}$ is the time operator as observed from the wave front $\Sigma(t)$ and

$$\bar{\lambda}_i = [U_{i,jk}]n_j n_k; \quad \bar{\xi} = [P_{,jk}]n_j n_k; \quad \bar{\zeta} = [\varrho_{,jk}]n_j n_k.$$

In view of the relation (11) we have

$$G\frac{\delta\zeta}{\delta t} = \varrho\frac{\delta\lambda}{\delta t}, \quad \frac{\delta\xi}{\delta t} = C^2\frac{\delta\zeta}{\delta t}. \tag{15}$$

Eliminating δ-time derivatives from equations (12), (13) and (14) with the help of (15) we get

$$G\bar{\zeta} - \varrho\bar{\lambda}_i n_i = \frac{1}{G^2+C^2}\{\mu(2\varepsilon_i H_i + H^\alpha H^\beta b_{\alpha\beta} - 2H^2\Omega)\lambda + \mu\varepsilon_i\varepsilon_i + G^2(2\lambda\zeta + 2\varrho\Omega\lambda) + 2C^2\lambda\zeta - (\nu+1)\lambda\xi\} \tag{16}$$

$$-G\bar{\xi} + (\varrho G^2 - \mu H^2)\bar{\lambda}_i n_i = \frac{1}{G^2+C^2}\{G^2(\nu+1)\lambda\xi - 2C^2G^2\lambda\zeta + \mu C^2(2\varepsilon_i H_i + H^\alpha H^\beta b_{\alpha\beta} - 2H^2\Omega)\lambda + C^2\mu\varepsilon_i\varepsilon_i + C^2G^2(2\lambda\zeta + 2\varrho\Omega\lambda)\}. \tag{17}$$

Let $\Sigma(t_0)$ represent the sonic wave surface at the initial time t_0 and let σ represent the distance measured from $\Sigma(t_0)$ along the normal trajectories. Then we have $\sigma = G(t-t_0)$ and the scalar functions λ, ξ, ζ can be regarded as functions of σ. Hence using (16), (17) and

(11) in (12), (13) and (14) we get

$$\frac{d\lambda}{d\sigma}+(A-\Omega)\lambda = B\lambda^2 \tag{18}$$

$$\frac{d\zeta}{d\sigma}+(A-\Omega)\zeta = \frac{G}{\varrho}B\zeta^2, \tag{19}$$

$$\frac{d\xi}{d\sigma}+(A-\Omega)\xi = \frac{G}{\varrho C^2}B\xi^2, \tag{20}$$

where

$$A = \frac{1}{2G^2}\frac{\mu}{\varrho}H^\alpha H^\beta b_{\alpha\beta}, \qquad B = \frac{1}{2G^3}\left\{3\frac{\mu H^2}{\varrho}+(\nu+1)C^2\right\}.$$

Differential equations (18), (19) and (20) govern the growth and decay of sonic waves during propagation. In view of relation (11) equations (18) and (20) are derivable from (19). Therefore, the equation (19) is sufficient to predict the growth and decay of sonic waves associated with the wave front $\Sigma(t)$. The mean curvature Ω of the wave $\Sigma(t)$ is a function of σ and is given by [13]

$$\Omega = \frac{\Omega_0 - K_0\sigma}{1-2\Omega_0\sigma+K_0\sigma^2},$$

where Ω_0 and K_0 are respectively the mean and Gaussian curvatures of the wave front at the initial time t_0. Substituting for Ω in (19) and integrating we get

$$\frac{1}{\zeta} = \frac{1}{\zeta_0}\sqrt{(1-2\Omega_0\sigma+K_0\sigma^2)}e^{A\sigma} - \frac{BG}{\varrho}e^{A\sigma}\sqrt{(1-2\Omega_0\sigma+K_0\sigma^2)}\int_0^\sigma \frac{e^{-A\sigma}d\sigma}{\sqrt{(1-2\Omega_0\sigma+K_0\sigma^2)}} \tag{21}$$

which determines the strength $\frac{\zeta}{\zeta_0}$ of the discontinuity as a function of σ. In order to predict the phenomenon more clearly we consider an interesting case of a plane wave front for which $\Omega = \Omega_0 = K_0 = 0$, $A = 0$. In this case equation (21) provides us the following relation

$$\frac{\zeta}{\zeta_0} = \left\{1-\frac{BG\zeta_0\sigma}{\varrho}\right\}^{-1} \tag{22}$$

which can be written in the dimensionless form as

$$\delta = \left\{1-\tfrac{1}{2}\left(\frac{3\alpha+\nu+1}{\alpha+1}\right)\eta\right\}^{-1},$$

where

$$\delta = \frac{\zeta}{\zeta_0}, \qquad \alpha = \frac{b^2}{C^2}, \qquad \eta = \frac{\zeta_0\sigma}{\varrho}, \qquad \nu = \frac{C_p}{C_v}.$$

From relation (22) we conclude that if the initial wave is a rarefaction wave ($\zeta_0 < 0$), the discontinuity ζ will go on decaying during propagation and will damp out ultimately. On the other hand if the initial wave is a compressive wave ($\zeta_0 > 0$), then the discontinuity ζ will grow continuously till it tends to infinity as

$$\sigma \to \frac{\varrho}{BG\zeta_0}.$$

In such a situation the continuity of the flow parameters will break down and consequently a strong discontinuity called shock wave will automatically appear. The critical time t_c for the termination of a sonic wave into a shockwave is given by

$$t_c = t_0 + \frac{\varrho}{B\zeta_0 G^2}$$

The other discontinuity parameters ξ and λ will also behave in a similar manner. In this case the magnetic field effects will cause earlier termination into a shock wave.

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