ON THE VELOCITY OPERATOR IN QUANTUM HYDRODYNAMICS

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In the usual form of quantum hydrodynamics there are two ways of introducing the velocity operator; one in terms of the inverse of the density operator $\varrho(\vec{x})^{-1}$ and other without using this inverse. Equivalence of these two approaches is investigated in this paper. An approach based on an integral equation is also shown to be equivalent to the above procedure. The velocity-velocity commutator is derived based on the non-relativistic current algebra. It is found that this commutator is not zero.

1. Introduction

In the process of quantization of classical hydrodynamics Landau [1] gave two definitions of the quantum velocity operator $\vec{v}(\vec{x})$:

$$\vec{v}(\vec{x}) = \tfrac{1}{2}\{\vec{J}(\vec{x})\varrho(\vec{x})^{-1}+\varrho(\vec{x})^{-1}\vec{J}(\vec{x})\}, \tag{1}$$

$$\vec{J}(\vec{x}) = \tfrac{1}{2}\{\varrho(\vec{x})\vec{v}(\vec{x})+\vec{v}(\vec{x})\varrho(\vec{x})\}, \tag{2}$$

where $\varrho(\vec{x})$ and $\vec{J}(\vec{x})$ are respectively the mass density and the mass current density operators satisfying non-relativistic current algebra [2]. Equations (1) and (2) may respectively be called the direct and the implicit definitions of the velocity operator. Landau has not established the equivalence of these two definitions, nor has he explicitely evaluated the velocity-velocity commutator which plays a significant role in the quantization of hydrodynamics. The velocity-velocity commutator has attracted a great deal of attention [3–5] and is still a puzzling problem. In this contribution an attempt has been made to understand this question.

Recently Yamasaki et al. [6, 7] have proposed a definition of the velocity operator in terms of an integral equation in the wave vector space and claim that the velocity-velocity commutator vanishes for a many-boson system while it has a non-zero value for a many-fermion system. This observation is a significant one, since on the basis of the non relativistic current algebra such a result is not anticipated. A study is made of this point as well.

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2. Equivalence of Landau's definitions

Assuming that the inverse of the density operator exists, it readily follows from (1) that

$$\varrho(\vec{x})\vec{v}(\vec{x})+\vec{v}(\vec{x})\varrho(\vec{x}) = \vec{J}(\vec{x})+\tfrac{1}{2}\{\varrho(\vec{x})\vec{J}(\vec{x})\varrho(\vec{x})^{-1}+\varrho(\vec{x})^{-1}\vec{J}(\vec{x})\varrho(\vec{x})\}. \tag{3}$$

From the algebra of currents [2] we have

$$\varrho(\vec{x})J_\mu(\vec{y})\varrho(\vec{x})^{-1} = -i\frac{\partial}{\partial x_\mu}[\delta(\vec{x}-\vec{y})\varrho(\vec{x})]\varrho(\vec{x})^{-1}+J_\mu(\vec{y}),$$

$$\varrho(\vec{x})^{-1}J_\mu(\vec{y})\varrho(\vec{x}) = i\frac{\partial}{\partial x_\mu}[\delta(\vec{x}-\vec{y})\varrho(\vec{x})]\varrho(\vec{x})^{-1}+J_\mu(\vec{y}),$$

where $\mu = 1, 2, 3$ denotes the three spatial indices. We thus have

$$\varrho(\vec{x})^{-1}J_\mu(\vec{y})\varrho(\vec{x})+\varrho(\vec{x})J_\mu(\vec{y})\varrho(\vec{x})^{-1} = 2J_\mu(\vec{y}),$$

where the limit $y \to x$ can be taken smoothly. Substituting this in (3) the implicit definition is obtained.

Starting from (2) we have

$$\varrho(\vec{x})^{-1}\vec{J}(\vec{x})+\vec{J}(\vec{x})\varrho(\vec{x})^{-1} = \vec{v}(\vec{x})+\tfrac{1}{2}\{\varrho(\vec{x})^{-1}\vec{v}(\vec{x})\varrho(\vec{x})+\varrho(\vec{x})\vec{v}(\vec{x})\varrho(\vec{x})^{-1}\}. \tag{4}$$

It is easy to verify that

$$\varrho(\vec{x})v_\mu(\vec{y})\varrho(\vec{x})^{-1}+\varrho(\vec{x})^{-1}v_\mu(\vec{y})\varrho(\vec{x}) = [[\varrho(\vec{x}), v_\mu(\vec{y})], \varrho(\vec{x})^{-1}]+2v_\mu(\vec{y}) \tag{5}$$

and if we assume that

$$[\varrho(\vec{z}), [\varrho(\vec{x}), v_\mu(\vec{y})]] = 0 \tag{6}$$

the limit $y \to x$ can be taken easily in (5) and on substituting this result in (4), the direct definition (1) is recovered.

Thus while definition (2) follows directly from (1), definition (2) implies (1) only if the condition (6) holds. It should also be remarked that assumption (6) is vital for the derivation of the velocity-velocity commutator from the implicit definition. However, it has not been realized that in such a derivation of the velocity-velocity commutator, one actually works in a frame work where (1) and (2) are equivalent [4, 5].

3. On the integral equation for the velocity operator

Consider the direct definition (1) and expand the operator $\varrho(\vec{x})^{-1}$ about the ground state expectation value $\langle\varrho\rangle$ of the operator $\varrho(\vec{x})$. Defining $\hat{\varrho}(\vec{x}) = \varrho(\vec{x})-\langle\varrho\rangle$ we have

$$2\vec{v}(\vec{x}) = \frac{1}{\langle\varrho\rangle}\left\{1-\frac{\hat{\varrho}(\vec{x})}{\langle\varrho\rangle}+\frac{\hat{\varrho}(\vec{x})\cdot\hat{\varrho}(\vec{x})}{\langle\varrho\rangle^2}-\dots\right\}\vec{J}(\vec{x})$$

$$+\frac{1}{\langle\varrho\rangle}\vec{J}(\vec{x})\left\{1-\frac{\hat{\varrho}(\vec{x})}{\langle\varrho\rangle}+\frac{\hat{\varrho}(\vec{x})\cdot\hat{\varrho}(\vec{x})}{\langle\varrho\rangle^2}-\dots\right\}. \tag{7}$$

By taking the Fourier transform we have

$$2\tilde{v}_\mu(\vec{k}) = \frac{2}{\langle\varrho\rangle}\tilde{J}_\mu(\vec{k}) - \frac{1}{\langle\varrho\rangle^2}\sum_{\vec{q}_1\neq 0}\{\tilde{\varrho}(\vec{q}_1)\tilde{J}_\mu(\vec{k}-\vec{q}_1)+\tilde{J}_\mu(\vec{k}-\vec{q}_1)\tilde{\varrho}(\vec{q}_1)\}$$

$$+\frac{1}{\langle\varrho\rangle^3}\sum_{\vec{q}_1\neq 0}\sum_{\vec{q}_2\neq 0}\{\tilde{\varrho}(\vec{q}_1)\tilde{\varrho}(\vec{q}_2)\tilde{J}_\mu(\vec{k}-\vec{q}_1-\vec{q}_2)+\tilde{J}_\mu(\vec{k}-\vec{q}_1-\vec{q}_2)\tilde{\varrho}(q_1)\tilde{\varrho}(\vec{q}_2)\} - \dots$$

This leads to the integral equation,

$$\tilde{v}_\mu(\vec{k}) = \frac{1}{\langle\varrho\rangle}\tilde{J}_\mu(\vec{k}) - \frac{1}{2\langle\varrho\rangle}\sum_{\vec{q}\neq 0}\{\tilde{\varrho}(\vec{q})\tilde{v}_\mu(\vec{k}-\vec{q})+\tilde{v}_\mu(\vec{k}-\vec{q})\tilde{\varrho}(\vec{q})\}, \tag{8}$$

which is analogous to the integral equation obtained by Yamasaki et al. [6].

It may be remarked that the operator within the summation sign over $\vec{q}$ in the above equation is in the symmetrized form. If this symmetrization is not done the velocity opertor is defined as in Ref. [7] viz. by an integral equation of the form

$$\tilde{v}_\mu(\vec{k}) = \frac{1}{\langle\varrho\rangle}\tilde{J}_\mu(\vec{k}) - \frac{1}{2\langle\varrho\rangle}\sum_{\vec{q}\neq 0}\tilde{\varrho}(\vec{q})\tilde{v}_\mu(\vec{k}-\vec{q}), \tag{9}$$

then the velocity operator $[\tilde{v}_\mu(\vec{k})]_L$ obtained from Landau's definition (1) and the operator obtained from (9) are related by

$$\tilde{v}_\mu(\vec{k}) = [\tilde{v}_\mu(\vec{k})]_L + \{\sum_{\vec{p}\neq 0} p_\mu\}\tilde{w}(\vec{k}), \tag{10}$$

where

$$\tilde{w}(\vec{k}) = \frac{\tilde{\varrho}(\vec{k})}{\langle\varrho\rangle^2} - \frac{2}{\langle\varrho\rangle^3}\sum_{\vec{q}_1\neq 0}\tilde{\varrho}(\vec{k}-\vec{q}_1)\tilde{\varrho}(\vec{q}_1)$$

$$+\frac{3}{\langle\varrho\rangle^4}\sum_{\vec{q}_1\neq 0}\sum_{\vec{q}_2\neq 0}\tilde{\varrho}(\vec{k}-\vec{q}_1-\vec{q}_2)\tilde{\varrho}(\vec{q}_1)\tilde{\varrho}(\vec{q}_2) - \dots$$

which in the configuration space becomes

$$w(\vec{x}) = \hat{\varrho}(\vec{x})\,[\varrho^{-1}(\vec{x})/\langle\varrho\rangle]^2.$$

The above relation is easily obtained by using non-relativistic current algebra in the momentum space:

$$[\tilde{\varrho}(\vec{p}), \tilde{\varrho}(\vec{q})] = 0$$

$$[\tilde{\varrho}(\vec{p}), \tilde{J}_\mu(\vec{q})] = -p_\mu\tilde{\varrho}(\vec{p}+\vec{q})$$

$$[\tilde{J}_\mu(\vec{p}), \tilde{J}_\nu(\vec{q})] = -p_\nu\tilde{J}_\mu(\vec{p}+\vec{q}) + q_\mu\tilde{J}_\nu(\vec{p}+\vec{q}).$$

Since definition (1) and (2) have been proved to be equivalent it follows that $[\tilde{v}_\mu(\vec{k})]_L$ has no singular behaviour. This establishes that $\tilde{v}(\vec{k})$ as defined by (9) is singular since it involves the term $\sum_{\vec{p}} p_\mu$. This is in agreement with the observation made by Yamasaki et al. [6].

4. *A comment on the velocity-velocity commutator*

Fanelli and Struzynski [3] have made an attempt to show that the velocity-velocity commutator is zero

$$[v_\mu(\vec{x}), v_\nu(\vec{y})] = 0. \tag{11}$$

In this derivation use has been made of the inverse of the quantum field operators ψ^{-1} and $(\psi^\dagger)^{-1}$ which do not exist. A later attempt of Yee [4] using the implicit definition does not subscribe to this result. Kobe and Coomer [5] obtained the result (11) by insisting on the additional requirement that $\vec{V}(x)$ should be a vector function which satisfies the condition $\vec{\nabla} \times \vec{V} = 0$ in which case the velocity operator has the following functional derivative representation:

$$v_\mu(\vec{x}) \to V_\mu(\vec{x}) - i\hbar \frac{\partial}{\partial x_\mu} \frac{\delta}{\delta \varrho(\vec{x})} \qquad (\mu = 1, 2, 3). \tag{12}$$

Yamasaki et al. [6, 7] obtained the result that (11) is valid for a many-boson system while it is not true for a many fermion system. However the approach based on non-relativistic current algebra does not support this point of view. It may be pointed out that an entirely independent approach by Varga and Eckstein [8] also differs from (11).

It should, however, be emphasised that no attempt has been made to evaluate $[v_\mu(\vec{x}), v_\nu(\vec{y})]$ using definition (1) and the non-relativistic current algebra. In view of the divergent results pointed out here, it is worthwhile to persue this problem from this point of view as well.

In this approach the commutator involving the inverse mass density operator plays a significant role [9].

$$[J_\mu(\vec{x}), \varrho(\vec{y})^{-1}] = -i \frac{\partial}{\partial y_\mu} [\delta(\vec{x}-\vec{y})\varrho(\vec{y})]\varrho^{-2}(\vec{y}). \tag{13}$$

On using this along with (1) we obtain.

$$[v_\mu(\vec{x}), v_\nu(\vec{y})] = i\delta(\vec{x}-\vec{y}) \left\{ \frac{\partial v_\nu(\vec{x})}{\partial x_\mu} - \frac{\partial v_\mu(\vec{x})}{\partial x_\nu} \right\} \varrho(\vec{x})^{-1}. \tag{14}$$

This is in agreement with the observation of Varga and Eckstein [8]. Equation (12) can be improved from a different observation as well. From equation (5) it follows that

$$\varrho(\vec{x})v_\mu(\vec{y})\varrho^{-1}(\vec{x}) + \varrho^{-1}(\vec{x})v_\mu(\vec{y})\varrho(\vec{x}) = 2v_\mu(\vec{y}). \tag{15}$$

The most general form of the velocity operator as obtained from this is

$$v_\mu(\vec{x}) \rightarrow F(\varrho) - i \frac{\partial}{\partial x_\mu} \frac{\delta}{\delta\varrho(\vec{x})},$$

$F(\varrho)$ being a suitably chosen functional of $\varrho(\vec{x})$. This observation is in agreement with that of Kobe and Commer [5].

5. *Conclusion*

An analysis of the definitions of the velocity operator as given by Landau has been made. It is established that these definitions are equivalent subject to the condition (6). An integral equation postulated by Yamasaki et al. [6] is also derived from these definitions. However, the velocity-velocity commutator is found to be different from zero in contradiction with some of the results obtained earlier [3, 6].

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