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ON NUMERICAL PROCESSING THE INTERFEROGRAMS OF AXIALLY SYMMETRICAL PLASMA DISCHARGES

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This paper presents a new procedure developed for the numerical processing the interferograms obtained by a side-on observation of axially symmetrical plasma discharges. Experimental results presented as a set of numerical data, are approximated by the least-squares method employing even Legendre polynomials only. The appropriate number of terms in the polynomial expansion is chosen by means of the statistical Fischer test adapted to the even polynomials employed. Since the resultant approximation function is also an even polynomial, the inverse Abel transformation is performed by means of a formula especially derived for the case considered. The application of the method developed is exemplified by calculations of the electron concentration distribution on the basis of a typical plasma interferogram. The results of calculations are compared with those obtained by means of other methods.

1. *Introduction*

In order to determine the electron concentration distribution for a quasi-symmetrical fully-ionized plasma on the basis of interferometric measurements performed perpendicularly to the axis of symmetry, it is necessary to solve the Abel integral equation

$$S(y) = \frac{\varrho\lambda}{\pi}\int\limits_y^R \frac{n_e(r)rdr}{\sqrt{r^2-y^2}}, \tag{1}$$

or to compute the inverse transformation

$$n_e(r) = -\frac{2}{\varrho\lambda}\int\limits_r^R \frac{dS(y)}{dy}\frac{dy}{\sqrt{y^2-r^2}} \tag{2}$$

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where $S(y)$ is the relative shift of the interference fringes, $n_e(r)$ is the electron concentration distribution within the region $0 \leqslant r \leqslant R$, λ is the wavelength of transmitted light, $\varrho = e^2/m_e c^2$ is the classical electron radius, and R is the over-all radius of a plasma column.

None of these equations can be analytically solved in general. An analytic solution can however be found if the unknown function $n_e(r)$ in Eq. (1), or the experimental data function $S(y)$ in Eq. (2), is a polynomial.

The first group of numerical methods used for solving the Abel equation is based on the division of the plasma cross-section into circular zones of equal width, and on interpolation of the data function $S(y)$ or an unknown function $n_e(r)$ in each zone appropriately. The step interpolation of the function $n_e(r)$ was used, e.g., by Maecker [1]; the linear interpolation of the same function has been employed by van Voorhis, and a parabolic interpolation of the data function $S(y)$ — by Weyl. The last two methods have been described by Weyl in the monography edited by Ladenburg [2]. The calculations in Weyl's paper contain however an error and in the final formula the factor two is omitted. The Weyl method has also been used by Nestor and Olsen [3], who avoided Weyl's error and tabulated the proper numerical coefficients. A parabolic interpolation of an unknown function was used by Frie [4]. Gribkov et al. [5] used another parabolic interpolation, but it gives correct results only in special cases. To increase the accuracy of interpolation, use was also made of higher-degree polynomials, e.g., Bockasten [6] has increased the degree of the interpolation polynomial up to three. Unfortunately, all those interpolation methods are sensitive to small random errors in the data. Since the equations used involve recursive relationships, such errors can propagate in succeeding calculations.

To reduce the influence of small random errors in the experimental readings, use can be made of a smoothing procedure based on the least-squares method, and an approximation polynomial can then be inserted into Eq. (2). In some earlier papers [7, 8] the mixed interpolation-approximation technique has been used; a polynomial of low degree has been fitted to a large number of experimental points and the smoothed data have been integrated in the zones, as described above.

In the recent papers [9, 10], to approximate the experimental readings, high-degree polynomials are used. The orthogonal polynomials are then preferred because their Gram matrix is diagonal, and coefficients in the polynomial expansion can be easily found. Accuracy of calculations can be also increased. Such a method was, e.g., employed by van Trigt [9], who used an approximation function in the form of an infinite series of the Legendre even polynomials. High-degree power polynomials have been employed by Mermet and Robin [10]. They have however used all (not only even) powers. For such a function the Abel inverse integral (2) can have a singularity at $r = 0$. The same objection can apply to the paper presented by Zaharenkov et al. [11] who have used polynomials orthogonal over a discrete set of data points. The main purpose of this paper was to eliminate the disadvantages discussed above. The paper presents a least-squares smoothing procedure which uses a set of the Legendre even polynomials only, and enables the number of terms in the approximation expansion to be fitted on the basis of the Fisher statistical test [12]. This test is especially adapted to the even polynomials employed. Moreover, the Abel inverse transformation formula for even polynomials is also derived.

2. *Smoothing procedure*

Experimental results are presented as a set of numerical data (y_i, S_i), where $i = 1, 2, \ldots n$, and n denotes the number of measurement points $(y_{i+1} > y_i)$. The actual axis of the discharge can however be different from the mechanical axis of the experimental chamber, and some perturbations in rotational symmetry of the discharge can also occur. Therefore the experimental distribution S_i can differ from the symmetrical one and the problem of locating the axis of the interferogram is of particular importance.

In this paper the axis of symmetry is chosen as the abscissa of the centre of gravity (the weighted mean value) of the experimental distribution S_i, and it is given by the formula

$$\bar{y} = \frac{\sum_{i=1}^{n} y_i S_i}{\sum_{i=1}^{n} S_i}. \tag{3}$$

All the coordinates y_i are then transformed into the interval $\xi \in (-1, 1)$ by the formula

$$\xi_i = \frac{y_i - \bar{y}}{R_p} \tag{4}$$

where

$$R_p = \sup[(y_n - \bar{y}), (\bar{y} - y_1)]. \tag{5}$$

2.1. Approximation

After the transformation described above the data are approximated by a function $W(\xi)$ expanded into the Legendre polynomials $P_k(\xi)$ [13]. On account of axial symmetry, the even polynomials only are taken into account, and the smoothing function is written in the form

$$W_m(\xi) = \sum_{h=0}^{m} a_{2h} P_{2h}(\xi). \tag{6}$$

For fitting the above function to the experimental data, the least-squares method is used. Then the coefficients in expansion (6) are given by

$$a_{2h} = \frac{1}{c_{2h}} \int_{-1}^{+1} S(y(\xi)) P_{2h}(\xi) d\xi \tag{7}$$

where

$$c_{2h} = \frac{1}{4h+1}.$$

In order to compute these integrals, the Lagrange parabolic interpolation of the experimental data is performed. For convenience of computations, the Legendre polynomials are presented in the form

$$P_k(\xi) = \sum_{l=0}^{k} p_l^{(k)} \xi^l \tag{8}$$

where the coefficients are

$$p_l^{(-1)} = p_{-1}^{(k)} = 0, \quad p_0^{(0)} = 1, \quad p_l^{(k+1)} = \frac{2k+1}{k+1} p_{l-1}^{(k)} - \frac{k}{k+1} p_l^{(k-1)},$$

$$l = 0, 1, \ldots, k+1. \tag{9}$$

Appropriately, the even Legendre polynomials are

$$P_{2h}(\xi) = \sum_{j=0}^{h} p_{2j}^{(2h)} \xi^{2j}. \tag{10}$$

2.2. Determination of the degree of the approximation polynomial

It should be noted that with an increase in the number of terms used in expansion (6), the accuracy of the approximation increases and so does the variance of the approximation function

$$D^2[W_m(\xi)] = \sigma^2 \sum_{h=0}^{m} \frac{P_{2h}^2(\xi)}{c_{2h}} \tag{11}$$

where σ is the standard deviation of any of the coefficients a_{2h}. This formula is easy obtained from Eq. (6) by applying the law of random errors propagation, and using the condition of orthogonality [12]. Therefore, it is necessary to determine how many terms can be reasonably used in expansion (6). To verify the significance of successive coefficients in Eq. (6), use can be made of Fisher's statistical test adapted to the polynomials employed.

Fisher's test for an approximating polynomial with all (not only even) terms can be written in the form

$$F = \frac{a_r^2 c_r}{\sum_{k=r+1}^{n-1} a_k^2 c_k} (n-1-r) \leqslant F_{1,n-1-r,\alpha} \tag{12}$$

which is a direct consequence of formula (7) given above and the relations (256), (250), and (253), given in the handbook by Hudson [12]. The index of the last term in the approximation polynomial can be equal to or less than $n-1$. The value $F_{1,n-1-r,\alpha}$ tabulated in Hudson's handbook and all modern statistical tables, e.g. [14], is chosen in such a way that the probability of exceeding it by F is less than α (e.g., $\alpha = 5\%$). If relation (12) is fulfilled the coefficient a_r can be neglected, and the calculation has to be repeated with r replaced by $r-1$, until the inequality is not satisfied. Then the reduced value of r is inserted in place of m. This procedure usually starts with $r = n-2$, since the coefficient a_r for $r = n-1$ is assumed a priori to be a negligible one.

For approximation by even polynomials only the index of the last term in the expansion can be $\left[\frac{n-1}{2}\right] \underset{\text{df}}{=} \text{entier}\left(\frac{n-1}{2}\right)$ instead of $(n-1)$ and the number of terms in the denominator of the expression in Eq. (12) is $\left[\frac{n-1}{2}\right]-p$ instead of $(n-1-r)$. Then Eq. (12) takes the form

$$F = \frac{a_{2p}^2 c_{2p}}{\sum\limits_{j=p+1}^{\left[\frac{n-1}{2}\right]} a_{2j}^2 c_{2j}} \left(\left[\frac{n-1}{2}\right]-p\right) \leqslant F_{1,\left[\frac{n-1}{2}\right]-p,\alpha} \tag{13}$$

where

$$p = \left[\frac{n-3}{2}\right], \left[\frac{n-5}{2}\right], \dots$$

3. Inverse Abel transformation

For the convenience of computing the integral in Eq. (2), the approximation function determined above is presented in the form

$$W_m(\xi) = \sum_{j=0}^{m} b_{2j}\xi^{2j} \tag{14}$$

with the coefficients

$$b_{2j} = \sum_{h=j}^{m} a_{2h} p_{2j}^{(2h)}. \tag{15}$$

The positive zeroes of this resultant polynomial can easily be found, and their minimum value is considered as the radius R_ξ of the plasma column (in the normalized scale). In rare cases $W(\xi)$ has no real roots in the interval $(-1, 1)$. Such a situation can be handled by choosing for R_ξ the point in the interval (0,1) which corresponds to the local minimum of $W_m(\xi)$.

Appropriately, the derivative $\frac{dW_m(\xi)}{d\xi}$ is expressed by

$$\frac{dW_m(\xi)}{d\xi} = \sum_{j=1}^{m} 2jb_{2j}\xi^{2j-1}. \tag{16}$$

After substitution of the above expression into Eq. (2), integration can be performed according to the formula

$$\int_r^R \frac{y^{2j-1}dy}{\sqrt{y^2-r^2}} = \sum_{h=0}^{j-1} \binom{j-1}{h} \frac{r^{2(j-1-h)}(R^2-r^2)^{h+\frac{1}{2}}}{2h+1} \stackrel{\mathrm{df}}{=} I_{2j}(R, r) \tag{17}$$

which is a special case ($j = s+1, c = 1, a = -r^2$) of the general relation

$$\int \frac{y^{2s+1}}{u} dy = \frac{u}{c^{s+1}} \sum_{h=0}^{s} \binom{s}{h} \frac{(-a)^{s-h}u^{2h}}{2h+1}; \quad u = \sqrt{a+cy^2} \tag{18}$$

which can be proved by means of mathematical induction. In the normalized scale given by Eq. (4), the radii R and r have to be replaced by R_ξ and r_ξ, respectively.

Consequently, the radial distribution of electron concentration can be expressed by the formula

$$n_e(r_\xi) = -\frac{2}{\varrho\lambda} \sum_{j=1}^{m} 2jb_{2j}I_{2j}(R_\xi, r_\xi) \tag{19}$$

where on account of the initial normalization $0 \leqslant r_\xi \leqslant R_\xi \leqslant 1$. To determine the real linear scale for the distribution computed one should also calculate the values

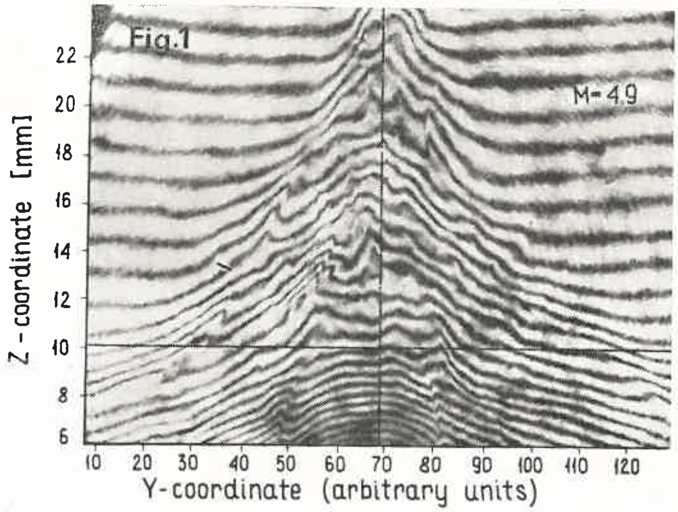
$$r = r_\xi R_p/M \quad \text{and} \quad R = R_\xi R_p/M \tag{20}$$

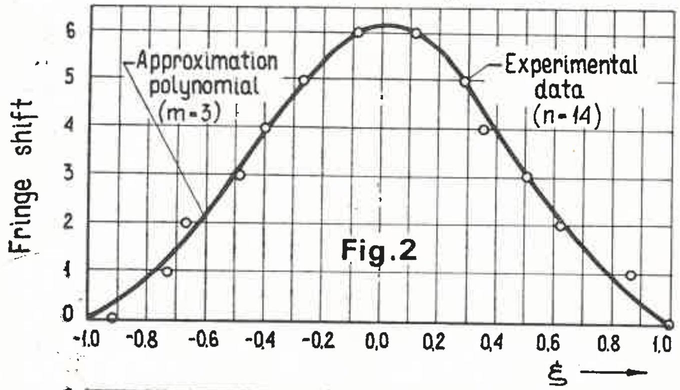
where M is the magnification coefficient for the interferogram processed, and R corresponds to the actual radius of the investigated plasma column. Finally, the radial electron concentration distribution is given by

$$n_e(r) = n_e(r_\xi)M/R_p. \tag{21}$$

4. *Applications*

The procedure described above has been used for numerical processing of the plasma interferograms obtained with the F-20 Plasma Focus Machine [15, 16]. An example of such an interferogram, taken with a Mach–Zehnder laser interferometer, is shown in Fig. 1. Fig. 2 shows the experimental readings of the interference fringe shifts at a distance 10 mm from the electrode edge, and the approximation curve corresponding to the smoothing polynomial $W(\xi)$ fitted by the method described above. The radial distribution of electron concentration computed by the method presented in this paper, and the results of calculations performed by other methods, are given in Fig. 3. All the figures shown together have been drawn according to the same scale. When the interpolation methods were employed, the computations for the left- and right-side of the interferogram have been performed separately. For such methods the results at $r = 0$ can of course be different for the left- and right-side of the interferogram, as shown in Fig. 3.





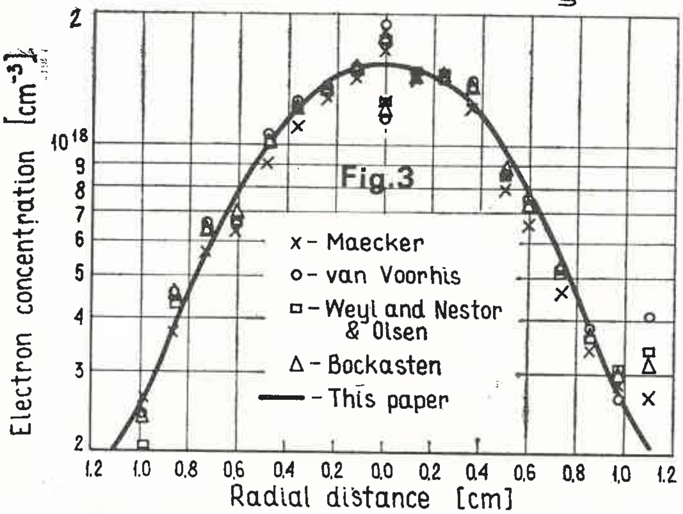


Fig. 1. Interferogram of the dense plasma produced in the F-20 Plasma-Focus Machine

Fig. 2. Relative fringe-shift and the approximation polynomial found for the plane $z = 10$ mm (from the electrode edge)

Fig. 3. Comparison of the results of calculations performed by various methods using the interferogram shown in Fig. 1

5. *Conclusions*

The procedure presented above is also suitable for axially symmetrical plasma discharges for as well as those, in which some deviations from axial symmetry are observed. For such quasi-symmetrical discharges the procedure discussed gives an averaged distribution symmetrized and fitted by the least-squares method. Moreover, the resultant distribution has no singularity at $r = 0$.

The procedure developed especially for interferometric studies, can also be used for calculations of plasma parameters on the basis of optical spectra obtained by a side-on observation of plasma discharges.

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