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RENORMALIZED MAGNONS AND PHONONS IN A STRONGLY ANHARMONIC HEISENBERG FERROMAGNETIC CRYSTAL. I

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The Matsubara thermodynamic perturbation calculus is used to describe the effects of coupling between spins and highly anharmonic lattice vibrations in a Heisenberg cubic ferromagnet. The Hartree-Fock approximation is employed to derive the system of self-consistent equations for the renormalized magnon and phonon average occupation numbers. Derivations are also presented for the renormalized Helmholtz free energy and the magnetization. It is shown that the spin-phonon interactions give rise to decrease of the spontaneous magnetization. The results obtained here are valid within a wide range of temperature (between absolute zero and the Curie point), wherein the rough harmonic theory of the lattice dynamics completely fails.

1. Introduction

The spin-phonon coupling in a Heisenberg magnetic crystal arises from the modulation of exchange integral by the thermal lattice vibrations, which vary the interatomic distances. Hence the exchange integral as a function of the instantaneous positions of atoms can be expanded formally in power series of the atom displacements from their equilibrium positions. A standard approximation truncates the series after the second-order term (harmonic approximation). By means of various methods, the harmonic model of the magnetic crystals has been studied intensively for thirty years, see e. g. [1–31]. This simple model is well justified at low temperatures, when the atoms execute only small oscillations about their rest positions. For cases, when the harmonicity condition is not satisfied, e. g. near the Curie point, it is necessary to take into account certain anharmonic terms, too.

Recently, the anharmonic effects in ferromagnetic crystals have been considered in several papers. Meissner [32] developed the method based on a functional-derivative Green's function approach combined with a cumulant technique. A variational calculation investigating the anharmonic Ising lattice was used by Horner [33]. Among these approaches, a special place is occupied by the self-consistent Green function theory pro-

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posed by Tyablikov and Konwent [34]. Their method was extensively developed and in some details is exploited to this day [35].

In this paper, we have carried out an analysis of the strongly anharmonic Heisenberg ferromagnetic lattice taking advantage of the renormalized spin-wave theory due to Szaniecki [36]. This theory is based on Matsubara's formalism [37], which yields the perturbation expansion of the partition function in terms of the Feynman graphs. We have restricted the calculations to the selected class of graphs, viz. to those deficient in the energy denominators (Hartree-Fock approximation). On availing ourselves of this approximation and having taken into consideration the anharmonicity effects of all orders, we have obtained the renormalized free energy and reduced magnetization of a ferromagnet.

2. *The Hamiltonian*

Throughout this paper we consider, for simplicity, the cubic ferromagnetic crystal consisting on N identical atoms. Moreover, we use the adiabatic approximation. The Hamiltonian under study is then

$$\mathscr{H} = \frac{1}{2M}\sum_{j} \vec{P}_j^2 + \tfrac{1}{2}\sum_{j\neq k} U(\vec{j}-\vec{k}) + L\sum_{j} S_j^z - \tfrac{1}{2}\sum_{j\neq k} J(\vec{j}-\vec{k})\vec{S}_j\cdot\vec{S}_k, \tag{2.1}$$

where $\vec{P}_j$ and $\vec{S}_j$ are the momentum and spin operators of an atom with the mass M at the point $\vec{j}$. The quantities $U(\vec{j}-\vec{k})$ and $J(\vec{j}-\vec{k})$ denote the interatomic potential energy and exchange integral, respectively, which depend on the difference between a pair of instantaneous atom sites $\vec{j}$ and $\vec{k}$. Moreover, $L = g\mu_B H$ with g being Lande's factor, μ_B-Bohr's magneton and H the external magnetic field applied along the z axis.

Let us make use of the Maleyev transformation [38]

$$S_j^+ = S_j^x + iS_j^y \to \sqrt{2S}\, a_j^*, \tag{2.2a}$$

$$S_j^- = S_j^x - iS_j^y \to \sqrt{2S}\left(1 - \frac{1}{2S} a_j^* a_j\right) a_j, \tag{2.2b}$$

$$S_j^z \to -S + a_j^* a_j, \tag{2.2c}$$

where S is the atomic spin number and a_j^*, a_j stand for the creation and annihilation boson operators for spin deviation. Utilizing the spatial periodicity condition we resort to the Fourier transformations

$$a_j^* = N^{-1/2}\sum_{\lambda} \alpha_\lambda^* e^{-i\vec{\lambda}\cdot\vec{j}}, \tag{2.3a}$$

$$a_j = N^{-1/2}\sum_{\lambda} \alpha_\lambda e^{i\vec{\lambda}\cdot\vec{j}}, \tag{2.3b}$$

$$J(\vec{j}-\vec{k}) = N^{-1}\sum_{\lambda} J_\lambda e^{i\vec{\lambda}\cdot(\vec{j}-\vec{k})}, \tag{2.3c}$$

$$U(\vec{j}-\vec{k}) = N^{-1}\sum_{\lambda} U_\lambda e^{i\vec{\lambda}\cdot(\vec{j}-\vec{k})}, \tag{2.3d}$$

where $\vec{\lambda}$ is the wave vector and $\alpha^*_\lambda, \alpha_\lambda, J_\lambda, U_\lambda$ are the Fourier components of $a^*_j, a_j, J(\vec{j}-\vec{k}), U(\vec{j}-\vec{k})$, respectively. The instantaneous atomic position vectors $\vec{j}$ and $\vec{k}$ in Eqs. (2.3a)-(2.3d) may be written as

$$\vec{j} = \vec{j}_0+\vec{\delta}_j, \quad \vec{k} = \vec{k}_0+\vec{\delta}_k, \tag{2.4}$$

with $\vec{\delta}_j$ and $\vec{\delta}_k$ being the displacements of the atoms from their equilibrium positions $\vec{j}_0$ and $\vec{k}_0$, respectively. We assume the atomic displacements and momenta in the conventional form

$$\vec{\delta}_j = iN^{-1/2}\sum_{\lambda s}\sqrt{\frac{\hbar}{2M\omega_{\lambda s}}}\,\vec{e}_{\lambda s}(\xi_{\lambda s}-\xi^*_{-\lambda s})e^{i\vec{\lambda}\cdot\vec{j}_0}, \tag{2.5a}$$

$$\vec{P}_j = N^{-1/2}\sum_{\lambda s}\sqrt{\frac{\hbar M\omega_{\lambda s}}{2}}\,\vec{e}_{\lambda s}(\xi_{\lambda s}+\xi^*_{-\lambda s})e^{i\vec{\lambda}\cdot\vec{j}_0}, \tag{2.5b}$$

where $\xi^*_{\lambda s}$ and $\xi_{\lambda s}$ are the creation and annihilation operators for a phonon of wave vector $\vec{\lambda}$, branch s and frequency $\omega_{\lambda s}$. The quantity $\vec{e}_{\lambda s}$ is the polarization vector associated with this phonon. As we assume the crystal lattice to be monoatomic, the acoustic branches appear only and the branch index is $s = 1, 2, 3$.

If we now expand the exponents on the right-hand sides of Eqs. (2.3a) — (2.3d) in power series of displacements $\vec{\delta}_j$ given by (2.5a), and make use of (2.2a) — (2.2c) and (2.5b), we obtain at last the Hamiltonian (2.1) in the form (for more details see [5])

$$\mathscr{H} = E_0+\mathscr{H}^{(1)}_0+\mathscr{H}^{(2)}_0+\mathscr{H}^{(1)}_{\mathrm{I}}+\mathscr{H}^{(2)}_{\mathrm{I}}+\mathscr{H}^{(3)}_{\mathrm{I}}+\mathscr{H}^{(4)}_{\mathrm{I}}, \tag{2.6}$$

$$E_0 = \tfrac{1}{2}NU_0-\tfrac{1}{2}NJ_0S^2+\tfrac{1}{2}\sum_\lambda \hbar\omega_{\lambda 3}, \tag{2.7}$$

$$\mathscr{H}^{(1)}_0 = \sum_\lambda \hbar\omega_{\lambda 3}\xi^*_{\lambda 3}\xi_{\lambda 3} \tag{2.8}$$

$$\mathscr{H}^{(2)}_0 = \sum_\lambda (L+\varepsilon_\lambda)\alpha^*_\lambda\alpha_\lambda, \tag{2.9}$$

$$\mathscr{H}^{(1)}_{\mathrm{I}} = \sum_{n=3}^{\infty}\sum_{\lambda_1 s_1}\cdots\sum_{\lambda_n s_n}\Phi^{\lambda_1 s_1,\ldots,\lambda_n s_n}\Delta(\vec{\lambda}_1+\ldots+\vec{\lambda}_n)\prod_{i=1}^{n}(\xi_{\lambda_i s_i}-\xi^*_{-\lambda_i s_i}), \tag{2.10}$$

$$\mathscr{H}^{(2)}_{\mathrm{I}} = -\tfrac{1}{4}N^{-1}\sum_{\lambda\varrho\sigma}\Gamma^{\lambda}_{\varrho,\sigma}\alpha^*_{\sigma+\lambda}\alpha^*_{\varrho-\lambda}\alpha_\varrho\alpha_\sigma, \tag{2.11}$$

$$\mathscr{H}^{(3)}_{\mathrm{I}} = \sum_{n=1}^{\infty}\sum_{\varrho\sigma}\sum_{\lambda_1 s_1}\cdots\sum_{\lambda_n s_n}\Phi^{\lambda_1 s_1,\ldots,\lambda_n s_n}_{\varrho,\sigma}\Delta(\vec{\sigma}-\vec{\varrho}+\vec{\lambda}_1+\ldots+\vec{\lambda}_n)$$
$$\times\alpha^*_\varrho\alpha_\sigma\prod_{i=1}^{n}(\xi_{\lambda_i s_i}-\xi^*_{-\lambda_i s_i}), \tag{2.12}$$

$$\mathscr{H}^{(4)}_{\mathrm{I}} = \sum_{n=1}^{\infty}\sum_{\varrho\sigma\mu\nu}\sum_{\lambda_1 s_1}\cdots\sum_{\lambda_n s_n}\Phi^{\lambda_1 s_1,\ldots,\lambda_n s_n}_{\varrho,\sigma,\mu,\nu}\Delta(\vec{\mu}+\vec{\nu}-\vec{\varrho}-\vec{\sigma}+\vec{\lambda}_1+\ldots+\vec{\lambda}_n)$$
$$\times\alpha^*_\varrho\alpha^*_\sigma\alpha_\mu\alpha_\nu\prod_{i=1}^{n}(\xi_{\lambda_i s_i}-\xi^*_{-\lambda_i s_i}), \tag{2.13}$$

where

$$\varepsilon_\lambda = S(J_0 - J_\lambda), \tag{2.14}$$

$$\Gamma^{\lambda}_{\varrho,\sigma} = J_\lambda + J_{\lambda+\sigma-\varrho} - J_{\lambda+\sigma} - J_{\lambda-\varrho}, \tag{2.15}$$

$$\Phi^{\lambda_1 s_1, \ldots, \lambda_n s_n} = -NM \sum_{k=1}^{k_{\max}} (1 - \tfrac{1}{2}\delta_{k,n/2}) \frac{(-1)^{n+k}}{k!(n-k)!} \omega^2_{\lambda_1 + \ldots + \lambda_k, 3} (\vec{\lambda}_1 + \ldots + \vec{\lambda}_k)^{-2}$$

$$\times \prod_{l=1}^{n} \left[\sqrt{\frac{\hbar}{2NM\omega_{\lambda_l s_l}}} (\vec{\lambda}_1 + \ldots + \vec{\lambda}_k) \cdot \vec{e}_{\lambda_l s_l} \right], \tag{2.16}$$

$$\Phi^{\lambda_1 s_1, \ldots, \lambda_n s_n}_{\varrho,\sigma} = \frac{(-1)^n}{n!} (L + S\Gamma^0_{\varrho,\sigma}) \prod_{l=1}^{n} \left[\sqrt{\frac{\hbar}{2NM\omega_{\lambda_l s_l}}} (\vec{\sigma} - \vec{\varrho}) \cdot \vec{e}_{\lambda_l s_l} \right]$$

$$+ S \sum_{k=1}^{k_{\max}} (1 - \tfrac{1}{2}\delta_{k,n/2}) \frac{(-1)^{n+k}}{k!(n-k)!} \Gamma^{\lambda_1 + \ldots + \lambda_k}_{\varrho,\sigma} \prod_{l=1}^{k} \left[\sqrt{\frac{\hbar}{2NM\omega_{\lambda_l s_l}}} (\vec{\lambda}_1 + \ldots + \vec{\lambda}_k) \cdot \vec{e}_{\lambda_l s_l} \right]$$

$$\times \prod_{m=k+1}^{n} \left[\sqrt{\frac{\hbar}{2NM\omega_{\lambda_m s_m}}} (\vec{\sigma} - \vec{\varrho} + \vec{\lambda}_1 + \ldots + \vec{\lambda}_k) \cdot \vec{e}_{\lambda_m s_m} \right], \tag{2.17}$$

$$\Phi^{\lambda_1 s_1, \ldots, \lambda_n s_n}_{\varrho,\sigma,\mu,\nu} = \tfrac{1}{4} N^{-1} \frac{(-1)^n}{n!} (\Gamma^{\varrho}_{\mu,-\nu} + \Gamma^{\sigma}_{\mu,-\nu}) \prod_{l=1}^{n} \left[\sqrt{\frac{\hbar}{2NM\omega_{\lambda_l s_l}}} (\vec{\mu} + \vec{\nu} - \vec{\varrho} - \vec{\sigma}) \cdot \vec{e}_{\lambda_l s_l} \right]$$

$$+ \sum_{k=1}^{k_{\max}} (1 - \tfrac{1}{2}\delta_{k,n/2}) \frac{(-1)^{n+k}}{k!(n-k)!} (\Gamma^{\varrho - \lambda_1 - \ldots - \lambda_k}_{\mu,-\nu} + \Gamma^{\sigma - \lambda_1 - \ldots - \lambda_k}_{\mu,-\nu})$$

$$\times \prod_{l=1}^{k} \left[\sqrt{\frac{\hbar}{2NM\omega_{\lambda_l s_l}}} (\vec{\lambda}_1 + \ldots + \vec{\lambda}_k) \cdot \vec{e}_{\lambda_l s_l} \right]$$

$$\times \prod_{m=k+1}^{n} \left[\sqrt{\frac{\hbar}{2NM\omega_{\lambda_m s_m}}} (\vec{\mu} + \vec{\nu} - \vec{\varrho} - \vec{\sigma} + \vec{\lambda}_1 + \ldots + \vec{\lambda}_k) \cdot \vec{e}_{\lambda_m s_m} \right], \tag{2.18}$$

$$\delta_{k,n/2} = \begin{cases} 1, & k = n/2, \\ 0, & k \neq n/2, \end{cases}$$

with $k_{\max}$ defined as

$$k_{\max} = \begin{cases} n/2 & \text{for even } n, \\ (n-1)/2 & \text{for odd } n, \end{cases}$$

and $\Delta(\vec{\lambda})$ being unity when $\vec{\lambda} = 0$ (we restrict $\vec{\lambda}$ to lie only inside the first Brillouin zone) and zero otherwise. Thus, we see that the first term in (2.6) is a constant, the second and third terms correspond to the free phonons and free magnons, respectively, the fourth term describes the highly anharmonic phonon-phonon interactions, the fifth term is responsible for magnon-magnon interactions (Dyson's interaction Hamiltonian), and the remaining two terms in (2.6) result from the interactions between the magnons and strongly anharmonic phonons. In the harmonic part $\mathscr{H}_0^{(1)}$ of the Hamiltonian (2.6), the branch index 3 appears only. It refers to the longitudinal acoustic mode, because the contributions from the transversal modes would contradict momentum conservation. According to this statement, the following equality

$$(J_\lambda S^2 - U_\lambda)\lambda^2 = M\omega_{\lambda 3}^2 \tag{2.19}$$

must hold. We have just utilized it while deriving (2.8).

Henceforth, we shall apply the approximation of nearest neighbours

$$J_\lambda \to J\gamma, \tag{2.20}$$

$$\gamma_\lambda = \sum_\eta e^{i\vec{\lambda}\cdot\vec{\eta}}, \tag{2.21}$$

where $\vec{\eta}$ denotes the vectors reaching to all nearest neighbours.

3. *Perturbation expansion and renormalization formulation*

Let us first recall, in general terms, the thermodynamic perturbation formalism [37] with due regard for our problem. As we deal with two different kinds of the boson quasi--particles (phonons and magnons) in the system discussed here, we define the set of orthonormal states as follows

$$|n) = |n^{(p)}, n^{(m)}) = \prod_\lambda \prod_s [(n_{\lambda s}^{(p)}!)^{-1/2}(\xi_{\lambda s}^*)^{n_{\lambda s}(p)}]$$
$$\times \prod_\lambda [(n_\lambda^{(m)}!)^{-1/2}(\alpha_\lambda^*)^{n_\lambda(m)}] \prod_s [|0_s^{(p)})] \, |0^{(m)}) \tag{3.1}$$

where $|0^{(m)})$ and $|0_s^{(p)})$ are the magnon and s-branch phonon vacuum states, respectively.

By Eqs. (2.6) — (2.18) and (3.1), the partition function can be written as

$$Z = \mathrm{Tr}\,(e^{-\beta\mathscr{H}}) = e^{-\beta E_0} \sum_n (n|e^{-\beta(\mathscr{H}_0+\mathscr{H}_1)}|n)$$
$$= e^{-\beta E_0} \prod_\lambda \prod_s [1-e^{-\beta\hbar\omega_{\lambda s}}]^{-1} \prod_\lambda [1-e^{-\beta(L+\varepsilon_\lambda)}]^{-1} \langle S(\beta)\rangle_0, \tag{3.2}$$

with

$$S(\beta) = T \exp\left[-\int_0^{\beta} d\tau \mathscr{H}_{\mathrm{I}}(\tau)\right], \tag{3.3}$$

$$\mathscr{H}_{\mathrm{I}}(\tau) = e^{\tau\mathscr{H}_0}\mathscr{H}_{\mathrm{I}}e^{-\tau\mathscr{H}_0}, \tag{3.4}$$

$$\mathscr{H}_0 = \mathscr{H}_0^{(0)} + \mathscr{H}_0^{(1)} + \mathscr{H}_0^{(2)}, \tag{3.5}$$

$$\mathscr{H}_{\mathrm{I}} = -\mathscr{H}_{\mathrm{I}}^{(0)} + \mathscr{H}_{\mathrm{I}}^{(1)} + \mathscr{H}_{\mathrm{I}}^{(2)} + \mathscr{H}_{\mathrm{I}}^{(3)} + \mathscr{H}_{\mathrm{I}}^{(4)}, \tag{3.6}$$

$$\mathscr{H}_0^{(0)} \equiv \mathscr{H}_{\mathrm{I}}^{(0)} = \sum_{\lambda; s=1,2} \hbar\omega_{\lambda s}\xi_{\lambda s}^{*}\xi_{\lambda s}, \tag{3.7}$$

where, for the purpose of our calculations, we have formally included the term $\mathscr{H}_0^{(0)}$ in the unperturbed Hamiltonian $\mathscr{H}_0$, and we added it with the opposite sign (for correctness) into the perturbed Hamiltonian $\mathscr{H}_{\mathrm{I}}$. T denotes Wick's ordering symbol. Using (3.2) and (3.3), we get

$$Z = \exp\left[-\beta E_0 + \sum_{\lambda s} \ln(1+\bar{n}_{\lambda s}^{(p)}) + \sum_{\lambda} \ln(1+\bar{n}_{\lambda}^{(m)}) + \sum_{n=1}^{\infty} D_n\right], \tag{3.8}$$

with

$$\bar{n}_{\lambda s}^{(p)} = [e^{\beta\hbar\omega_{\lambda s}} - 1]^{-1}, \tag{3.9}$$

$$\bar{n}_{\lambda}^{(m)} = [e^{\beta(L+\varepsilon_\lambda)} - 1]^{-1}, \tag{3.10}$$

and

$$D_n = \frac{(-1)^n}{n!}\int_0^{\beta} d\tau_1 \int_0^{\beta} d\tau_2 \ldots \int_0^{\beta} d\tau_n \langle T[\mathscr{H}_{\mathrm{I}}(\tau_1)\mathscr{H}_{\mathrm{I}}(\tau_2) \ldots \mathscr{H}_{\mathrm{I}}(\tau_n)]\rangle_{oc}, \tag{3.11}$$

where the subscript c denotes that only connected diagrams have to be taken, as the disconnected ones result exactly from the exponentiation. In order to apply Wick's theorem [39], we follow the treatment of Matsubara [37] and introduce the contractions

$$\underbrace{\xi_{\lambda_i s_i}^{*}(\tau_i)\xi_{\lambda_j s_j}(\tau_j)} = \delta_{\lambda_i,\lambda_j}\delta_{s_i,s_j}e^{\hbar\omega_{\lambda_i s_i}(\tau_i-\tau_j)}[\theta_{i,j}\bar{n}_{\lambda_i s_i}^{(p)} + \theta_{j,i}(\bar{n}_{\lambda_i s_i}^{(p)}+1)], \tag{3.12a}$$

$$\underbrace{\xi_{\lambda_i s_i}(\tau_i)\xi_{\lambda_j s_j}^{*}(\tau_j)} = \delta_{\lambda_i,\lambda_j}\delta_{s_i,s_j}e^{-\hbar\omega_{\lambda_i s_i}(\tau_i-\tau_j)}[\theta_{i,j}(\bar{n}_{\lambda_i s_i}^{(p)}+1) + \theta_{j,i}\bar{n}_{\lambda_i s_i}^{(p)}], \tag{3.12b}$$

$$\underbrace{\xi_{\lambda_i s_i}^{*}(\tau)\xi_{\lambda_j s_j}(\tau)} = \delta_{\lambda_i,\lambda_j}\delta_{s_i,s_j}\bar{n}_{\lambda_i s_i}^{(p)}, \tag{3.12c}$$

$$\underbrace{\xi_{\lambda_i s_i}(\tau)\xi_{\lambda_j s_j}^{*}(\tau)} = \delta_{\lambda_i,\lambda_j}\delta_{s_i,s_j}(\bar{n}_{\lambda_i s_i}^{(p)}+1), \tag{3.12d}$$

$$\underbrace{\alpha_\varrho^{*}(\tau_i)\alpha_\sigma(\tau_j)} = \delta_{\varrho,\sigma}e^{(L+\varepsilon_\varrho)(\tau_i-\tau_j)}[\theta_{i,j}\bar{n}_\varrho^{(m)} + \theta_{j,i}(\bar{n}_\varrho^{(m)}+1)], \tag{3.12e}$$

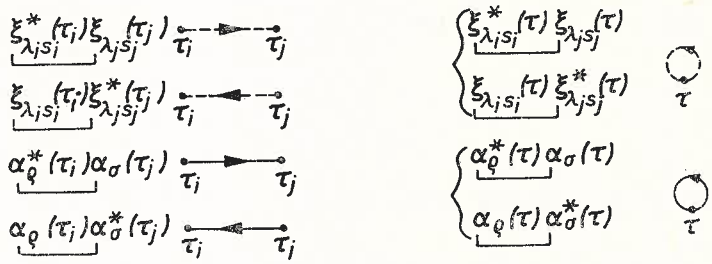
$$\underbrace{\alpha_\varrho(\tau_i)\alpha_\sigma(\tau_j)} = \delta_{\varrho,\sigma}e^{-(L+\varepsilon_\varrho)(\tau_i-\tau_j)}[\theta_{i,j}(\bar{n}_\varrho^{(m)}+1) + \theta_{j,i}\bar{n}_\varrho^{(m)}], \tag{2.12f}$$

$$\underbrace{\alpha_\varrho^{*}(\tau)\alpha_\sigma(\tau)} = \delta_{\varrho,\sigma}\bar{n}_\varrho^{(m)}, \tag{3.12g}$$

$$\underset{\sqcup}{\alpha_\varrho(\tau)\alpha^*_\sigma(\tau)} = \delta_{\varrho,\sigma}(\overline{n}_\varrho^{(m)}+1), \tag{3.12h}$$

$$\theta_{i,j} \equiv \theta(\tau_i-\tau_j) = \begin{cases} 1, & \tau > \tau_j \\ 0, & \tau_i < \tau_j. \end{cases}$$

With the aid of the following graphical interpretation



each term in D_n, Eq. (3.11), can be represented diagrammatically.

In this paper, we neglect the graphs which have energy denominators, i.e. we confine ourselves to the bubble diagrams (Hartree-Fock approximation), cf. [40, 41]. For this purpose, we have to omit in Hamiltonian, Eqs. (2.10), (2.12), (2.13), all the terms with n being odd, as only the ones with even n can yield the graphs deficient in energy denominators.

Unfortunately, the Hartree-Fock approximation is not sufficiently exact for temperatures from immediate vicinity of the Curie point, see [40]. Thus, our considerations should be extended to include some classes of the graphs which have energy denominators, see for example [42], but in this case the numerical computations would become very intricate. For this reason, here we refrain from examination of such classes of graphs.

We proceed now to carry out the summation of graphs and thereby to the renormalization of magnons and phonons. For the sake of clarity, we denote the diagram of n-order of the perturbational expansion by $D_n^{(i_1,j_1;\dots;i_n,j_n)}$, where the superscripts $i_l = 0, 1, 2, 3, 4$ denote that the terms $\mathscr{H}_I^{(i_l)}$ from the perturbation Hamiltonian $\mathscr{H}_I$, Eq. (3.6), are taken into account. The indices j_l refer to the terms with $i_l = 1, 3, 4$ and point out the anharmonicity orders of the components of $\mathscr{H}_I^{(i_l)}$.

To calculate the graphs we use (2.8) — (2.18), (2.20), (2.21), (3.1) — (3.12) and apply Wick's [39] and Thouless' [43] theorems. The first-order graphs are drawn in Fig. 1. The respective expressions are

$$D_1^{(0)} = \beta \sum_{\lambda;s=1,2} \hbar\omega_{\lambda s}\overline{n}_{\lambda s}^{(p)}, \tag{3.13}$$

$$D_1^{(2)} = \tfrac{1}{2}\beta NJ\gamma_0[N^{-1}\sum_\varrho (1-x_\varrho)\overline{n}_\varrho^{(m)}]^2, \tag{3.14}$$

and, by Eqs. (A.1) — (A.5) given in Appendix,

$$\sum_{n=4}^{\infty} D_1^{(1,n)} = -\tfrac{1}{2}\beta\sum_\lambda \hbar\omega_{\lambda 3}(\overline{n}_{\lambda 3}^{(p)}+\tfrac{1}{2})\left[e^{\frac{\hbar}{NM}\sum_{\mu s}\frac{(\vec{\lambda}\cdot\vec{e}_{\mu s})^2}{\omega_{\mu s}}(\overline{n}_{\mu s}^{(p)}+\frac{1}{2})}-1\right], \tag{3.15}$$

$$\sum_{n=2}^{\infty} D_1^{(3,n)} = \beta \frac{JS\hbar}{M} \sum_{\lambda} \frac{\gamma_\lambda \lambda^2}{\omega_{\lambda 3}} (\bar{n}_{\lambda 3}^{(p)} + \tfrac{1}{2}) e^{\frac{\hbar}{NM} \sum_{\mu s} \frac{(\vec{\lambda} \cdot \vec{e}_{\mu s})^2}{\omega_{\mu s}} (\bar{n}_{\mu s}(p) + \frac{1}{2})} N^{-1} \sum_{\varrho} (1-x_\varrho) \bar{n}_\varrho^{(m)}, \tag{3.16}$$

$$\sum_{n=2}^{\infty} D_1^{(4,n)} = -\beta \frac{JS\hbar}{M} \sum_{\lambda} \frac{\gamma_\lambda \lambda^2}{\omega_{\lambda 3}} (\bar{n}_{\lambda 3}^{(p)} + \tfrac{1}{2}) e^{\frac{\hbar}{NM} \sum_{\mu s} \frac{(\vec{\lambda} \cdot \vec{e}_{\mu s})^2}{\omega_{\mu s}} (\bar{n}_{\mu s}(p) + \frac{1}{2})}$$

$$\times \left[\frac{1}{2S} N^{-1} \sum_{\varrho} (1-x_\varrho) \bar{n}_\varrho^{(m)} \right]^2, \tag{3.17}$$

where, allowing for the symmetry of the three cubic lattices, we used the equality

$$\Gamma_{\varrho,\sigma}^{\lambda} = J\gamma_\lambda (1-x_\varrho)(1-x_\sigma), \tag{3.18}$$

valid on the average, with

$$x_\varrho = \gamma_\varrho / \gamma_0. \tag{3.19}$$

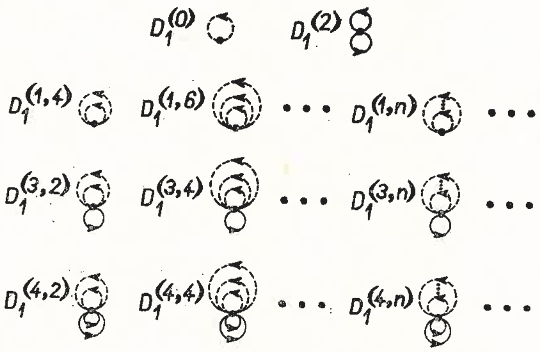


Fig. 1. Graphical representation of the first-order diagrams

The representative second-order graphs are plotted in Fig. 2. As an example, we have written explicitely some of the simplest diagrams shown in Fig. 2

$$D_2^{(0;0)} = \tfrac{1}{2} \beta^2 \sum_{\lambda; s=1,2} (\hbar\omega_{\lambda s})^2 \bar{n}_{\lambda s}^{(p)} (\bar{n}_{\lambda s}^{(p)} + 1), \tag{3.20}$$

$$D_2^{(2;2)} = \tfrac{1}{2} \beta^2 J^2 \gamma_0^2 \sum_{\varrho} (1-x_\varrho)^2 \bar{n}_\varrho^{(m)} (\bar{n}_\varrho^{(m)} + 1) [N^{-1} \sum_{\sigma} (1-x_\sigma) \bar{n}_\sigma^{(m)}]^2, \tag{3.21}$$

$$D_2^{(1,n;1,m)}$$

$$= \tfrac{1}{2} \beta^2 \sum_{\lambda s} (\hbar\omega_{\lambda s})^2 \bar{n}_{\lambda s}^{(p)} (\bar{n}_{\lambda s}^{(p)} + 1) \left\{ \tfrac{1}{2} \delta_{s,3} \left[\left(\frac{n-2}{2} \right)! \right]^{-1} \left[\frac{h}{NM} \sum_{\mu s} \frac{(\vec{\lambda} \cdot \vec{e}_{\mu s})^2}{\omega_{\mu s}} (\bar{n}_{\mu s}^{(p)} + \tfrac{1}{2}) \right]^{\frac{n-2}{2}} \right.$$

$$+\tfrac{1}{2}\frac{1}{NM\omega_{\lambda s}^2}\sum_{\mu}\hbar\omega_{\mu 3}(\bar{n}_{\mu 3}^{(p)}+\tfrac{1}{2})(\vec{\mu}\cdot\vec{e}_{\lambda s})^2\left[\left(\frac{n-4}{2}\right)!\right]^{-1}\left[\frac{\hbar}{NM}\sum_{\nu s}\frac{(\vec{\mu}\cdot\vec{e}_{\nu s})^2}{\omega_{\nu s}}(\bar{n}_{\nu s}^{(p)}+\tfrac{1}{2})\right]^{\frac{n-4}{2}}\Bigg\}$$

$$\times\Bigg\{\tfrac{1}{2}\delta_{s,3}\left[\left(\frac{m-2}{2}\right)!\right]^{-1}\left[\frac{\hbar}{NM}\sum_{\mu s}\frac{(\vec{\lambda}\cdot\vec{e}_{\mu s})^2}{\omega_{\mu s}}(\bar{n}_{\mu s}^{(p)}+\tfrac{1}{2})\right]^{\frac{m-2}{2}}+\tfrac{1}{2}\frac{1}{NM\omega_{\lambda s}^2}\sum_{\mu}\hbar\omega_{\mu 3}(\bar{n}_{\mu 3}^{(p)}+\tfrac{1}{2})$$

$$\times(\vec{\mu}\cdot\vec{e}_{\lambda s})^2\left[\left(\frac{m-4}{2}\right)!\right]^{-1}\left[\frac{\hbar}{NM}\sum_{\nu s}\frac{(\vec{\mu}\cdot\vec{e}_{\nu s})^2}{\omega_{\nu s}}(\bar{n}_{\nu s}^{(p)}+\tfrac{1}{2})\right]^{\frac{m-4}{2}}\Bigg\}. \tag{3.22}$$

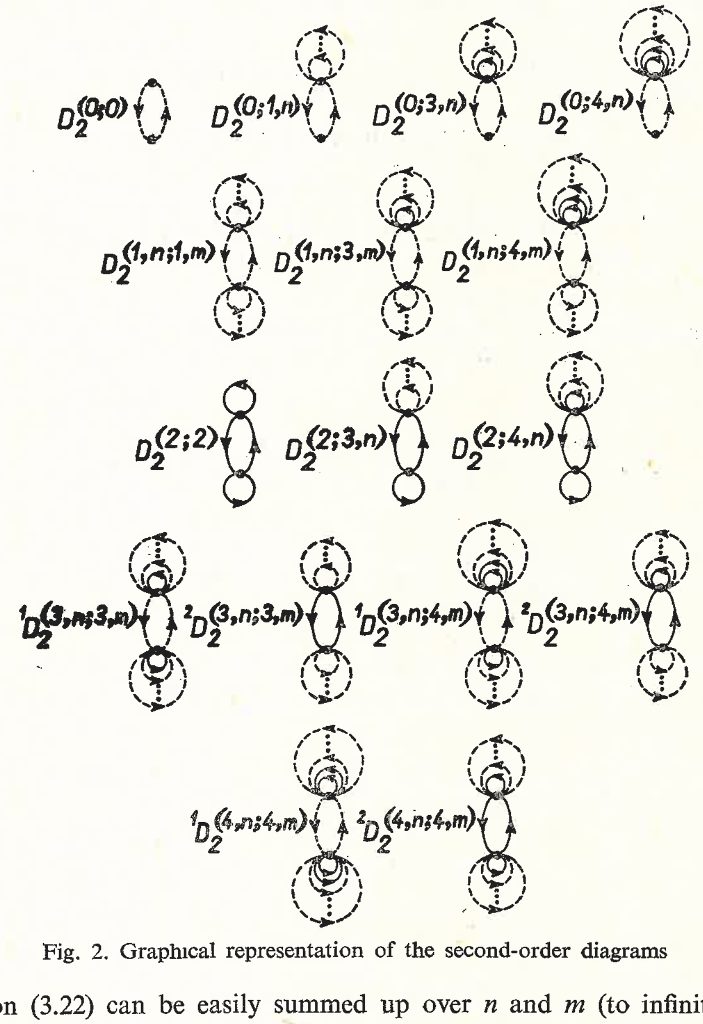


Fig. 2. Graphical representation of the second-order diagrams

The expression (3.22) can be easily summed up over n and m (to infinite anharmonic orders) yielding

$$\sum_{n=4}^{\infty}\sum_{m=4}^{\infty}D_2^{(1,n;1,m)}=\tfrac{1}{2}\beta^2\sum_{\lambda s}(\hbar\omega_{\lambda s})^2\bar{n}_{\lambda s}^{(p)}(\bar{n}_{\lambda s}^{(p)}+1)$$

$$\times\left\{\tfrac{1}{2}\,\delta_{s,3}\left[e^{\frac{\hbar}{NM}\sum\limits_{\mu s}\frac{(\vec{\lambda}\cdot\vec{e}_{\mu s})^2}{\omega_{\mu s}}(\bar{n}_{\mu s}{}^{(p)}+\frac{1}{2})}-1\right]+\tfrac{1}{2}\,\frac{1}{NM\omega_{\lambda s}^2}\sum_{\mu}\hbar\omega_{\mu 3}\right.$$

$$\left.\times(\bar{n}_{\mu 3}^{(p)}+\tfrac{1}{2})\,(\vec{\mu}\cdot\vec{e}_{\lambda s})^2 e^{\frac{\hbar}{NM}\sum\limits_{\nu s}\frac{(\vec{\mu}\cdot\vec{e}_{\nu s})^2}{\omega_{\nu s}}(\bar{n}_{\nu s}{}^{(p)}+\frac{1}{2})}\right\}^2. \tag{3.23}$$

In the similar way, we can continue our consideration concerning other diagrams, but the procedure becomes very tedious and we refrain from adducing it here.

Taking into account successive orders of perturbational expansion, we can carry out the summation of all bubble diagrams in the following manner

$$\sum_{n=1}^{\infty}\sum_{i_1 j_1}\cdots\sum_{i_n j_n} D_n^{(i_1,j_1;\,\ldots;\,i_n,j_n)} = -\sum_{\lambda s}\ln(1+\bar{n}_{\lambda s}^{(p)})+\sum_{\lambda s}\ln(1+\tilde{n}_{\lambda s}^{(p)})$$

$$-\sum_{\lambda}\ln(1+\bar{n}_{\lambda}^{(m)})+\sum_{\lambda}\ln(1+\tilde{n}_{\lambda}^{(m)})-\tfrac{1}{2}\,\beta\sum_{\lambda}\hbar\omega_{\lambda 3}a_{\lambda 3}^{(1)}$$

$$+\beta\sum_{\lambda s}\hbar\omega_{\lambda s}a_{\lambda s}^{(2)}\tilde{n}_{\lambda s}^{(p)}+\tfrac{1}{2}\,\beta\sum_{\lambda}\hbar\omega_{\lambda 3}c_{\lambda 3}^{(1)}Y^{(m)}\left(1-\frac{Y^{(m)}}{2S}\right)$$

$$-\beta\sum_{\lambda s}\hbar\omega_{\lambda s}c_{\lambda s}^{(2)}\tilde{n}_{\lambda s}^{(p)}Y^{(m)}\left(1-\frac{Y^{(m)}}{2S}\right)-\beta NJ\gamma_0\left[Y^{(p)}Y^{(m)}\left(1-\frac{Y^{(m)}}{S}\right)+\tfrac{1}{2}\,(Y^{(m)})^2\right], \tag{3.24}$$

with

$$a_{\lambda 3}^{(1)} = \tfrac{1}{2}\left\{\exp\left[\frac{\hbar}{NM}\sum_{\mu s}\frac{(\vec{\lambda}\cdot\vec{e}_{\mu s})^2}{\omega_{\mu s}}(\tilde{n}_{\mu s}^{(p)}+\tfrac{1}{2})\right]-1\right\}, \tag{3.25}$$

$$a_{\lambda s}^{(2)} = \frac{1}{NM\omega_{\lambda s}^2}\sum_{\mu}\hbar\omega_{\mu 3}(\tilde{n}_{\mu 3}^{(p)}+\tfrac{1}{2})\,(\vec{\mu}\cdot\vec{e}_{\lambda s})^2(a_{\mu 3}^{(1)}+\tfrac{1}{2}), \tag{3.26}$$

$$c_{\lambda 3}^{(1)} = \frac{JS\gamma_\lambda\lambda^2}{M\omega_{\lambda 3}^2}\exp\left[\frac{\hbar}{NM}\sum_{\mu s}\frac{(\vec{\lambda}\cdot\vec{e}_{\mu s})^2}{\omega_{\mu s}}(\tilde{n}_{\mu s}^{(p)}+\tfrac{1}{2})\right], \tag{3.27}$$

$$c_{\lambda s}^{(2)} = \frac{1}{NM\omega_{\lambda s}^2}\sum_{\mu}\hbar\omega_{\mu 3}(\tilde{n}_{\mu 3}^{(p)}+\tfrac{1}{2})\,(\vec{\mu}\cdot\vec{e}_{\lambda s})^2 c_{\mu 3}^{(1)}, \tag{3.28}$$

$$Y^{(m)} = N^{-1}\sum_{\lambda}(1-x_\lambda)\tilde{n}_{\lambda}^{(m)}, \tag{3.29}$$

$$\tilde{n}_{\lambda}^{(m)} = \frac{1}{e^{\beta\left[L+\varepsilon_\lambda\left(1-\frac{Y^{(p)}}{S}\right)\left(1-\frac{Y^{(m)}}{S}\right)\right]}-1}, \tag{3.30}$$

$$Y^{(p)} = \frac{1}{NJ\gamma_0}\sum_{\lambda}\hbar\omega_{\lambda 3}(\tilde{n}_{\lambda 3}^{(p)}+\tfrac{1}{2})c_{\lambda 3}^{(1)}, \tag{3.31}$$

$$\tilde{n}_{\lambda}^{(p)} = \frac{1}{e^{\beta\hbar\omega_{\lambda s}\left[a_{\lambda s}-c_{\lambda s}Y^{(m)}\left(1-\frac{Y^{(m)}}{2S}\right)\right]}-1}, \tag{3.32}$$

$$a_{\lambda s} = (a_{\lambda 3}^{(1)}+1)\delta_{s,3}+a_{\lambda s}^{(2)}, \tag{3.33}$$

$$c_{\lambda s} = c_{\lambda 3}^{(1)}\delta_{s,3}+c_{\lambda s}^{(2)}, \tag{3.34}$$

where the tilde above $n_{\lambda}^{(m)}$ and $n_{\lambda s}^{(p)}$ denotes that we are now dealing with the renormalized magnon and phonon average occupation numbers.

We see that Eqs. (3.25) — (3.32) together with the abbreviations (3.33) and (3.34) represent the closed system of self-consistent equations with respect to the renormalized average phonon and magnon occupation numbers which can be determined numerically by the iteration procedure. However, these computations turn out to be by no means easy and we postpone them to the subsequent paper.

4. *Free energy and reduced magnetization*

We are now able to derive the free energy of the phonon-magnon system. With the help of Eqs. (3.8) and (3.24), it can be easily verified that the free energy

$$F = -\beta^{-1}\ln Z \tag{4.1}$$

assumes the form

$$\begin{aligned} F = {} & E_0+\sum_{\lambda}\hbar\omega_{\lambda 3}\tilde{n}_{\lambda 3}^{(p)}+\sum_{\lambda}(L+\varepsilon_{\lambda})\tilde{n}_{\lambda}^{(m)} \\ & +\sum_{\lambda}\hbar\omega_{\lambda 3}(\tilde{n}_{\lambda 3}^{(p)}+\tfrac{1}{2})a_{\lambda 3}^{(1)}-NJ\gamma_0\left[Y^{(p)}Y^{(m)}\left(1-\frac{Y^{(m)}}{2S}\right)+\tfrac{1}{2}(Y^{(m)})^2\right] \\ & +\beta^{-1}\sum_{\lambda s}[\tilde{n}_{\lambda s}^{(p)}\ln\tilde{n}_{\lambda s}^{(p)}-(1+\tilde{n}_{\lambda s}^{(p)})\ln(1+\tilde{n}_{\lambda s}^{(p)})] \\ & +\beta^{-1}\sum_{\lambda}[\tilde{n}_{\lambda}^{(m)}\ln\tilde{n}_{\lambda}^{(m)}-(1+\tilde{n}_{\lambda}^{(m)})\ln(1+\tilde{n}_{\lambda}^{(m)})]. \end{aligned} \tag{4.2}$$

As far as the magnetization is concerned, on applying the well known formula for its average reduced value

$$\mu(T) = (SN)^{-1}\frac{\partial F}{\partial L}, \tag{4.3}$$

and on using (4.2), we get

$$\mu(T) = 1-(SN)^{-1}\sum_{\lambda}\tilde{n}_{\lambda}^{(m)}. \tag{4.4}$$

Without recourse to the numerical calculations, we can draw the general conclusion that because of $Y^{(p)}\geqslant 0$ (see Eq. (3.31)), the magnon-phonon interactions give rise to decrease of the reduced magnetization (see Eqs. (4.4) and (3.30)).

5. Conclusions

Throughout this paper we applied the Matsubara perturbation calculus to elucidate how in a highly anharmonic Heisenberg ferromagnetic crystal the spin--phonon interactions affect the reduced magnetization. The anharmonic components of the Hamiltonian, up to infinite orders, were taken into account in the Hartree-Fock approximation and the graphs due to them were summed up yielding a set of self-consistent equations. It should be emphasized that the procedure outlined and applied in this paper because of it self-consistency holds true for a wide range of temperature below the Curie point.

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APPENDIX

Let us consider the graphs of the first-order perturbational expansion taking into account the successive anharmonic components of $\mathscr{H}_I^{(1)}$, $\mathscr{H}_I^{(3)}$ and $\mathscr{H}_I^{(4)}$ (see (2.10), (2.12) and (2.13)).

With respect to $\mathscr{H}_I^{(1)}$, the straightforward calculations lead to the following results

$$D_1^{(1,4)} = -\tfrac{1}{2}\beta \sum_{\lambda} \hbar\omega_{\lambda 3}(\bar{n}_{\lambda 3}^{(p)}+\tfrac{1}{2}) \frac{1}{1!}\frac{\hbar}{NM}\sum_{\mu s}\frac{(\vec{\lambda}\cdot\vec{e}_{\mu s})^2}{\omega_{\mu s}}(\bar{n}_{\mu s}^{(p)}+\tfrac{1}{2})$$

$$+\tfrac{1}{2}\beta\sum_{\lambda\mu}\sum_{ss'}\hbar\omega_{\lambda 3}^2\lambda^{-2}\frac{(\vec{\lambda}\cdot\vec{e}_{\mu s})^2}{\omega_{\mu s}}(\bar{n}_{\mu s}^{(p)}+\tfrac{1}{2})\frac{1}{2!}\frac{\hbar}{NM}\frac{(\vec{\lambda}\cdot\vec{e}_{\lambda+\mu,s'})^2}{\omega_{\lambda+\mu,s'}}(\bar{n}_{\lambda+\mu,s'}^{(p)}+\tfrac{1}{2}), \tag{A.1}$$

$$D_1^{(1,6)} = -\tfrac{1}{2}\beta\sum_{\lambda}\hbar\omega_{\lambda 3}(\bar{n}_{\lambda 3}^{(p)}+\tfrac{1}{2})\frac{1}{2!}\left[\frac{\hbar}{NM}\sum_{\mu s}\frac{(\vec{\lambda}\cdot\vec{e}_{\mu s})^2}{\omega_{\mu s}}(\bar{n}_{\mu s}^{(p)}+\tfrac{1}{2})\right]^2$$

$$+\tfrac{1}{2}\beta\sum_{\lambda\mu}\sum_{ss'}\hbar\omega_{\lambda 3}^2\lambda^{-2}\frac{(\vec{\lambda}\cdot\vec{e}_{\mu s})^2}{\omega_{\mu s}}(\bar{n}_{\mu s}^{(p)}+\tfrac{1}{2})\frac{1}{2!}\frac{\hbar}{NM}\frac{(\vec{\lambda}\cdot\vec{e}_{\lambda+\mu,s'})^2}{\omega_{\lambda+\mu,s'}}(\bar{n}_{\lambda+\mu,s'}^{(p)}+\tfrac{1}{2})$$

$$\times\frac{1}{1!}\frac{\hbar}{NM}\sum_{\mu s}\frac{(\vec{\lambda}\cdot\vec{e}_{\mu s})^2}{\omega_{\mu s}}(\bar{n}_{\mu s}^{(p)}+\tfrac{1}{2})$$

$$-\tfrac{1}{2}\beta\sum_{\lambda\mu\nu}\sum_{ss's''}\hbar\omega_{\lambda 3}^2\lambda^{-2}\frac{(\vec{\lambda}\cdot\vec{e}_{\mu s})^2}{\omega_{\mu s}}(\bar{n}_{\mu s}^{(p)}+\tfrac{1}{2})\frac{\hbar}{NM}\frac{(\vec{\lambda}\cdot\vec{e}_{\nu s'})^2}{\omega_{\nu s'}}(\bar{n}_{\nu s'}^{(p)}+\tfrac{1}{2})$$

$$\times\frac{1}{3!}\frac{\hbar}{NM}\frac{(\vec{\lambda}\cdot\vec{e}_{\lambda+\mu+\nu,s''})^2}{\omega_{\lambda+\mu+\nu,s''}}(\bar{n}_{\lambda+\mu+\nu,s''}^{(p)}+\tfrac{1}{2}), \tag{A.2}$$

etc. It can be easily seen that on performing the summation of all anharmonic diagrams $D_1^{(1,n)}$ (to infinite anharmonic order), one gets

$$\sum_{n=4}^{\infty} D_1^{(1,n)} = -\tfrac{1}{2}\beta\sum_{\lambda}\hbar\omega_{\lambda 3}(\bar{n}_{\lambda 3}^{(p)}+\tfrac{1}{2})\left[e^{\frac{\hbar}{NM}\sum_{\mu s}\frac{(\vec{\lambda}\cdot\vec{e}_{\mu s})^2}{\omega_{\mu s}}(\bar{n}_{\mu s}{}^{(p)}+\frac{1}{2})}-1\right]$$

$$+\tfrac{1}{2}\beta\sum_{\lambda\mu}\sum_{ss'}\hbar\omega_{\lambda 3}^2\lambda^{-2}\frac{(\vec{\lambda}\cdot\vec{e}_{\mu s})^2}{\omega_{\mu s}}(\bar{n}_{\mu s}^{(p)}+\tfrac{1}{2})\frac{1}{2!}\frac{\hbar}{NM}\frac{(\vec{\lambda}\cdot\vec{e}_{\lambda+\mu,s'})^2}{\omega_{\lambda+\mu,s'}}(\bar{n}_{\lambda+\mu,s'}^{(p)}+\tfrac{1}{2})$$

$$\times e^{\frac{\hbar}{NM}\sum_{\mu s}\frac{(\vec{\lambda}\cdot\vec{e}_{\mu s})^2}{\omega_{\mu s}}(\bar{n}_{\mu s}{}^{(p)}+\frac{1}{2})}$$

$$-\tfrac{1}{2}\beta\sum_{\lambda\mu\nu}\sum_{ss's''}\hbar\omega_{\lambda 3}^2\lambda^{-2}\frac{(\vec{\lambda}\cdot\vec{e}_{\mu s})^2}{\omega_{\mu s}}(\bar{n}_{\mu s}^{(p)}+\tfrac{1}{2})\frac{\hbar}{NM}\frac{(\vec{\lambda}\cdot\vec{e}_{\nu s'})^2}{\omega_{\nu s'}}(\bar{n}_{\nu s'}^{(p)}+\tfrac{1}{2})$$

$$\times\frac{1}{3!}\frac{\hbar}{NM}\frac{(\vec{\lambda}\cdot\vec{e}_{\lambda+\mu+\nu,s''})^2}{\omega_{\lambda+\mu+\nu,s''}}(\bar{n}_{\lambda+\mu+\nu,s''}^{(p)}+\tfrac{1}{2})e^{\frac{\hbar}{NM}\sum_{\mu s}\frac{(\vec{\lambda}\cdot\vec{e}_{\mu s})^2}{\omega_{\mu s}}(\bar{n}_{\mu s}{}^{(p)}+\frac{1}{2})}+\ldots, \qquad \text{(A.3)}$$

where $n = 4, 6, 8, \ldots$.

Since for all ferromagnets $\dfrac{\hbar\lambda^2}{M\omega_{\lambda s}} \ll 1$, it can be shown that only the first term on the right hand side in (A.3) is essential in our procedure. Thus, in further considerations we shall neglect all terms comprising the expressions like $\bar{n}_{\lambda+\mu,s}^{(p)}$, $\bar{n}_{\lambda+\mu+\nu,s}^{(p)}$, $\bar{n}_{\lambda+\mu+\nu+\kappa,s}^{(p)}$, etc.

In the same way, we obtain for $\mathscr{H}_I^{(3)}$

$$\sum_{n=2}^{\infty} D_1^{(3,n)} = \tfrac{1}{2}\beta\frac{S\hbar}{NM}\sum_{\lambda}\frac{\lambda^2}{\omega_{\lambda 3}}(\bar{n}_{\lambda 3}^{(p)}+\tfrac{1}{2})e^{\frac{\hbar}{NM}\sum_{\mu s}\frac{(\vec{\lambda}\cdot\vec{e}_{\mu s})^2}{\omega_{\mu s}}(\bar{n}_{\mu s}{}^{(p)}+\frac{1}{2})}N^{-1}\sum_{\varrho}\Gamma_{\varrho,\varrho}^{\lambda}\bar{n}_{\varrho}^{(m)}, \qquad \text{(A.4)}$$

and for $\mathscr{H}_I^{(4)}$

$$\sum_{n=2}^{\infty} D_1^{(4,n)} = \tfrac{1}{2}\beta\frac{S\hbar}{NM}\sum_{\lambda}\frac{\lambda^2}{\omega_{\lambda 3}}(\bar{n}_{\lambda 3}^{(p)}+\tfrac{1}{2})e^{\frac{\hbar}{NM}\sum_{\mu s}\frac{(\vec{\lambda}\cdot\vec{e}_{\mu s})^2}{\omega_{\mu s}}(\bar{n}_{\mu s}{}^{(p)}+\frac{1}{2})}$$

$$\times\frac{1}{2S}N^{-2}\sum_{\varrho\sigma}(\Gamma_{\varrho,-\sigma}^{\varrho-\lambda}+\Gamma_{\varrho,-\sigma}^{\sigma-\lambda})\bar{n}_{\varrho}^{(m)}\bar{n}_{\sigma}^{(m)}, \qquad \text{(A.5)}$$

where $n = 2, 4, 6, \ldots$.

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