Vol. A44 (1973) ACTA PHYSICA POLONICA No 4

REFLECTION OF SCALAR WAVE FROM A PLANE WITH RANDOM IMPEDANCE

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(Received January 12, 1973)

The problem of reflection of scalar harmonic wave from a plane whose impedance is described by a random function of position is considered. It is assumed that the fluctuations of random impedance are sufficiently small and the idea of a smoothing method developed by J. B. Keller and others is applied. The reflection coefficient of a mean (or coherent) wave field over the plane is determined for a statistically homogeneous random impedance of the boundary.

1. Introduction

Recently, an increasing amount of attention has been devoted to different problems associated with propagation of random waves. In particular, the problem of wave propagation in a random medium has been the subject of numerous investigations. When a wave propagates in a random medium, energy scatters out of the mean (or coherent) wave. As a result the mean wave is attenuated and its propagation speed is altered. This is the reason why the mean solution of a stochastic differential equation describing the wave propagation is of practical interest becouse it leads to the determination of "effective" propagation constants.

In the last years Keller [1] and Karal and Keller [2] gave a rather general method of studying mean waves in a random medium. On the base of the perturbation approach they have derived the equation for a mean solution of a broad class of stochastic differential equations. The method based on the equations for a mean solution (smoothing method) has been successfully applied to the study of electromagnetic, elastic and magnetohydrodynamic waves in infinite random media. It is natural to try to extend this fruitful idea on some stochastic boundary value problems. Such an extension has lately been suggested by Keller [3]. In this paper the smoothing method formulated for boundary problem is used for the study of the reflection of a scalar wave from a statistically inhomogeneous plane.

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2. General formulation

Let us consider a linear stochastic differential equation in infinite space

$$L(\gamma)u = g \tag{1}$$

where $L = L(\gamma)$ is a random operator; γ is an element of the space of elementary events Γ on which probability is determined. The forcing function g is assumed here to be deterministic. On the assumption that the random variations of the coefficients (of the medium properties) are sufficiently small one can introduce a small parameter ε and expand the operator $L = L(\gamma, \varepsilon)$ in powers of ε

$$[L_0+\varepsilon L_1(\gamma)+\varepsilon^2 L_2(\gamma)+O(\varepsilon^3)]u(\gamma, \varepsilon) = g. \tag{2}$$

For $\varepsilon = 0$, the operator $L(\gamma, \varepsilon)$ reduces to a deterministic (mean) operator L_0. Keller [1] and Karal and Keller [2] derived the following equation for a mean solution of the equation (1):

$$L_0\langle u\rangle+\varepsilon\langle L_1\rangle \langle u\rangle+\varepsilon^2\{\langle L_1\rangle L_0^{-1}\langle L_1\rangle-\langle L_1 L_0^{-1}L_1\rangle+\langle L_2\rangle\} \langle u\rangle = g+O(\varepsilon^3), \tag{3}$$

where the angular bracket $\langle\ \rangle$ denotes average value. If $\langle L_1\rangle = 0$, which is a common case, we have

$$(L_0+\varepsilon^2\{\langle L_2\rangle-\langle L_1 L_0^{-1}L_1\rangle\}) \langle u\rangle = g+O(\varepsilon^3). \tag{4}$$

Let us assume that L is the sum of a non-random invertible operator M and a relatively small random operator $V(\gamma)$

$$L(\gamma) = M+V(\gamma). \tag{5}$$

In this case we have: $M = L_0$, $V(\gamma) = L_1(\gamma)$, $L_2 = L_3 = \dots = 0$. We shall assume that $\langle V\rangle = 0$. As a particular case of the equation (4) we obtain

$$[M-\varepsilon^2\langle VM^{-1}V\rangle]\langle u\rangle = g+O(\varepsilon^3). \tag{6}$$

Let us consider a domain $\mathscr{D}$ with the boundary S. Let D be a matrix linear, deterministic, differential operator. Let $B = B(\gamma)$ be a matrix linear, random differential operator which takes its values on the boundary S. Let us formulate the following boundary problem

$$\begin{aligned} Dv(x) &= f_1, \quad \text{in } \mathscr{D} \\ Bv(x) &= f_2, \quad \text{on } S; \quad x \in \mathscr{D} \cup S \end{aligned} \tag{7}$$

where f_1 and f_2 are assumed to be deterministic. Further, it shall be denoted: $v_1 = v(x)$ if $x \in \mathscr{D}$; $v_2 = v(x)$ if $x \in S$. Let us introduce two following random operators:

B_1 — is an operator that acts on functions determined in $\mathscr{D}$ and yields functions determined on S;

B_2 — is an operator that acts on functions determined on S and yields functions determined on S.

We shall assume that

$$B_1(\gamma) = B_1^0+V_1(\gamma), \quad B_2(\gamma) = B_2^0+V_2(\gamma) \tag{8}$$

where the random terms $V_1(\gamma)$ and $V_2(\gamma)$ are sufficiently small and their means are equal to zero. In order to reduce our boundary problem to the form of (1) and (6) the following denotations are introduced

$$L = \begin{Vmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{Vmatrix} = \begin{Vmatrix} D & O \\ B_1(\gamma) & B_2(\gamma) \end{Vmatrix}; \quad u = \begin{Vmatrix} v_1 \\ v_2 \end{Vmatrix}, \quad g = \begin{Vmatrix} f_1 \\ f_2 \end{Vmatrix} \tag{9}$$

$$M = \begin{Vmatrix} D & O \\ B_1^0 & B_2^0 \end{Vmatrix}, \quad V = \begin{Vmatrix} O & O \\ V_1(\gamma) & V_2(\gamma) \end{Vmatrix} \tag{10}$$

Operator L is a differential operator, so the operator M occuring in (6) can be represented as integral operator whose kernal is a Green matrix $G(x, x')$. Performing all the transformations indicated in equation (6) one obtaines the following deterministic boundary value problem for a mean solution:

$$\begin{gathered} D\langle v_1\rangle = f_1 \\ B_1^0\langle v_1\rangle + B_2^0\langle v_2\rangle - \int_{\mathscr{D}} (G_{12}R_{11} + G_{22}R_{21})\langle v_1\rangle d\mathscr{D} - \\ - \int_S (G_{12}R_{12} + G_{22}R_{22})\langle v_2\rangle dS = f_2 \end{gathered} \tag{11}$$

where $G_{ij}(x, x')$ are components of the Green matrix and $R_{ij}(x, x')$ are components of the correlation matrix, that is

$$R_{ij}(x, x') = \langle V_i(x, \gamma)V_j(x', \gamma)\rangle. \tag{12}$$

Assuming that $V_1(\gamma) = 0$ the boundary problem (11) reduces to the following one

$$\begin{gathered} D\langle v_1\rangle = f_1 \\ B_1^0\langle v_1\rangle + B_2^0\langle v_2\rangle - \int_S G_{22}R_{22}\langle v_2\rangle dS = f_2. \end{gathered} \tag{13}$$

We shall show how the idea briefly reported above works in application to a certain physical situation.

3. Reflection coefficient of a mean field

Let us consider the problem of reflection of a plane harmonic wave from a plane boundary of half-space [4], [5]. The system of coordinates is chosen in such a way that the field is parallel to the x–y plane. The plane $y = 0$ is considered as a reflection plane. The angle of incidence is denoted by α. We shall consider a situation, when the impedance of the boundary is described by a homogeneous random function $Z(x, \gamma)$, whose correlation function is given. It is additionally assumed that the fluctuations of the impedance are sufficiently small. So, we may write

$$Z(x; \gamma) = Z_0 + V(x; \gamma), \quad \langle V(x; \gamma)\rangle = 0.$$

The mathematical base of considerations is the following stochastic boundary value problem $\left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$

$$\begin{gathered}(\Delta+k^2)v = 0, \quad y \geqslant 0 \\ \left[\frac{\partial}{\partial y} + Z(x, y)\right] v = 0, \quad y = 0.\end{gathered} \tag{14}$$

To simplify the computations we shall suppose here that $Z_0 = 0$. With reference to previous section here

$$D = \Delta+k^2, \quad B_1^0 = \frac{\partial}{\partial y}, \quad B_2^0 = Z_0 = 0$$

$$V_1(y) = 0, \quad V_2(y) = V(x; y) \tag{15}$$

and the equations (13) take the form

$$(\Delta+k^2)\langle v\rangle = 0, \quad y > 0 \tag{16}$$

$$\frac{\partial}{\partial y}\langle v\rangle - \int G_{22}(x, x')R_{22}(x, x')\langle v(x')\rangle dx' = 0, \quad y = 0.$$

The equations (16) constitute the boundary value problem for a mean field with non-local boundary condition. A we know [4], in the classical case when the plane $y = 0$ is perfectly rigid ($Z = 0$) the reflection coefficient is equal to one. In our situation, energy is scattered by small inhomogenities of the boundary, so it is natural to seek some "effective" reflection coefficient of the average field.

We shall look for a solution of (16) in the form of a sum of incident wave and reflected wave with some reflection coefficient Λ of the average wave

$$\langle v\rangle = e^{ik(x\cos\alpha - y\sin\alpha)} + \Lambda e^{ik(x\cos\alpha + y\sin\alpha)}. \tag{17}$$

Substitution of (17) into the boundary condition (16) yields

$$\Lambda = \frac{ik\sin\alpha - I(\sin\alpha)}{ik\sin\alpha + I(\sin\alpha)} \tag{18}$$

where

$$I(\sin\alpha) = -\int_{-\infty}^{+\infty} G_{22}(x, x')R_{22}(x, x')e^{-ik(x-x')\cos\alpha}dx'. \tag{19}$$

Now, we have to find $G_{22}(x, x')$. To do that we use the Green formula and look for the Green function $\mathscr{G}(x, y; x', y')$ associated with operator M, that is

$$\begin{gathered}(\Delta+k^2)\mathscr{G} = \delta, \quad y > 0 \\ \frac{\partial \mathscr{G}}{\partial y} + Z_0\mathscr{G} = 0, \quad y = 0 \quad (Z_0 = 0).\end{gathered} \tag{20}$$

Using the image method we obtain

$$\mathscr{G}(x, y; x', y') = \frac{i}{4}[H_0^{(1)}(kR)+H_0^{(1)}(kR_0)]$$

where $H_0^{(1)}$ is the Hankel function and $R = [(x-x')^2+(y-y')^2]^{\frac{1}{2}}$, $R_0 = [(x-x')^2+(y-y')^2]^{\frac{1}{2}}$. Since $G_{22}(x, x') = \mathscr{G}(x, 0; x', 0)$

$$G_{22} = -\frac{i}{2}H_0^{(1)}(k|x-x'|). \tag{21}$$

The formulae (18), (19) and (21) give the solution of our problem; they express the "effective" reflection coefficient in terms of the correlation function of random inhomogenities of the plane. In order to obtain more explicit results it is necessary to make some assumptions concerning the form of the correlation function $R_V(x, x')$. Let us assume that

$$R_V(x, x') = \sigma^2 e^{-\beta|x-x'|} \tag{22}$$

where σ^2 denotes the variance of the random function $V(x; \gamma)$ and β is a positive constant which measures the statistical dependence of the values of the function $V(x, \gamma)$.

The integral (19) takes the form

$$I(\sin\alpha) = \frac{i\sigma^2}{2}\int_{-\infty}^{+\infty} H_0^{(1)}(k|x-x'|)e^{-\beta|x-x'|}e^{-ik(x-x')\cos\alpha}dx'. \tag{23}$$

After apropriate transformations this integral reduces to two Laplace transforms of the function $H_0^{(1)}(ku)$ and its value can be found explicitly. Namely,

$$I(\sin\alpha) = P+iQ \tag{24}$$

where

$$P = \frac{2\sigma^2(cs_1+ds_2)}{\pi(c^2+d^2)(s_1^2+s_2^2)}, \quad Q = \frac{c\sigma^2}{\pi(c^2+d^2)} \tag{25}$$

$$s_1 = \sin h\frac{\beta}{k}\cos h\,(\cos\alpha),$$

$$s_2 = \cos h\frac{\beta}{k}\sin h\,(\cos\alpha),$$

$$a = \beta^2+k^2\sin^2\alpha, \quad b = 2\beta k\cos\alpha$$

$$c = \frac{b}{2\sqrt{-\frac{a}{2}+\frac{1}{2}\sqrt{a^2+b^2}}}, \quad d = -\sqrt{-\frac{a}{2}+\frac{1}{2}\sqrt{a^2+b^2}}. \tag{26}$$

The modulus of the reflection coefficient is given by the formula

$$|A|^2 = AA^* = \frac{k^2 \sin^2 \alpha + II^* - 2kQ \sin \alpha}{k^2 \sin^2 \alpha + II^* + 2kQ \sin \alpha} \leqslant 1. \tag{27}$$

The modulus of the "effective" reflection coefficient is less than unity, as could have been expected, since a part of the energy is scattered. If $\sigma = 0$ (deterministic case — a rigid surface), than $A = 1$.

Let us assume, that $\alpha \to \frac{\pi}{2}$ (normal incidence). In this case:

$$s_1 = \sin h \frac{\beta}{k}, \quad s_2 = 0, \quad a = \beta^2 + k^2, \quad b = 0, \quad d = 0.$$

The parameter c is not defined $\left(c = \frac{0}{0}\right)$. Using the de l'Hospital rule and performing some approximations one obtains finally, that $c = \beta^{1/2}$. So, in the case of normal incidence we have (for this case we denote $I = I_1$)

$$I_1 = \frac{2\sigma^2 \sqrt{\beta}}{\pi\beta \sin h \frac{\beta}{k}} + i \frac{\sigma^2 \sqrt{\beta}}{\pi\beta} \tag{28}$$

and

$$|A|^2 = \frac{k^2 + I_1 I_1^* - 2k \operatorname{Im} I_1}{k^2 + I_1 I_1^* + 2k \operatorname{Im} I_1}. \tag{29}$$

This expression makes it possible to study the dependence of $|A|^2$ upon ratio $\frac{\beta}{k}$ (relation between the length of incident wave and radius of correlation of random impedance). For example, when $\beta k \ll 1$ (that is, the length of incident wave is greater less than the radius of correlation) then $\sin h^2 \frac{\beta}{k} \approx 0$ and $|A|^2 = 1$, which is in agreement with intuition and experiment.

The author wishes to express his appreciation to Professor J. B. Keller for suggesting the topic of this study and to Dr G. Papanicolau for fruitful discussions.

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