INVARIANT PHENOMENOLOGICAL THEORY OF THE MAGNETIZATION DENSITY OF INHOMOGENEOUS PRECESSION IN FERROMAGNETIC SUBSTANCES OF DOMAIN STRUCTURE

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The dispersion relations for the spin waves were derived, applying the classical phenomenological theory of spin waves to the case of uniaxial ferromagnetic substance of flat lamellar domain structure. Thus obtained results coincide with the ones obtained with the help of the formalism of approximate second quantization.

1. Introduction

The aim of the work is to determine the influence of ferromagnetic domain structure on the excitation spectrum, using the classical phenomenological theory of spin waves. This theory which is well developed for single-domain ferromagnetic substances [1, 2] requires that certain favoured coordinate systems are chosen and the crystal symmetry is specified. Invariant theories of the homogeneous and inhomogeneous precession of the magnetization density vector which were developed in papers [3, 4] still do not take into account the existence of the domain structure. On the other hand the influence of the domain structure on the spin waves energy has been treated in many papers, *e. g.* [5–9], using the method of approximate second quantization.

In this work it is shown that the majority of results (and first of all, the character of the dispersion relations) obtained with the method of the second quantization can be derived on the basis of the classical spin waves theory.

The underlying assumption is made that the magnetic energy of the crystal can be expressed in terms of the magnetization density vectors and their derivatives with respect to the space coordinates. The exchange energy, the energy of anisotropy and that of crystal in the external homogeneous magnetic field were taken into account. Energy of demagnetization was neglected. The conditions for the minimum of magnetic energy are given by

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the set of Euler-Lagrange equations. The solution of equations with the additional requirements of periodicity determines the domain structure and enables one to find the local effective field [11]. In the ground state, the magnetization density vectors are parallel to the direction of the local effective field. Excitations (precession of the magnetization density vector around the position of equilibrium) were described with the help of classical equation of motion, neglecting the attenuation. Calculations were carried out taking into account first terms of the Taylor expansion.

2. *General assumptions*

It is assumed that the magnetic energy F of the ferromagnetic substance is given by the expression:

$$F = \int_V f\{M_\alpha; M_{\alpha,\mu}\}dx \tag{1}$$

where $f\{...\}$ is the energy density, V-crystal volume, $M_\alpha = M_\alpha(x)$ — vector of the magnetization density, $M_{\alpha,\mu} = \frac{\partial M_\alpha}{\partial x_\mu}$ ($\alpha, \mu, ..., = 1, 2, 3$). All through the paper the convention of summation over the vector indices α, β, μ is applied. According to [10, 11] the local effective field is given by the formula:

$$H_\alpha^{\text{ef}} = -\frac{\delta F}{\delta M_\alpha} = -\frac{\partial f}{\partial M_\alpha} + \frac{\partial}{\partial x_\mu}\frac{\partial f}{\partial M_{\alpha,\mu}}. \tag{2}$$

The necessary condition for the functional minimum (1) with the side condition:

$$M_\alpha(x)M_\alpha(x) = M_0^2 \tag{3}$$

is, that the Euler-Lagrange equation is satisfied:

$$-\frac{\delta F}{\delta M_\alpha} = \lambda M_\alpha \tag{4}$$

or

$$H_\alpha^{\text{ef}} = \lambda\gamma_\alpha M_0, \tag{5}$$

where $\lambda = \lambda(x)$, $M_\alpha = M_\alpha(x) = \gamma(x)\, M_0$. Unit vector oriented along the direction of the magnetization density vector at the point x is denoted by $\gamma_\alpha(x)$. The additional periodicity condition is imposed on function $\gamma_\alpha(x)$ in the form:

$$\gamma_\alpha(x+\Delta) = \gamma_\alpha(x). \tag{6}$$

In the case, when vectors M_α undergo precession around the position of equilibrium (expressed by condition (4)) the following denotation is introduced:

$$M_\alpha \to M_\alpha(x, t) = \gamma_\alpha(x)M_0 + m_\alpha(x, t), \tag{7}$$

when

$$\gamma_\alpha(x)m_\alpha(x, t) = 0. \tag{8}$$

In this case the local effective field is a function of spatial variables x and of time t. In the first approximation of Taylor's expansion (linear terms in m_α, $m_{\alpha,\mu}$) the effective field can be expressed in the following way:

$$H_\alpha^{ef}(x, t) = \lambda\gamma_\alpha M_0 + P^{\alpha\beta} m_\beta + R_{\mu\nu}^{\alpha\beta} m_{\beta,\mu\nu}, \tag{9}$$

where

$$P^{\alpha\beta} = -\left.\frac{\partial^2 f}{\partial M_\alpha \partial M_\beta}\right|_{M_\alpha = \gamma_\alpha M_0}$$

$$R_{\mu\nu}^{\alpha\beta} = \left.\frac{\partial^2 f}{\partial M_{\alpha,\mu} \partial M_{\beta,\nu}}\right|_{M_{\varrho,\sigma} = \gamma_{\varrho,\sigma} M_0}. \tag{10}$$

Let us introduce denotation

$$h_\alpha = h_\alpha(x, t) = P^{\alpha\beta} m_\beta + R_{\mu\nu}^{\alpha\beta} m_{\beta,\mu\nu}, \tag{11}$$

then

$$H_\alpha^{ef}(x, t) = \lambda\gamma_\alpha M_0 + h_\alpha. \tag{12}$$

So far, the dependence of the energy density $f\{...\}$ on vectors M_α, $M_{\alpha,\mu}$ was not given. Taking into account the exchange energy, energy of anisotropy and Zeeman's energy this dependence for the uniaxial ferromagnetic substances will have the form

$$f\{M_\alpha; M_{\alpha,\mu}\} = A^{\alpha\beta} M_\alpha M_\beta + B_{\mu\nu}^{\alpha\beta} M_{\alpha,\mu} M_{\beta,\nu} - H_\alpha^0 M_\alpha. \tag{13}$$

Assuming that the axis x_3 is the magnetic easy axis of uniaxial ferromagnetic substance one can define tensors $A^{\alpha\beta}$, $B_{\mu\nu}^{\alpha\beta}$ in the following way:

$$A^{\alpha\beta} = -\tfrac{1}{2} K_1 \delta_{\alpha 3} \delta_{\beta 3}; \quad B_{\mu\nu}^{\alpha\beta} = \tfrac{1}{2} C_\mu \delta_{\alpha\beta} \delta_{\mu\nu}. \tag{14}$$

In order to be able to solve the Euler-Lagrange equations one has to specify the type of the observed domain structure in the ferromagnetic substance. Let us assume a flat lamellar structure of Shirobokov type [12]. For this type of structure vectors $M_\alpha = \gamma_\alpha M_0$ remain in the plane $(x_2 0 x_3)$ and their rotation takes place around axis x_1. Let us denote by $\varphi = \varphi(x_1)$ the angle between the vector M_α and the magnetic easy axis x_3. Then

$$\gamma_\alpha(x) = \gamma_\alpha(x_1) = (0, \gamma_2, \gamma_3); \quad \gamma_2 = \sin\varphi; \quad \gamma_3 = \cos\varphi,$$

$$\varphi(x_1 + \Delta) = \varphi(x_1) + \pi. \tag{15}$$

Solution of the Euler-Lagrange equations for this type of the domain structure has been discussed in details in papers [12–15] and in the case of $H_\alpha^0 = 0$ has the form:

$$\gamma_3 = \cos\varphi(x_1) = \operatorname{sn} qx_1, \gamma_2 = \sin\varphi(x_1) = \operatorname{cn} qx_1 \tag{16}$$

where

$$q = k^{-1}\sqrt{\frac{K_1}{C_1}}. \tag{17}$$

Modulus of the eliptical integral k appearing in (17) is given by the relation:

$$K(k) = \frac{\Delta}{2k}\sqrt{\frac{K_1}{C_1}}. \tag{18}$$

In (17) the complete eliptical integral of the first order is denoted by $K(k)$ and by Δ — the width of domain. Function $\lambda = \lambda(x_1)$ defined by formula (4) and (5), in the discussed case has the form:

$$\lambda = \lambda(x_1) = K_1(2\cos^2\varphi - k^{-2}) = K_1(2\,\mathrm{sn}^2\, qx_1 - k^{-2}). \tag{19}$$

Moreover

$$\dot{\varphi}^2 = \left(\frac{d\varphi}{dx_1}\right)^2 = \frac{K_1}{C_1}(k^{-2} - \mathrm{sn}^2\, qx_1). \tag{20}$$

3. Equation of motion

Precession of the magnetization density vector is given by the equation:

$$-\frac{\partial M_\alpha}{\partial t} = g\varepsilon_{\alpha\beta\varrho}H^{\mathrm{ef}}_\beta M_\varrho, \tag{21}$$

where $\varepsilon_{\alpha\beta\varrho}$ is the Levi-Civita tensor, g — gyromagnetic coefficient. Considering (7), (9), (11) and (12) and taking into account the linear terms in m_α, $m_{\alpha,\mu}$ one gets:

$$-g^{-1}\frac{dm_\alpha}{dt} = M_0\varepsilon_{\alpha\beta\sigma}(\lambda\gamma_\beta m_\sigma + P^{\beta\varrho}m_\varrho\gamma_\sigma + R^{\beta\varrho}_{\mu\nu}m_{\varrho,\mu\nu}\gamma_\sigma). \tag{22}$$

Let us make Fourier transformation of vector m_α components:

$$m_\alpha = \int d\omega \int d\kappa \tilde{m}_\alpha e^{i(\omega t - \kappa\varrho)} \tag{23}$$

where

$$\tilde{m}_\alpha = \tilde{m}_\alpha(x_1; \kappa, \omega); \qquad \kappa = (0, k_2, k_3), \qquad \varrho = (0, x_2, x_3). \tag{24}$$

Substituting (23) into (22) and taking into account (13)–(16) one gets the set of equations:

$$\begin{aligned}
&-i\omega g^{-1}\tilde{m}_1 + M_0\gamma_3(\lambda + C_\perp\kappa^2)\tilde{m}_2 - M_0\gamma_2(\lambda + C_\perp\kappa^2)\tilde{m}_3 + \\
&+\frac{1}{2}K_1\gamma_2\tilde{m}_3 = M_0C_1\left(\gamma_3\frac{d^2\tilde{m}_2}{dx_1^2} - \gamma_2\frac{d^2\tilde{m}_3}{dx_1^2}\right) \\
&M_0\gamma_3(\lambda + C_\perp\kappa^2)\tilde{m}_1 + i\omega g^{-1}\tilde{m}_2 = M_0\gamma_3C_1\frac{d^2\tilde{m}_1}{dx_1^2} \\
&M_0\gamma_3(\lambda + C_\perp\kappa^2)\tilde{m}_1 - i\omega g^{-1}\tilde{m}_3 = M_0\gamma_2C_1\frac{d^2\tilde{m}_1}{dx_1^2}.
\end{aligned} \tag{25}$$

In (25) a denotation $C_1\kappa^2 = C_2k_2^2 + C_3k_3^2$ is used. Because of the condition (8) not all components $\tilde{m}_\alpha$ in (25) are linearly independent. In order to eliminate one of the com-

ponents $\tilde{m}_\alpha$ let us shift to the local coordinates system, for which axis x_3 is parallel to the unit vector γ_α. With this aim let us rotate $\tilde{m}_\alpha$ around axis x_1

$$\tilde{m}'_\alpha = T_{\alpha\beta}\tilde{m}_\beta. \tag{26}$$

Because of (15) and (16) matrix $T_{\alpha\beta}$ has the form

$$T_{\alpha\beta} = T_{\alpha\beta}(x_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma_3 & -\gamma_2 \\ 0 & \gamma_2 & \gamma_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \operatorname{sn} qx_1 & -\operatorname{cn} qx_1 \\ 0 & \operatorname{cn} qx_1 & \operatorname{sn} qx_1 \end{pmatrix}. \tag{27}$$

Condition (8) is equivalent to the condition $\tilde{m}'_3 = 0$ in the local coordinates system. Equations of motion (25) in the local coordinate system have the form:

$$\frac{d^2\tilde{m}'_1}{dx_1^2} - C_1^{-1}\{\lambda + C_\perp\kappa^2\}\tilde{m}'_1 - i\frac{\omega}{gM_0C_1}\tilde{m}'_2 = 0$$

$$\frac{d^2\tilde{m}'_2}{dx_1^2} - C_1^{-1}\{\lambda + C_\perp\kappa^2 + K_1\gamma_2^2 - C_1\dot{\varphi}^2\}\tilde{m}'_2 + i\frac{\omega}{gM_0C_1}\tilde{m}'_1 = 0. \tag{28}$$

Substituting into (28) relations (16), (19) and (20) and changing variables:

$$y = qx_1 = k^{-1}\sqrt{\frac{K_1}{C_1}}\,x_1, \tag{29}$$

one gets:

$$\frac{d^2\tilde{m}'_1}{dy^2} - \{2k^2\operatorname{sn}^2 y + k^2K_1^{-1}C_\perp\kappa^2 - 1\}\tilde{m}'_1 - i\Omega\tilde{m}'_2 = 0$$

$$\frac{d^2\tilde{m}'_2}{dy^2} - \{2k^2\operatorname{sn}^2 y + k^2K_1^{-1}C_\perp\kappa^2 - 1 + k_0^2\}\tilde{m}'_2 + i\Omega\tilde{m}'_1 = 0 \tag{30}$$

where

$$\Omega = \frac{k^2}{gM_0K_1}\omega; \quad k_0^2 = 1 - k^2. \tag{31}$$

Due to the fact that both equations of set (30) have analogical form their solutions can differ only by the constant factor *i. e.*

$$\tilde{m}'_1 = S_1 g(y); \quad \tilde{m}'_2 = S_2 g(y). \tag{32}$$

After substitution of (32) into (30) each equation of set (30) transforms into the form of Lame equation

$$\frac{d^2g}{dy^2} - \{2k^2\operatorname{sn}^2 y + A_0\}g = 0 \tag{33}$$

where parameter A_0 in equation (33) is related to S_1 and S_2 in the following way:

$$A_0 = k^2 K_1^{-1} C_\perp \kappa^2 - 1 + i \frac{S_2}{S_1} \Omega$$

$$A_0 = k^2 K_1^{-1} C_\perp \kappa^2 - 1 + k_0^2 - i \frac{S_1}{S_2} \Omega. \tag{34}$$

4. Evaluation of the dispersion relations

Expressions (34) represent a set of equations with respect to S_1 and S_2. This set can have non-trivial solutions only when its determinant is equal zero. Such condition allows one to determine Ω:

$$\Omega = \sqrt{(k^2 K_1^{-1} C_\perp \kappa^2 - 1 - A_0)(k^2 K_1^{-1} C_\perp \kappa^2 - 1 + k_0^2 - A_0)}. \tag{35}$$

In order to determine parameter A_0 appearing in (35) it is necessary to know the solution of Lame equation (33). Solution of this equation has the form [16–18]:

$$g(y) = \frac{\vartheta_1 \left\{ (y \pm y_1) \frac{\pi}{2K} \right\}}{\vartheta_4 \left\{ \frac{y\pi}{2K} \right\}} \exp \{ \pm y Z(y_1) \}, \tag{36}$$

where

$$Z(y_1) = E(y_1) - \frac{E}{K} y_1, \tag{37}$$

while ϑ_1, ϑ_4 are the Weierstrasse theta functions. Symbols $E(y_1)$, E denote incomplete and complete eliptical integrals of the second kind and K stands for the complete eliptical integral of the first kind.

Parameter A_0 can be expressed in terms of y_1 as follows:

$$A_0 = -1 - k^2 \operatorname{cn}^2 y_1. \tag{38}$$

Substituting (38) into (35) one gets

$$\Omega = \sqrt{(k^2 K_1^{-1} C_\perp \kappa^2 + k^2 \operatorname{cn}^2 y_1)(k_0^2 + k^2 K_1^{-1} C_\perp \kappa^2 + k^2 \operatorname{cn}^2 y_1)}. \tag{39}$$

Only these solutions of equation (33) in the form of (36) have physical sense, which are the continuous and enique functions of variable y. Such conditions determine the permissible values of parameter y_1. Let parameter y_1 be expressed as follows:

$$y_1 = u + iv. \tag{40}$$

A condition for solution (36) to be limited is:

$$\operatorname{Re} Z(u + iv) = 0. \tag{41}$$

Condition (41) is satisfied for:

$$u = nK; n = 0, 1, 2, \ldots \tag{42}$$

When u takes values given in (42), parameter A_0 is a function of the imaginary part v of parameter y_1. This dependence is different for the even and odd values of n in (42). Let us consider the case:

A) $u = 2nk$;

In this case one gets:

$$\mathrm{cn}^2\, y_1 = \mathrm{nc}^2(v, k_0) = 1 + \frac{\mathrm{sn}^2\,(v, k_0)}{\mathrm{cn}\,(v, k_0)} \tag{43}$$

$$\Omega_A = \left[\left(k^2 K_1^{-1} C_\perp \kappa^2 + k^2 + k^2 \frac{\mathrm{sn}^2\,(v, k_0)}{\mathrm{cn}^2\,(v, k_0)}\right)\left(k^2 K_1^{-1} C_\perp \kappa^2 + 1 + k^2 \frac{\mathrm{sn}^2\,(v, k_0)}{\mathrm{cn}^2\,(v, k_0)}\right)\right]^{1/2} \tag{44}$$

$$g_A(y) = \frac{\vartheta_1\left\{(y + iv)\dfrac{\pi}{2K}\right\}}{\vartheta_4\left\{\dfrac{y\pi}{2K}\right\}} \exp\left\{iy\left[\frac{\mathrm{sn}\,(v, k_0)}{\mathrm{cn}\,(v, k_0)} - \frac{\pi v}{2KK'} - Z(v, k_0)\right]\right\}. \tag{45}$$

In the real situation [12–15] the modulus of the eliptical integral k is very close to unity ($k \approx 1$, $k_0 \approx 0$, $\mathrm{sn}(v, k_0) \to \mathrm{sn}\, v$). Taking into account this fact and the relation (29) one gets the approximate expression for Ω_A, $g_A(y)$

$$\Omega_A = K_1^{-1}(C_1 k_1^2 + C_2 k_2^2 + C_3 k_3^2) + 1, \tag{46}$$

$$g_A(x_1) = \frac{\vartheta_1\left\{(x_1\sqrt{K_1/C_1} + iv)\dfrac{\pi}{2K}\right\}}{\vartheta_4\left\{x_1\sqrt{K_1/C_1}\,\dfrac{\pi}{2K}\right\}} \exp(ik_1 x_1), \tag{47}$$

where the component k_1 of the wave vector is defined as:

$$k_1 = \sqrt{\frac{K_1}{C_1}}\,\frac{\mathrm{sn}\,(v, k_0)}{\mathrm{cn}\,(v, k_0)} \approx \sqrt{\frac{K_1}{C_1}}\,\mathrm{tg}\, v. \tag{48}$$

With the accuracy to the first order with respect to k_0 function $g_A(x_1)$ can be written in the form:

$$g_A(x_1) = \{\mathrm{sn}\,(k^{-1}\sqrt{C_1/K_1}\, x_1) - i\sqrt{C_1/K_1}\, k_1\} \exp(ik_1 x_1). \tag{49}$$

Let us now consider the case

B) $u = (2n+1)k$

In this case one has

$$\mathrm{cn}^2\, y_1 = -k_0^2 \frac{\mathrm{sn}^2\,(v, k_0)}{\mathrm{dn}^2\,(v, k_0)} = -k_0^2\,\mathrm{sd}^2\,(v, k_0), \tag{50}$$

$$\Omega_B = \sqrt{(k^2 K_1^{-1} C_\perp \kappa^2 - k^2 k_0^2 \operatorname{sd}^2(v, k_0))(k^2 K_1^{-1} C_\perp \kappa^2 + k_0^2 - k^2 k_0^2 \operatorname{sd}^2(v, k_0))}, \quad (51)$$

$$g_B(y) = -\frac{\vartheta_1\left\{(y+iv)\frac{\pi}{2K}\right\}}{\vartheta_4\left\{\frac{y\pi}{2K}\right\}} \exp\left\{iy\left[k_0^2 \operatorname{sd}(v, k_0) - \frac{\pi v}{2KK'} - Z(v, k_0)\right]\right\}. \quad (52)$$

For the modulus value $k \to 1$ one gets the approximate expressions:

$$\Omega_B = K_1^{-1} C_\perp \kappa^2$$

$$g_B(y) = e^{\pm iv} \operatorname{cn} y. \quad (53)$$

5. *Concluding remarks*

The classical spin wave theory was applied to the case of ferromagnetic substances of flat lamellar domain structure. The dependence of the inhomogeneous precession frequency of magnetization density vector on the wave vector was derived. The obtained results coincide with the results obtained with the help of the second quantization method [6, 7, 9] in the lowest approximation of the Holstein-Primakoff representation *i. e.* neglecting the spin waves interaction.

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