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WEIGHT AND STEP OPERATORS FOR FINITE GROUPS*

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Weight and step operators for finite groups with properties analogous to those of Lie group infinitesimal operators are defined. A method for the determination of such operators as eigenvectors of the commutator eigenvalue problem in the group algebra is given. General results are demonstrated in the case of the point group D_3.

1. Introduction

In order to evaluate various quantities of the representation theory of finite groups (representation matrices, Clebsch-Gordan coefficients, reduction coefficients etc.) the projection operators [1] are used. The analytic formulae for these operators depend in general not only on properties of a given group G, but also on the choice of bases of irreducible representations, and hence allow for large arbitrariness [2] which makes deriving any general relation between various coefficients of the theory in a compact form difficult.

On the other hand, in the representation theory of Lie groups, the canonical basis of infinitesimal operators algebra consisting of weight and step operators is in common use [3]. E. g. for the $SU(2)$ group we have the well known relations

$$j^z|jm\rangle = m|jm\rangle, \quad j^{\pm}|jm\rangle = [(j\mp m)(j\pm m+1)]^{1/2}|jm\pm 1\rangle, \tag{1}$$

which determine a basis for any irreducible representation $D^{(j)}$ of this group with an accuracy of one phase factor for the whole $(2j+1)$-dimensional carrier space. Such relations replace in an efficient and elegant way the projection operators of finite groups, and enable one to derive analytic expressions and recurrence formulas for appropriate coupling coefficients of the theory [1, 3].

Gamba [4] has shown that a finite group G can be associated in a natural way with a Lie algebra through a definition of the commutator product

$$[g, g'] = gg'-g'g, \quad g, g'\in G \tag{2}$$

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in an ordinary associative group algebra. The aim of the present paper is to develope further this idea, namely by application of generators of the Lie algebra to the determination of matrices of irreducible representations of the associated finite group. We also propose a method for the determination of these generators in terms of elements of the group G, and demonstrate it in the case of the point group D_3.

2. *Algebras and bases for a finite group*

An ordinary group algebra $A(G)$ over a finite group G (the Fröbenius algebra) with the group multiplication is a semisimple associative algebra, and its constituent simple algebras $A_\Gamma(G)$ can be labelled by irreducible representations Γ of the group G [5]

$$A(G) = \sum_\Gamma \oplus A_\Gamma(G), \tag{3}$$

where $\oplus$ denotes the direct sum. Each $A_\Gamma(G)$ is a simple matrix algebra, i. e. the algebra of all $[\Gamma]\times[\Gamma]$ matrices ($[\Gamma]$ denotes the dimension of Γ). Elements

$$S^{(\Gamma)}_{\gamma\to\gamma'} = \frac{[\Gamma]}{n(G)} \sum_{g\in G} D^{(\Gamma)}_{\gamma'\gamma}(g)^* g, \tag{4}$$

where $D^{(\Gamma)}_{\gamma\gamma'}(g)$ are the matrix elements of the irreducible representation Γ, and $n(G)$ the number of elements in G, form a basis of the algebra $A_\Gamma(G)$ [1, 5]. Substituting the last factor g in this formula by the corresponding operator of any representation T we obtain for $\gamma = \gamma'$ usual operators projecting any vector of a carrier space of T into a subspace of vectors having the same transformational properties as the basis vector $|\Gamma\gamma\rangle$ of the irredducible representation Γ, and for $\gamma \neq \gamma'$ the transition operators which map $|\Gamma\gamma\rangle$ into $|\Gamma\gamma'\rangle$ [1]. Therefore, one can choose a basis for the Fröbenius algebra $A(G)$ consisting of $q_1 = \sum_\Gamma [\Gamma]$ idempotents $S^{(\Gamma)}_{\gamma\to\gamma}$ and $q_2 = \sum_\Gamma ([\Gamma]^2-1)$ nilpotents $S^{(\Gamma)}_{\gamma\to\gamma'} (\gamma \neq \gamma')$.

The maximal Abelian subalgebra of the Fröbenius algebra $A(G)$ is the character algebra $A_\chi(G)$, which can be spanned over a basis of idempotents

$$X^{(\Gamma)} = \sum_\gamma S^{(\Gamma)}_{\gamma\to\gamma} = \frac{[\Gamma]}{n(G)} \sum_K \chi^{(\Gamma)}(K)K, \tag{5}$$

where K is the sum over a class of mutually conjugate elements of the group G, and $\chi^{(\Gamma)}(K)$ is the character of any element of the class K in Γ.

Replacing the group multiplication gg' in the Fröbenius algebra $A(G)$ by the commutator product $[g, g']$ defined by Eq. (2) we obtain a Lie algebra $B(G)$ [4], referred to in the following as the commutator algebra. Since the $S^{(\Gamma)}_{\gamma\to\gamma'}$ satisfy the commutation relations of generators of the unitary group $U([\Gamma])$ [3], i. e.

$$[S^{(\Gamma)}_{\gamma_1\to\gamma_2}, S^{(\Gamma')}_{\gamma_1'\to\gamma_2'}] = \delta_{\Gamma\Gamma'}(\delta_{\gamma_1\gamma_2'} S^{(\Gamma)}_{\gamma_1'\to\gamma_2} - \delta_{\gamma_1'\gamma_2} S^{(\Gamma)}_{\gamma_1\to\gamma_2'}), \tag{6}$$

then we have

$$B(G) = \sum_{\Gamma} \oplus B_\Gamma(G), \tag{7}$$

$$B_\Gamma(G) = B_\Gamma^{(1)}(G) \oplus B_\Gamma^{(2)}(G), \tag{8}$$

with subalgebras $B_\Gamma(G)$, $B_\Gamma^{(1)}(G)$, and $B_\Gamma^{(2)}(G)$ isomorphic with $U([\Gamma])$, $SU([\Gamma])$, and $U(1)$, respectively.

The character algebra $A_\chi(G)$ after introducing the commutator product (2) becomes a nilpotent algebra $B_\chi(G)$, and

$$B_\chi(G) = \sum_{\Gamma} \oplus B_\Gamma^{(2)}(G). \tag{9}$$

Evidently, the basis for each one-dimensional algebra $B_\Gamma^{(2)}(G)$ is the element $X^{(\Gamma)}$ given by Eq. (5).

Subtracting the nilpotent subalgebra $B_\chi(G)$ from the commutator algebra $B(G)$ we obtain a semisimple algebra

$$B^{(1)}(G) = \sum_{\Gamma} \oplus B_\Gamma^{(1)}(G) \tag{10}$$

which includes all consequences of non-commutivity of multiplication in the group G. For each simple subalgebra $B_\Gamma^{(1)}(G)$ we can choose the canonical Cartan-Weyl basis [3] consisting of weight operators H_i and step operators E_α which satisfy standard commutation relations

$$\begin{gathered}
[H_i, H_j] = 0 \quad (i, j = 1, 2, \ldots, [\Gamma]-1), \\
[H_i, E_\alpha] = \alpha_i E_\alpha, \\
[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta} \quad \text{for } \beta \neq -\alpha \\
[E_\alpha, E_{-\alpha}] = \sum_i \alpha^i H_i,
\end{gathered} \tag{11}$$

where α^i and α_i are covariant and contravariant components of the root vector α labelling the step operator E_α; the corresponding metric tensor g_{ij} as well as numbers $N_{\alpha\beta}$ can be determined according to the Wybourne monograph [3]. The basis functions $|\Gamma\gamma\rangle$ can be labelled by means of the set of eigenvalues of weight operators $H_i(\Gamma)$.

The Casimir operator

$$C(\Gamma) = \sum_{i,j} g^{ij} H_i(\Gamma) H_j(\Gamma) + \sum_{\alpha} E_\alpha(\Gamma) E_{-\alpha}(\Gamma) \tag{12}$$

commutes with each element of $B(G)$. Since in each algebra $B_\Gamma(G)$ only one independent element, namely $X^{(\Gamma)}$, exhibits this property, we find that

$$C(\Gamma) = \kappa(\Gamma) X^{(\Gamma)}, \tag{13}$$

where $\kappa(\Gamma)$ is a number.

We can therefore define a basis in the Fröbenius algebra $A(G)$ consisting of operators $X^{(\Gamma)}$, $H_i(\Gamma)$, and $E_\alpha(\Gamma)$ instead of $S^{(\Gamma)}_{\gamma\to\gamma'}$. In the case of non-Abelian groups, all operators related to representations with $[\Gamma] > 1$ can be derived from the semisimple Lie algebra $B^{(1)}(G)$.

3. Derivation of matrices of irreducible representations

Putting

$$X^{(\Gamma')} = \check{I}\delta_{\Gamma\Gamma'}, H_i(\Gamma') = \check{H}_i\delta_{\Gamma\Gamma'}, E_\alpha(\Gamma') = \check{E}_\alpha\delta_{\Gamma\Gamma'}, \tag{14}$$

where $\check{I}$ is the unit $[\Gamma]\times[\Gamma]$ matrix, $\check{H}_i$ and $\check{E}_\alpha$ are the matrices of weight and step elements in the group $SU([\Gamma])$, and writting down the elements $X^{(\Gamma)}$, $H_i(\Gamma')$, and $E_\alpha(\Gamma')$ in terms of elements of the group G we get a complete set of linear equations for the matrix elements $D^{(\Gamma)}_{\gamma\gamma'}(g)$. Hence if we know the weight, step, and character operators we are able to determine matrices of any irreducible representation Γ of the group G.

Weight and step operators can be found as solutions of the commutator eigenvalue problem [3]

$$[F, Y] = \varrho Y, \tag{15}$$

where F is an arbitrary element of $B^{(1)}$, and ϱ and Y is an eigenvalue (a number) and an eigenvector (an element of $B^{(1)}$), respectively. If we choose F in such a way that the corresponding secular equation has the maximum number of different roots ϱ, then only the root $\varrho = 0$ can be degenerated. Eigenvectors corresponding to $\varrho = 0$ and $\varrho \neq 0$ are the weight and step operators, respectively.

If we look for solutions of the commutator secular equation within the whole algebra $B(G)$, we get, besides weight and step operators, the elements of the algebra $B^{(2)}(G)$ as well. The latter belong obviously to the eigenvalue $\varrho = 0$. Since the solution is independent of the choice of basis of the algebra $B(G)$ we can use the basis consisting of the group elements $g \in G$. Putting

$$F = \sum_{g\in G} f_g g, \qquad Y = \sum_{g\in G} y_g g \tag{16}$$

we obtain from (15)

$$\sum_{g'} (f_{gg'^{-1}} - f_{g'^{-1}g}) y_{g'} = \varrho y_g. \tag{17}$$

Hence the weight and step operators are determined by a solution of the eigenvalue problem for the $n(G)\times n(G)$ matrix $\check{F}$ with elements

$$F_{g,g'} = f_{gg'^{-1}} - f_{g'^{-1}g}. \tag{18}$$

4. An example: the group D_3

The dihedral group D_3 consists of 6 elements which we arrange in a sequence $\{E\}$, $\{C_3, C_3^{-1}\}$, $\{u_1, u_2, u_3\}$, where the Schöenflies notation is used, and curly brackets indicate classes of conjugate elements. When we establish the group multiplication table by $C_3 u_i = u_{i+1(\text{mode } 3)}$, the matrix $\check{F}$ determined by Eq. (18) has in this sequence the form

$$\check{F} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 & B_2 & B_3 \\ 0 & 0 & 0 & -B_1 & -B_2 & -B_3 \\ 0 & B_1 & -B_1 & 0 & -A & A \\ 0 & B_2 & -B_2 & A & 0 & -A \\ 0 & B_3 & -B_3 & -A & A & 0 \end{bmatrix} \quad \begin{aligned} A &= f_{C_3} - f_{C_3^{-1}} \\ B_1 &= f_{u_2} - f_{u_3} \\ B_2 &= f_{u_3} - f_{u_1} \\ B_3 &= -B_1 - B_2 \end{aligned} \tag{19}$$

This matrix has, for arbitrary A, B_i, at most two nonzero roots. Assuming e. g. $F = C_3$ we get $\varrho_\pm = \pm i\sqrt{3}$, and $\varrho = 0$ is a four-fold root. We can choose basis elements of the algebra $B^{(2)}(D_3)$ (isomorphic with $SU(2)$) in a form

$$H = \frac{-i}{2\sqrt{3}}(C_3 - C_3^{-1})$$

$$E_\pm = \tfrac{1}{3}(u_1 + \omega^{\mp 1} u_2 + \omega^{\pm 1} u_3), \qquad \omega = e^{\frac{2\pi i}{3}}. \tag{20}$$

These elements satisfy commutation rules

$$[H, E_\pm] = \pm E_\pm, \qquad [E_+, E_-] = 2H \tag{21}$$

identical with those for angular momentum operators j^z, $j^\pm$, used in Eq. (1). The Casimir operator for the algebra $B^{(2)}(D_3)$ has the form

$$C(E) = H^2 + \tfrac{1}{2}(E_+E_- + E_-E_+) = \tfrac{1}{2}E - \tfrac{1}{4}(C_3 + C_3^{-1}) = \tfrac{3}{4}X^{(E)}, \tag{22}$$

where $X^{(E)}$ is the character operator for the two-dimensional irreducible representation E of D_3. The complete set of linear equations for matrices of this representation can be written as

$$X^{(A_1)} \equiv \tfrac{1}{6}(E + C_3 + C_3^{-1} + u_1 + u_2 + u_3) = \check{O}$$

$$X^{(A_2)} \equiv \tfrac{1}{6}(E + C_3 + C_3^{-1} - u_1 - u_2 - u_3) = \check{O}$$

$$X^{(E)} \equiv \tfrac{1}{3}(2E - C_3 - C_3^{-1}) = \check{I}$$

$$H(E) \equiv \frac{-i}{2\sqrt{3}}(C_3 - C_3^{-1}) = \tfrac{1}{2}\check{\sigma}_z$$

$$E_\pm(E) \equiv \tfrac{1}{3}(u_1 + \omega^{\mp 1} u_2 + \omega^{\pm 1} u_3) = \tfrac{1}{2}(\check{\sigma}_x \pm i\check{\sigma}_y), \tag{23}$$

where A_1, A_2 are Mulliken symbols for one-dimensional representations of D_3, and $\check{\sigma}_x$, $\check{\sigma}_y$ and $\check{\sigma}_z$ are the Pauli matrices. The solution is given by the following matrices $D^{(E)}(g)$ arranged in the same sequence as rows and columns of the matrix $\check{F}$ of Eq. (19)

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}. \tag{24}$$

5. *Final remarks and conclusions*

We have shown in this paper that the concept of commutator product for elements of a finite group opens the way, in agreement with the predictions of Gamba [4], to an application of the methods characteristic of representation theory of Lie groups in the corresponding theory for the finite group. Though finite groups do not have infinitesimal operators, they can be endowed with weight, step, and Casimir operators. The latter play essentially the same role in the representation theory as the former for the case of Lie groups, i. e. they define the basis functions for irreducible representations of a finite group in analogy with formula (1). A specific feature of finite groups is that generators of appropriate Lie group are associated with a particular representation Γ (generators associated with $\Gamma' \neq \Gamma$ are nilpotent in the representation Γ). The determination of generators of Lie groups for a finite group consists in solving the eigenvalue problem for the matrix $\check{F}$ given by Eq. (18).

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