BOUND STATES FOR SINGULAR POTENTIALS

M. DATTA

Centre of Advanced Study in Applied Mathematics, University of Calcutta*

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The WKB approximation has been used to find the bound state energy levels for the sum of two singular potentials $\left(V(r) = \frac{\alpha}{r^4} - \frac{\beta}{r^2}\right)$.

1. Introduction

The bound state problem for the case of singular potentials has been discussed by many authors [1]. It has been observed that there does not exist physically meaningful bound states for attractive potentials. Different prescriptions which are completely arbitrary in nature, lead to different set of bound states and in general these are not bounded below. This is true also for the transition potential β/r^2. Case [2] and Meetz [3] have explicitly found such bound state spectra for the strongly attractive transition potential β/r^2 $\left(\text{i.e. for } \frac{2\mu\beta}{\hbar^2} < -\frac{1}{4}\right)$. Scarf [4] obtained similar results for the potential $\frac{\beta}{r^2} + \frac{\alpha}{r}$. Scarf also obtained an infinite number of bound states for $-\frac{1}{4} < \frac{2\mu\beta}{\hbar^2} < 0$ with finite lower bound. Tietz [5] obtained the same result by making the Wronskian vanishing for two independent solutions, one of which is bounded at infinity and the other unbounded. Landau and Lifshitz [6] have shown that if the potential ($V(r) = -\beta/r^2$) takes any other form at small distances such that the particle does not fall to the centre, then for $0 > \gamma\left(= \frac{2\mu\beta}{\hbar^2}\right) > -\frac{1}{4}$ a finite number of bound states may exist while for $\gamma < -\frac{1}{4}$ an infinite number of bound states exist. It follows from all these considerations that a unique bound state problem does not exist for a simple singular power potential. It is likely that the sum of two singular potentials, one of which is attractive and the other repulsive, may support in a unique way, in general, some bound states if the repulsive potential is

* Address: Centre of Advanced Study in Applied Mathematics, University of Calcutta, 92 Acharya Prafulla Chandra Road, Calcutta – 700 009, India.

more singular at the origin. In this article we shall explore one such case. We choose $V(r) = \frac{\alpha}{r^4} - \frac{\beta}{r^2}$ and find the possible energy levels in the s-state using the WKB approximation.

2. Energy levels

The energy levels of the bound-states for the potential $V(r) = \frac{\alpha}{r^4} - \frac{\beta}{r^2}$ are determined from the following WKB approximation formula

$$\int_{r_1}^{r_2} Q(r)dr = (n+\tfrac{1}{2})\pi, \quad n = 0, 1, 2 \ldots, \tag{1}$$

where $Q(r) = \left[\frac{2m}{\hbar^2}(E-V)\right]^{1/2}$ and $r_1 < r_2$ are the two turning points.

For the bounding energy we put

$$E = -|E| = -\frac{\hbar^2}{2m}k^2. \tag{2}$$

Substituting $a = \frac{\hbar^2}{m\beta}$, $b^2 = \frac{\hbar^2}{2m\alpha}$ equation (1) takes the form

$$k\int_{r_1}^{r_2} \frac{\sqrt{(r^2-r_1^2)(r_2^2-r^2)}}{r^2}\,dr = (n+\tfrac{1}{2})\pi, \tag{3a}$$

where

$$r_1 = \left[\frac{1}{k^2a}\left(1-\sqrt{1-\frac{k^2a^2}{b^2}}\right)\right]^{1/2} \tag{3b}$$

$$r_2 = \left[\frac{1}{k^2a}\left(1+\sqrt{1-\frac{k^2a^2}{b^2}}\right)\right]^{1/2} \tag{3c}$$

The integral on the left hand side of equation (3a) may be evaluated in terms of standard elliptic integrals. The result is

$$kr_2[(2-p)K(p)-2E(p)] = (n+\tfrac{1}{2})\pi, \tag{4a}$$

$$K(p) = \int_0^{\pi/2} \frac{d\Phi}{\sqrt{1-p\sin^2\Phi}}, \tag{4b}$$

$$E(p) = \int_0^{\pi/2} \sqrt{1-p\sin^2\Phi}\,d\Phi, \tag{4c}$$

and

$$p = 2\sqrt{1-k^2a^2/b^2}/(1+\sqrt{1-k^2a^2/b^2}). \tag{4d}$$

Now given the parameters a, b, k^2 may be determined from equation (4a) and finally the energy levels from equation (2). The equations are such that we may choose one of the parameters, say, a arbitrarily and then the energy levels come out in the combination

$$E = -\frac{\hbar^2}{2m}\kappa^2(a)\frac{b^2}{a^2},$$

where $\kappa^2(a) = \dfrac{k^2a^2}{b^2}$.

We have displayed in Table I the quantity κ^2 for some low-lying states for various choice of the parameter a. For fixed a and b there is an infinite set of levels having an accumulation point at $E = 0$. From the Table it is also evident that for small a the lower

TABLE I

The parameter κ^2 corresponding to the energy levels for different values of a

n \ a	0.0005	0.001	0.005	0.01	0.1
0	0.95	0.94	0.88	0.82	0.52
1	0.87	0.82	0.66	0.54	0.14
2	0.80	0.73	0.48	0.35	<0.04
3	0.73	0.64	0.36	0.22	
4	0.67	0.56	0.23	0.15	
5	0.60	0.49	0.19	0.10	
6	0.55	0.44	0.15	0.07	
7	0.51	0.38	0.11	0.04	
8	0.46	0.33	0.08		
9	0.42	0.27	0.06		
10	0.38	0.23	0.04		
11	0.34	0.20			
12	0.31	0.19			
13	0.29	0.17			
14	0.26	0.15			
15	0.23	0.13			
16	0.21	0.11			
17	0.19	0.10			
18	0.18	0.09			
19	0.16	0.075			
20	0.15	0.07			
21	0.14	0.06			
22	0.12	0.05			
23	0.11	0.04			
24	0.10				
25	0.09				
26	0.08				
27	0.07				
28	0.066				
29	0.06				
30	0.05				

levels are close to each other in comparison with the low-lying hydrogen levels. With an increase in the parameter a the low-lying levels become more separated. For some well chosen parameters a, b the bound states may be made to correspond with some physical systems. We now justify in the next section the use of WKB approximation in our present analysis.

3. *Justification of the use of* WKB *approximation*

The use of WKB approximation is justified in cases where $\left|\frac{Q'}{2Q^2}\right| \ll 1$. In the present case for $r_1 \leqslant r \leqslant r_2$, $\frac{Q'}{2Q^2} \sim \frac{\sqrt{a}}{\left(1-\frac{k^2a^2}{b^2}\right)^{1/2}}$. Now for small a (say $\ll 1$) and those cases for which $\frac{k^2a^2}{b^2}$ is appreciably less than unity, the right hand side is also appreciably less than unity and the WKB results are quite accurate. Thus all the results presented in Table I are very accurate.

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